Emergence of non-abelian statistics from an abelian model

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EPSRC Engineering and Physical Sciences Research Council

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Simulating Anyons

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- **Anyons** can be encoded in 2D systems:
  - **Superconducting** electrons in a strong magnetic field (Fractional Quantum Hall Effect)
  - **Lattice** systems (Kitaev's toric code/hexagonal lattice, Wen, Ioffe, Freedman-Nayak-Shtengel, Bombin-Delgado)

- Anyons are **quasiparticles** that can be identified and transported by **local operators**.
- The quantum states of the corresponding systems are **highly entangled** with long range correlations.
- Anyonic statistics are possible due to entanglement in the underlying system.

Topological properties of Toric Code

Particles: 1, e (abelian), m (abelian), ϵ (fermion)

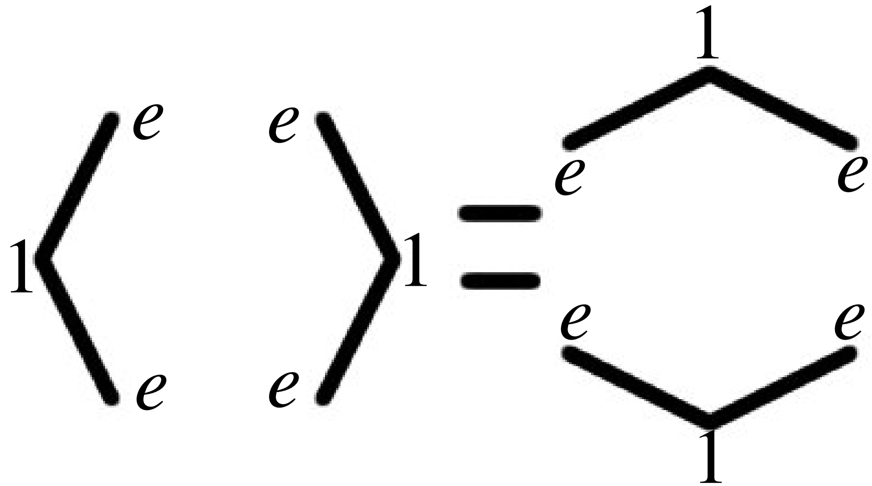
Fusion rules:

$$m\times m=e\times e=\epsilon\times\epsilon=1, e\times m=\epsilon\ , e\times\epsilon=m\ , m\times\epsilon=e$$

Braiding:

$$R^{1}_{\epsilon\,\epsilon}=-1, (R^{m}_{\epsilon\,e})^{2}=(R^{e}_{\epsilon\,m})^{2}=(R^{\epsilon}_{e\,m})^{2}=-1, R^{1}_{ee}=R^{1}_{mm}=1$$

F-matrices:



Topological properties of Ising model

Particles: 1, σ (non-abelian), ψ (fermion)

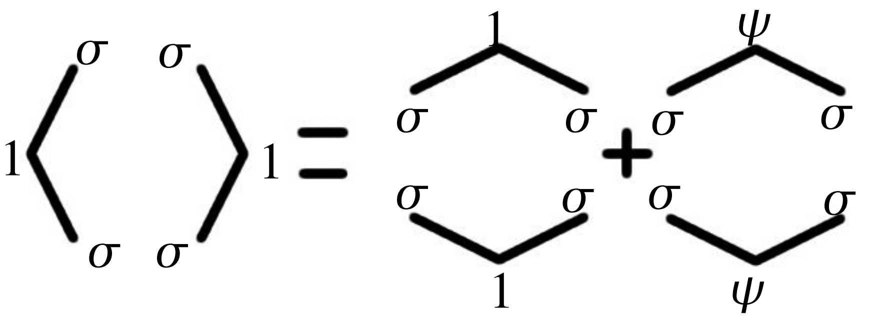
Fusion:

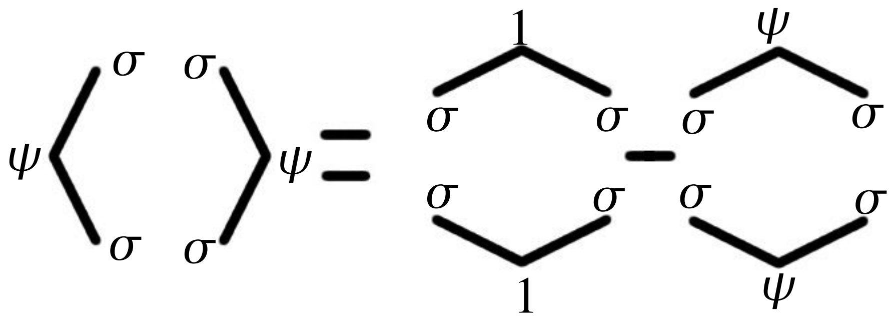
$$\sigma\times\sigma=1+\psi\,,\,\psi\times\psi=1,\sigma\times\psi=\sigma$$

Braiding:

$$R^1_{\sigma\sigma}=e^{-i\pi/4},\,(R^{\psi}_{\sigma\sigma})^2=-e^{-i\pi/4},\,(R^{\sigma}_{\psi\sigma})^2=-1,R^1_{\psi\psi}=-1$$

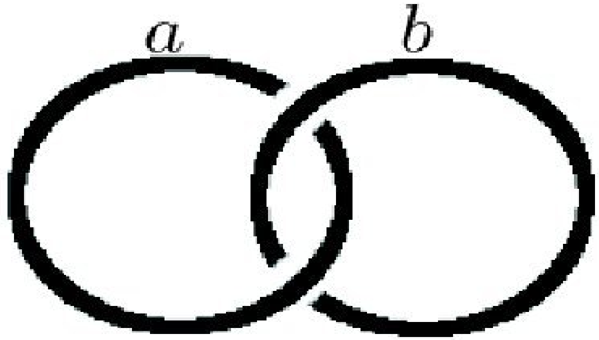
F-matrices:





S-matrices

- The S-matrix of a model has elements defined by

$$S_{ab} = \frac{1}{D}$$ 

- For the toric code and Ising model these are

$$S^{Ising} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \qquad S^{Toric} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

S-matrices

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- These S-matrices may be related by

$$S_{11}^{Ising} = S_{11}^{Toric} \qquad S_{\psi\psi}^{Ising} = S_{\epsilon\epsilon}^{Toric} \qquad S_{\psi\sigma}^{Ising} = \frac{S_{\epsilon e}^{Toric} + S_{\epsilon m}^{Toric}}{\sqrt{2}}$$

$$S_{1\sigma}^{Ising} = \frac{S_{1e}^{Toric} + S_{1m}^{Toric}}{\sqrt{2}} \qquad S_{\sigma\sigma}^{Ising} = \frac{S_{ee}^{Toric} + S_{em}^{Toric} + S_{me}^{Toric} + S_{mm}^{Toric}}{\sqrt{2}}$$

- These relation suggest that σ's are, in some way, like superpositions of e's and m's.
- We can find a similar result by looking at the lattices on which these anyon models reside.

Spin Lattice Models

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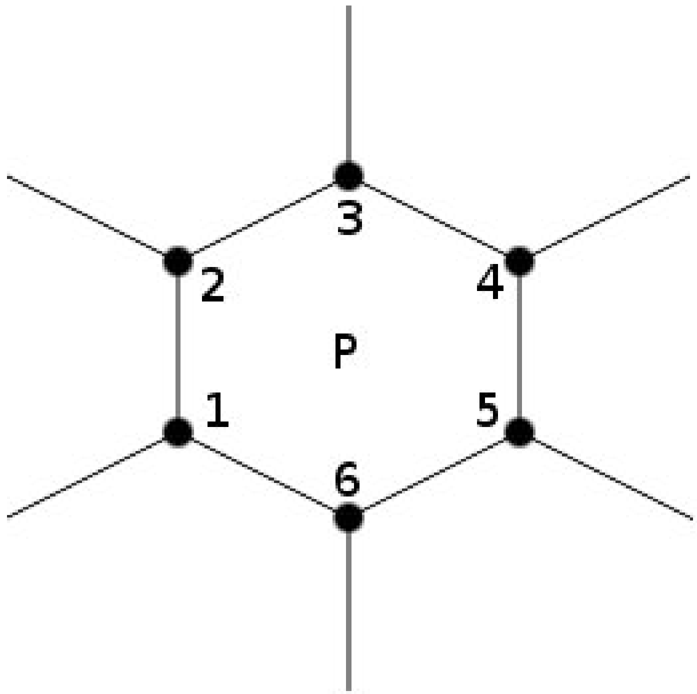
- **Alexei Kitaev's** honeycomb lattice model:
  - Defined on a honeycomb lattice with qubits at the vertices;
  - Plaquette operators

$$W_P = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z$$



[A. Kitaev, Ann. Phys. 321, 2 (2006)]

Spin Lattice Models

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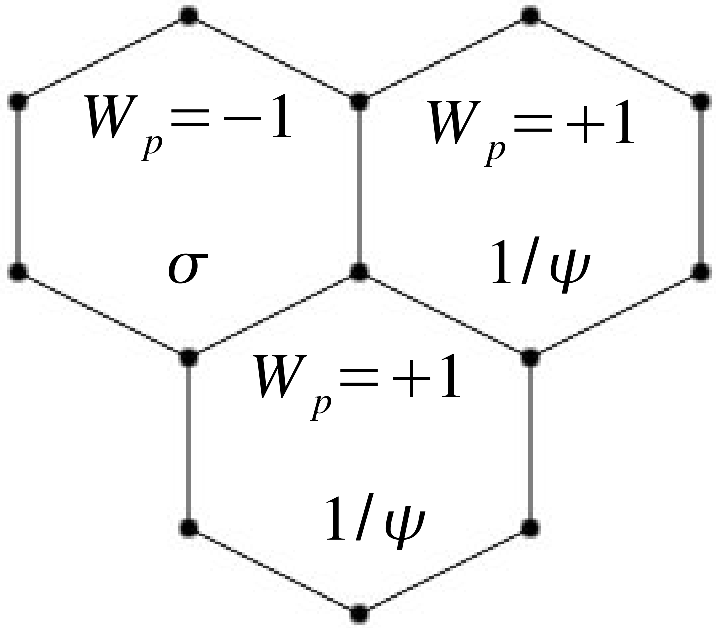
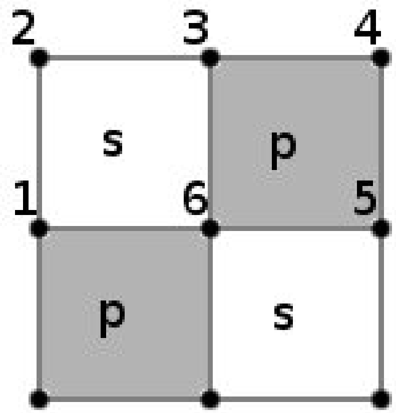
- The honeycomb lattice is expected to support the Ising model, but the details on the spin representation are illusive.
- We do know that σ particles should be detected by the plaquette operators. ψ particles should not.



Spin Lattice Models

- Wen plaquette model:
  - Defined on a bicoloured square lattice with qubits at the vertices;
  - Supports the abelian toric code model;
  - Plaquette operators



$$A_s = \sigma_1^x \sigma_2^y \sigma_3^x \sigma_6^y$$
$$B_p = \sigma_6^x \sigma_3^y \sigma_4^x \sigma_5^y$$

detect anyons on each plaquette;

  - e anyons live on s plaquettes, m anyons on p plaquettes.

[X. G. Wen, Phys. Rev. Lett. 90, 016803 (2003).]

Spin Lattice Models

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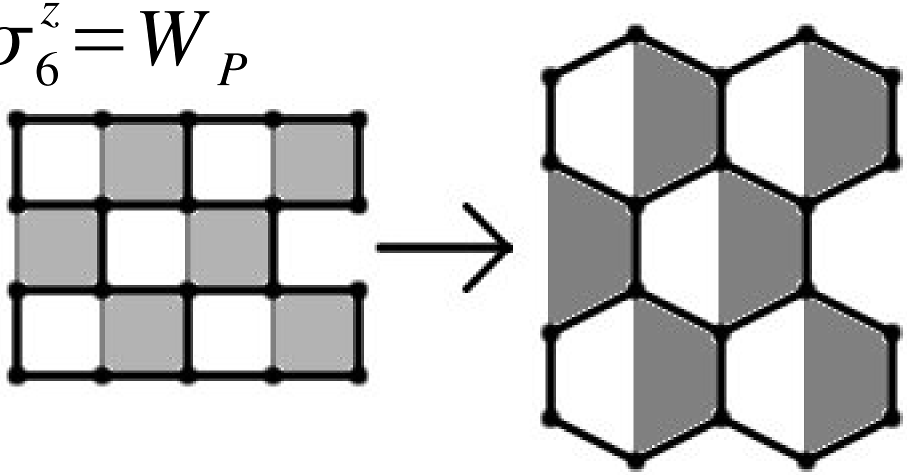
- Consider forming composite plaquettes.
- The composite plaquettes form a honeycomb lattice.
- We define the plaquette operator to be the product of those for the component plaquettes

$$A_s B_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z = W_P$$



Spin Lattice Models

$$A_s B_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z = W_P$$

- This shows an interesting relation between the honeycomb lattice model and Wen plaquette model:
  - σ particles can be expressed in terms of e's and m's;
  - The toric code fermion, like the Ising fermion, cannot be detected by W_P.
- So perhaps it is possible to build a lattice representation of Ising model particles from toric code particles.

Superposition principle: Particles

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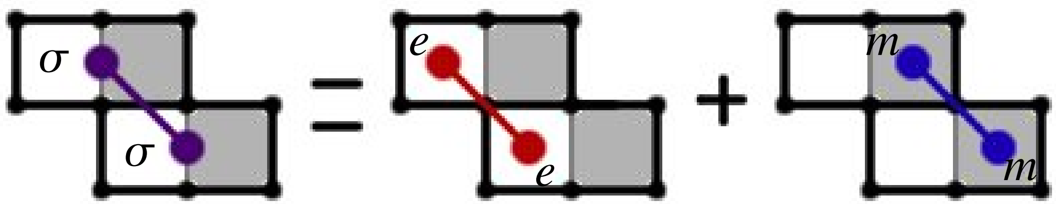
- We will now attempt to do this
  - We identify the fermions of the toric code with those of the Ising model, $\psi \equiv \epsilon$;
  - We identify a pair of σ's with the superposition

$$|\sigma\,\sigma;j\rangle \equiv \frac{1}{\sqrt{2}}\left(|ee\rangle + j|mm\rangle\right)$$

  the relative sign j=+/-1 is a non-local property of the pair. What is its significance?



Superposition principle: Fusion

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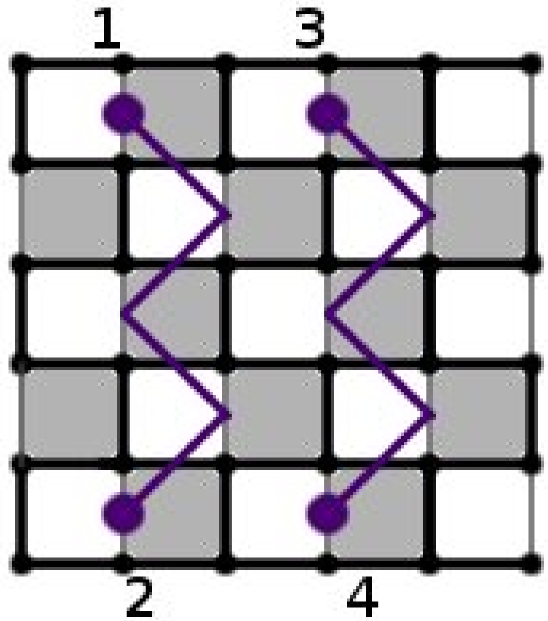
- The state of two pairs is

$$|\sigma_1\sigma_2;j\rangle|\sigma_3\sigma_4;j\rangle=\frac{1}{2}(|e_1e_2e_3e_4\rangle+|m_1m_2m_3m_4\rangle+j|e_1e_2m_3m_4\rangle+j|m_1m_2e_3e_4\rangle)$$

- Changing the fusion basis

$$|\sigma_1\sigma_2;j\rangle|\sigma_3\sigma_4;j\rangle=\frac{1}{\sqrt{2}}(|\sigma_1\sigma_3;1\rangle|\sigma_2\sigma_4;1\rangle+j|\sigma_1\sigma_3;\psi\rangle|\sigma_2\sigma_4;\psi\rangle)$$

- Note that the relative sign is determined by those of the initial pairs.



Superposition principle: Fusion

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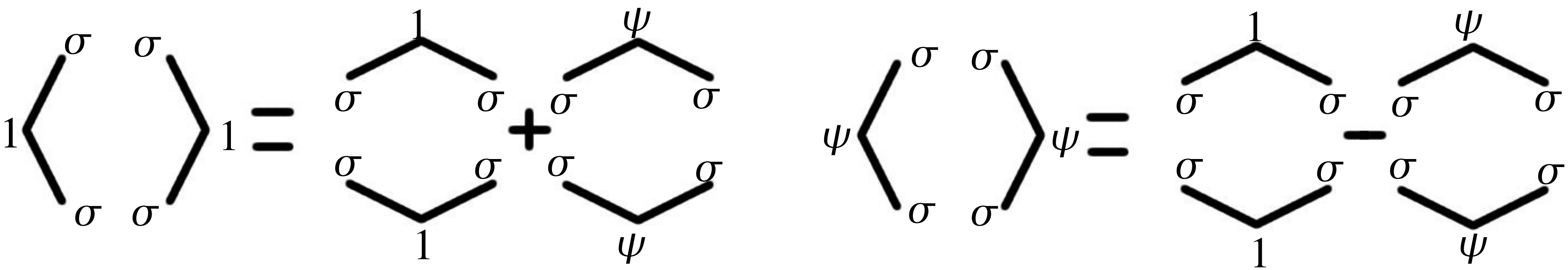
$$|\sigma_1\sigma_2;j\rangle|\sigma_3\sigma_4;j\rangle=\frac{1}{\sqrt{2}}\left(|\sigma_1\sigma_3;1\rangle|\sigma_2\sigma_4;1\rangle+j|\sigma_1\sigma_3;\psi\rangle|\sigma_2\sigma_4;\psi\rangle\right)$$

- From the F-matrices we know



So states with j=+1 act like a pair created from the vacuum and those with j=-1 act like a pair created from a fermion.

Superposition principle: Braiding

- We need to move σ particles.

- They cannot simply be moved by Pauli operations.

- Instead we use

$$C_s = (1 + A_s) \otimes 1_q + (1 + A_s) \otimes \sigma_q^x$$

and

$$D_i = \sigma_i^x \otimes |0\rangle\langle 0|_q + \sigma_i^y |1\rangle\langle 1|_q$$

Superposition principle: Braiding

- The state of two strings is

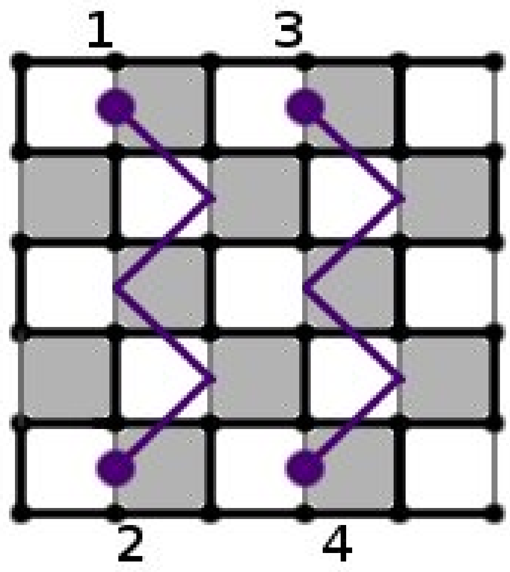
$$|\sigma_1\sigma_2; j\rangle|\sigma_3\sigma_4; j\rangle = \frac{1}{2}(|e_1e_2e_3e_4\rangle + |m_1m_2m_3m_4\rangle + j|e_1e_2m_3m_4\rangle + j|m_1m_2e_3e_4\rangle)$$

- Braid the σ at 1 with that at 3.

$$(R^1_{\sigma\sigma})^2 = 1, (R^\psi_{\sigma\sigma})^2 = -1$$

so

$$R^2_{1,3}|\sigma_1\sigma_2; j\rangle|\sigma_3\sigma_4; j\rangle = |\sigma_1\sigma_2; -j\rangle|\sigma_3\sigma_4; -j\rangle$$



- It is easy to verify that

$$(R^\sigma_{\sigma\psi})^2 = -1, R_{\psi\psi}1 = -1$$

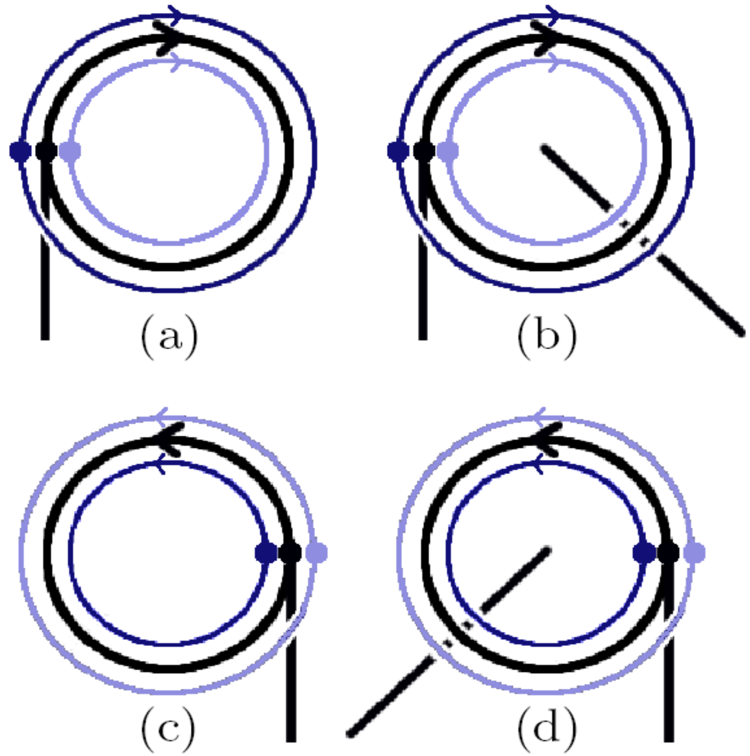
Framing

- Phase can be added using framing.

- This uses the operations
  $E_i = 1_i \otimes |+\rangle\langle+| + i\sigma_i^z \otimes |-\rangle\langle-|$
  applied either side of particles.

- A phase of $e^{i\pi/8}$ or $e^{-i\pi/8}$ can then be applied.

- This accounts for braiding and a twist.



Conclusions

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- From the S-matrices and lattice models, we see that relations exist between the toric code and Ising anyons model.

- Using a superposition principle, we can demonstrate braiding and fusion properties of the Ising model with toric code particles. This demonstrates and uses non-local properties encoded with the toric code states.

- Could a similar thing be done for other anyon models?