Pattern of zeros-
a joint work with X.-G. Wen

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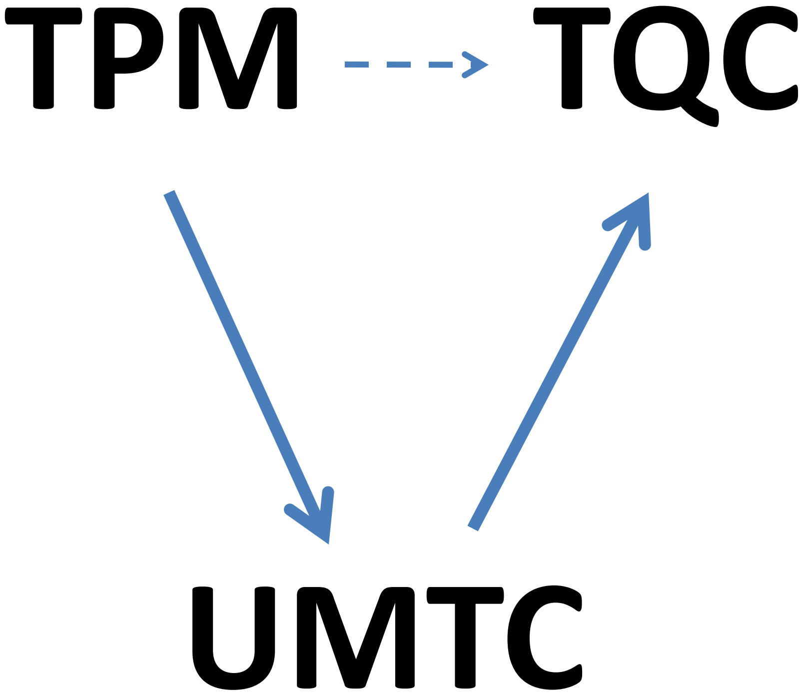
Microsoft Station Q

UCSB


TPM
TQC
UMTC

Two Kinds of Model Systems

- String-net condensation---doubled MTCs

  Mathematically well-understood, Physically not clear

- Trial wave functions---chiral MTCs

  Physically in better shape (FQH liquids)

Electrons on S^2

Thomson's Problem:

Configuration of N-electrons on S^2 minimizing the total potential energy

$$E_N = \sum 1/d_{ij}, \quad d_{ij} = \text{distance between } i,j$$

What happens if $N \to \infty$? $\Psi(z_i, \bar{z}_i)$

Quantum phases of matter

Given a set of wave functions W.F.={$\Psi(s_i)$}

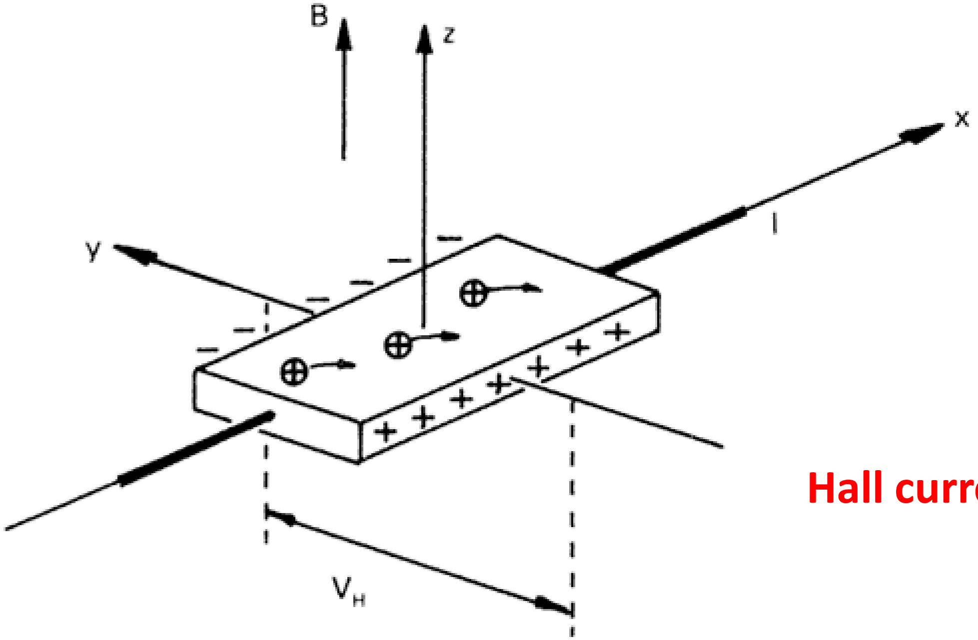
When does W.F. represent a topological phase of matter?

At least thermodynamic limit exists

with an energy gap

If so, which one?

Electrons in a flatland

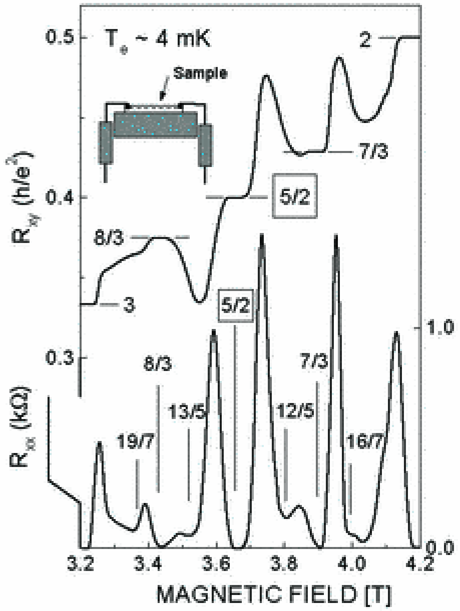


Hall current I_H

Energy levels for electrons are called Landau levels,
the filling fraction ν = # of electrons/# flux

Non-Abelian anyons in real life: FQHE?

Fig. 1, Pan et al



What are electrons doing at the plateaus?

ν=1/3

$$\psi_{1/3} = \prod_{i<j} (z_i - z_j)^3 e^{-\sum z_i \bar{z}_i / 4}$$

R. Laughlin

ν=5/2 ?

$$\psi_{5/2} = Pf\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^2 e^{-\sum z_i \bar{z}_j / 4}$$

Moore-Read

FQH liquids in LLL

- Chirality:

  $\Psi(z_1,\ldots,z_N)$ is a polynomial (Ignore Gaussian)
- Statistics:

  symmetric for bosons or anti-symmetric for fermions
- Translation invariant
- Filling fraction:

  $\nu=\lim N/N_\phi$, where N_ϕ is flux or maximum degree of any z_i

Pattern of zeros

W.F.s vanish at certain powers when particles in clusters approach to each other, and when clusters approach to each other.

These powers should be consistent to represent the same local physics, and encode many, possibly all, topological properties of the system.

Bosonic Laughlin States

$q\ even$ $\Psi_{1/q} = \prod_{i<j} (z_i - z_j)^q$

Then $S_a = qa(a-1)/2$

All zeros live on particles

Fuse a-particles

Given a-particles at $\{z_i, \ i=1,\ldots,a\}$

write $z_i = z_1^{(a)} + \lambda\, \xi_i$, where $z_1^{(a)} = (\sum\, z_i)/a$

, and normalize $\sum |\xi_i|^2 = 1$

Imagine z_i as vertices of a simplex, then $z_1^{(a)}$ is the barycenter of the simplex. As λ ->0,

$z_i \to z_1^{(a)}$ keeping the same shape.

Sphere S^{2a-1} (S^{2a-3} as $\sum \xi_i = 0$) of $\{\xi_i\}$ parameterizes the shape of the simplices (or a-particles).

Given W.F.={$\Psi(z_1,\ldots,z_N)$}, translation invariant symmetric polynomials of z_i

Substitute $z_1^{(a)}+\lambda\, \xi_i$ into $\Psi\,(z_i)$, expand the polynomial into a polynomial of λ,

$\Psi(z_i)=\lambda^{S_a}\, \Psi(z_1^{(a)},\xi_1,\ldots,\xi_a;\, z_{a+1},\ldots,z_N)$ +o(λ^{S_a}),

where S_a is the minimal power of λ.

The infinite sequence {S_a} will be called the pattern of zeros of the W.F. Note S_1=0

Relation to CFT

In CFT approach to FQH,

Let V_e be the electron operator and $V_a=(V_e)^a$ with scaling dimension h_a, then

$$S_a= h_a-a\, h_1$$

Unique fusion condition

Take a-variables z_i fusing them to $z_1^{(a)}$

The resulting polynomials (coefficients of λ^k)

$$\Psi^k(z_1^{(a)},\xi_1,\ldots,\xi_a;z_{a+1},\ldots,z_N)$$

depend on the shape of $\{z_i\}$, ie $\{\xi_i\} \in S^{2a-3}$

If the resulting polynomials of $z_1^{(a)},z_{a+1},\ldots,z_N$ for each degree k of λ span $\leq$ 1-dim vector spaces for all choices, then we say the W.F. satisfies the UFC.

Derived Polynomials

Given $\Psi(z_1,\ldots,z_N)$, if all variables are fused to new variables $z_i^{(a)}$. If UFC is satisfied, then the resulting new polynomial $P(z_i^{(a)})$ is well-defined, and called the Derived polynomials.

Derived polynomials for Laughlin states:

$$P_{1/q} = \prod_{a<b} \prod_{i,j} (z_i^{(a)} - z_j^{(b)})^{qab} \prod_{a} \prod_{i<j} (z_i^{(a)} - z_j^{(a)})^{qa^2}$$

n-cluster form

If there exists an n>0 such that for any n|N,

$$\Psi(z_i)=\prod\_\{k<l\} (z_k^{(n)}-z_l^{(n)})^q$$

Then W.F. has the n-cluster form (nCF)

nCF reduces pattern of zeros to a finite problem:

$S_{a+kn}=S_a+kS_n+kma+k(k-1)mn/2$, where $m=\nu^{-1}$ n

Main Theorem

If translation invariant symmetric polynomials W.F.={Ψ (z_i)} satisfy both UFC and nCF, then

1) S_{a+b}-S_a-$S_b \geq 0$
2) S_{a+b+c}-S_{a+b}-S_{b+c}-S_{c+a} +S_a +S_b +$S_c \geq 0$
3) S_{2a} even
4) mn even
5) $2S_n$=0 mod n
6) S_{3a}-S_a even

D_{ab} labeling of Pattern of zeros

For any a, b, fuse a-variables to $z_1^{(a)}$, and b-variables to $z_1^{(b)}$, then fuse $z_1^{(a)}$ and $z_1^{(b)}$

$$\Psi \sim (z_1^{(a)}-z_1^{(b)})^{D_{ab}}\, \Psi',$$

where $\sim$ means up to a non-zero scalar and higher order zeros

Pattern of zeros $\{S_a\}$ can be labeled equivalently by $\{D_{a,b}\}$

Outline of Proof

$\{D_{a,b}\}$ and $\{S_a\}$ are equivalent:

$$D_{ab}=S_{a+b}-S_a-S_b,\ S_a=\sum_1^{a-1} D_{b,1}$$

Properties of D_{ab}

1) $D_{ab}=D_{ba}$
2) $D_{ab} \geq 0$
3) D_{aa} even
4) $D_{a+b,c} \geq D_{a,c}+D_{b,c}$

Laughlin states satuate the equalities

Classification of W.F.’s

Find all possible patterns of zeros

Realize each with polynomials

Stability

Topological properties

General Structures

- S_k for k>n is determined by S_i, i=1,..,n
- If two families are multiplied, then their pattern of zeros are additive, and their filling fractions are inversely additive
- Search for primitive solutions for each n
- Notation for a solution:

  $m=D_{n,1}$, $\nu=n/m$

  $(m; S_2,\ldots,S_m)$ $(S_1=0)$

Laughlin states

Laughlin states $\Psi_{1/q}=\prod_{i<j}(z_i-z_j)^q$

have UFC and n-cluster form for each $n \geq 1$

As an n-cluster solution,

m=nq, ν=1/q, (m; q,…, qn(n-1)/2)

In general, an n-cluster state is always a kn cluster state, where $S_{n+1},\ldots, S_{kn}$ can be computed as above.

n=1

Only Bosonic Laughlin states

Notation

m=q, ν=1/q,

(q;)

D_{ab}=qab

n=2

Two primitive solutions denoted as $(m;S_2)$:

(1;0) and (4;2)

By ad hoc argument, (1;0) does not exist.

So we IMPOSED a new condition from NOW:

Δ_3 (a,b,c)=

$S_{a+b+c} - S_{a+b} - S_{b+c} - S_{c+a} + S_a + S_b + S_c$ is even.

By using CFT, we believe this is a unitarity condition or spin-statistics consistency

n=2

- (2;0)---Bosonic ν=1 Pfaffian state q=1

  Ψ= Pfaffian $(1/(z_i-z_j))$ $\prod$ $(z_i-z_j)^q$

  S_a=a(a-1)/2-[a/2], $S_1=S_2=0$

  D_{ab}=ab-(ab mod 2), $D_{11}=0$, $D_{12}=2$, $D_{22}=4$

- (4;2)---Laughlin Ψ_{1/2}

n=3

Two primitive solutions $(m;S_2,S_3)$

- (2;0,0)---Z_3 Read-Rezayi parafermion state
- (6;2,6)---Laughlin state Ψ_{1/2}

n=7, 5 primitive solutions

- (2;0,0,0,0,0,0)---Z_7 RR parafermion state
- (8;0,0,2,6,10,14)---generalized Z_7 Parafermion
- (18;0,4,10,18,30,42)---generalized Z_7
- (14;0,2,6,12,20,28),

  THIS state exists, yet a CFT construction is unknown
- Laughlin ½ state

n=9,   6 primitive solutions

Among the 6 solutions, one solution

(12;0,2,4,8,14,20,28,36)

is NOT known to us if it can be realized by symmetric polynomials.

Anyons

- Suppose there exists a q.p. γ above the groundstate at z=0, then translation symmetry is broken. If we bring particles to z=0, we will have different pattern of zeros. This pattern of zeros $\{S_{\gamma;a}\}$ will characterize the q.p. γ

- Given S_a, we have similar equations to solve for all q.p.'s

Quarsi-particles

- $S_{\gamma;a} \geq S_a$
- $S_{\gamma;a+b} - S_{\gamma;a} - S_b \geq 0$
- $S_{\gamma;a+b+c} - S_{\gamma;a+b} - S_{\gamma;a+c} - S_{b+c} + S_{\gamma;a} + S_b + S_c \geq 0$
- $S_{\gamma;a+kn} = S_{\gamma;a} + k(S_{\gamma;n} + ma) + k(k-1)mn/2$

A q.p. γ is determined by $\{S_{\gamma;i}\}$; i=1,2,…,n

Relation to CFT

Let V_γ be the q.p. operator with scaling dimension h_γ, then

$$S_{\gamma;a} = h_{\gamma+a} - h_\gamma - a h_1,$$

where $h_{\gamma+a}$ is the scaling dimension of $V_\gamma V_a$

Orbit occupation numbers

Orbitals are labeled $0,1,\ldots,N_\phi$

The a-th particle occupies the l_a–th orbit,

where $l_a = S_a - S_{a-1}$

Let n_l be the number of particles occupying the l-th orbit. n_l is periodic with period=m. There are n particles in each period. Hence the same state can be labeled as $[n_0,\ldots,n_{m-1}]$

For q.h. $\{S_{\gamma;a}\}$, $l_{\gamma;a} = S_{\gamma;a} - S_{\gamma;a-1}$

Examples

- Laughlin states: [1,0,…,0], n=1, m=q
- Pfaffian: [2,0]
- Z_k Parafermion: [k,0]

- n=7, m=14, CFT? [2,0,1,0,1,0,1,0,2,0,0,0,0,0]
- n=9, m=12, unknown: [2,0,2,0,1,0,2,0,2,0,0,0]

Topological properties

- Degeneracy on T^2, which is the # of q.p. types
- Fusion rules
- Charge of q.p.:

$$Q_\gamma = \sum_0^{km}(n_l - n_{\gamma;l}) - 1/m \sum_{km-m}^{km-1}(n_l - n_{\gamma;l})l$$

Particle types

- n=1 Laughlin ν=1/2,

  [10] Q=0, [01] Q=1/2
- n=2 Pfaffian

  [20] Q=0, [02] Q=1, [11] Q=1/2
- n=3 Z_3 parafermion

  [30] Q=0, [03] Q=3/2

  [12] Q=1, [21] Q=1/2

Modular Category Structure

Consider the Hamiltonian of FQH system on T^2 and the magnetic translation operator,

we get information of the modular S-matrix if we assume the resulting theory is a topological theory.

Recall the modular S-matrix determines all quantum dimensions and fusion rules.

Open Questions

- Twist
- UFC
- Uniqueness:

  There are different CFTs with the simple currents having same scaling dimensions by ZF. They are examples of same pattern of zeros. How are they related?

Stability

How to deicide if the W.F. indeed represents a topological phase?

- Energy gap

- Non-unitary CFT W.F.’s

Conclusions

Study FQH liquids using pattern of zeros as an alternative to CFTs. Maybe lead to deeper understanding of CFTs.

References:

1. PRB 77, 235108 (2008), arxiv 0801.3291
2. PRB (to appear), arxiv 0803.1016