PHYSICAL IMPLEMENTATION AND RESIDUAL DECOHERENCE OF PROTECTED QUBITS

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Principle of qubit protection I

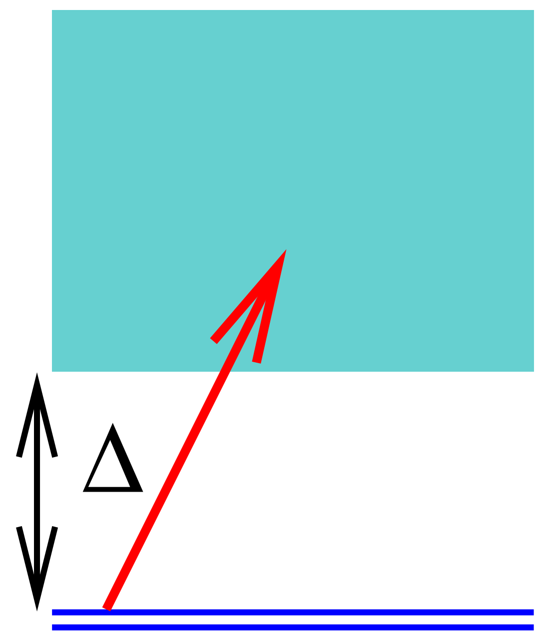
A. Kitaev, Ann. Phys. **303**, 2, (2003)

Coding space separated from non-coding one by large gap Δ.

Absence of decoherence to first order:

$$PH_cP = P \otimes \tilde{H}^{(1)}_{\text{env}}$$

Single error that can be detected.



Principle of qubit protection II

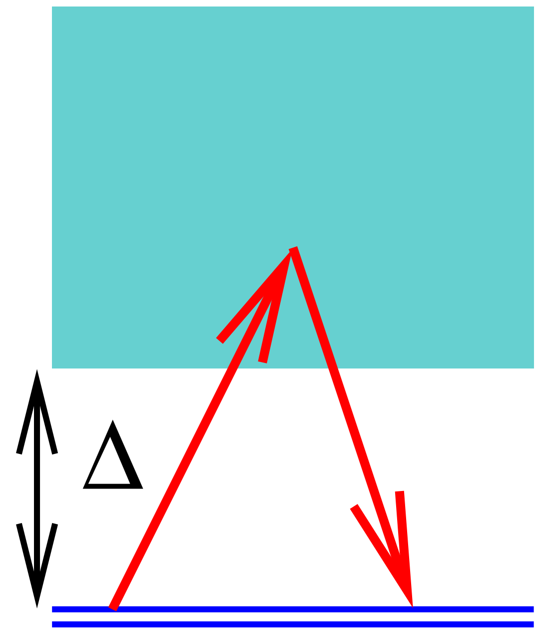
Absence of decoherence to second order:

$$PH_cQ\frac{1}{H_0}QH_cP = P \otimes \tilde{H}^{(2)}_{\text{env}}$$

Can be generalized to arbitrary order in H_c $\rightarrow$ notion of protected system at order N.

Can we achieve N large in a physical system ?

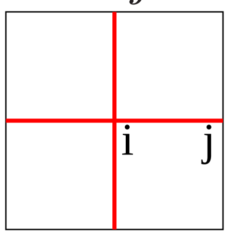
Potentially dangerous double error.



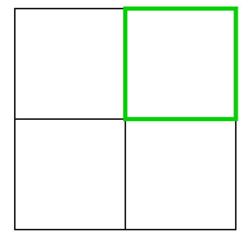
Lattice gauge theory in deconfined regime I

A. Kitaev, Ann. Phys. **303**, 2, (2003)

Z_2 charge $U_i = \prod_j^{(i)} \sigma_{ij}^x$



Z_2 flux $B_\square = \prod_{ij\in\square} \sigma_{ij}^z$



$$H_{\text{Kitaev}} = -\frac{\Delta_{\text{c}}}{2}\sum_i U_i - \frac{\Delta_{\text{f}}}{2}\sum_\square B_\square$$

Localized excitations with finite energy gap

Ground-state degeneracy depends on global topology of the lattice.

Lattice gauge theory in deconfined regime II

Two-fold degenerate ground-state on a cylinder

Degeneracy enforced by non-local symmetries:

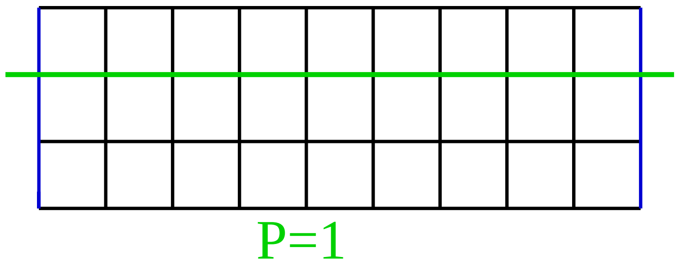
row operators:

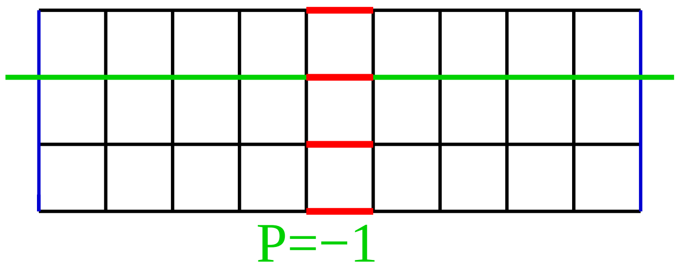
$$P_i = \prod_j \sigma^z_{ij}$$

column operators:

$$Q_j = \prod_i \sigma^x_{ij}$$

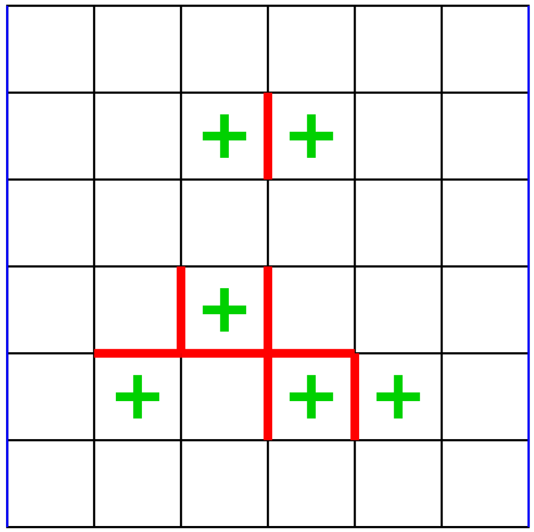
$$\{P_i, Q_j\} = 0$$





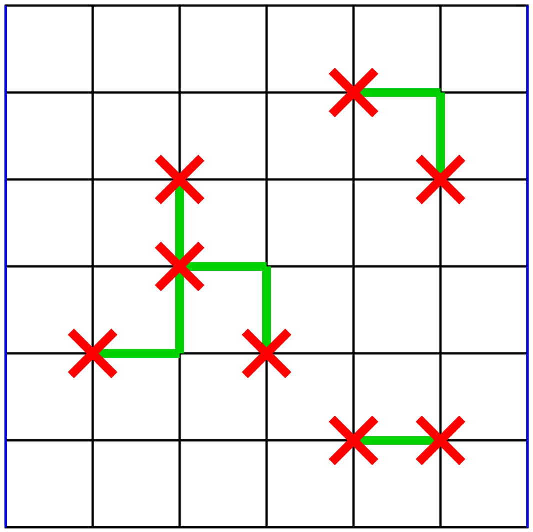
Local errors are harmless

Electric noise: $\prod_{ij \in \text{cluster}} \sigma_{ij}^x$



Creates localized Z_2 fluxes.

Magnetic noise: $\prod_{ij \in \text{cluster}} \sigma_{ij}^z$



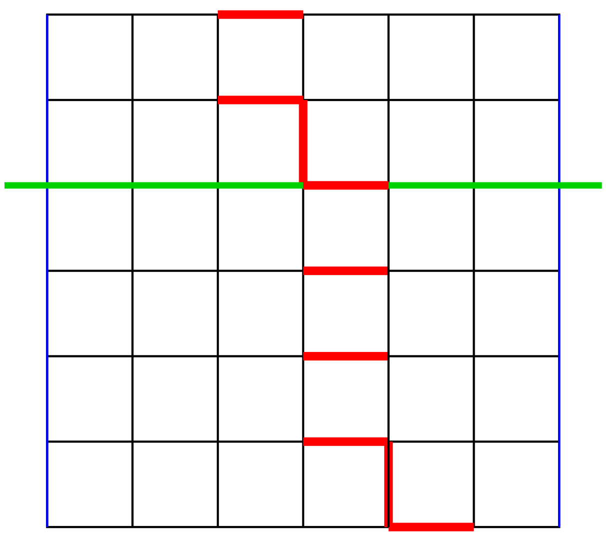
Creates local Z_2 charges.

These errors create only virtual states above finite energy gap.

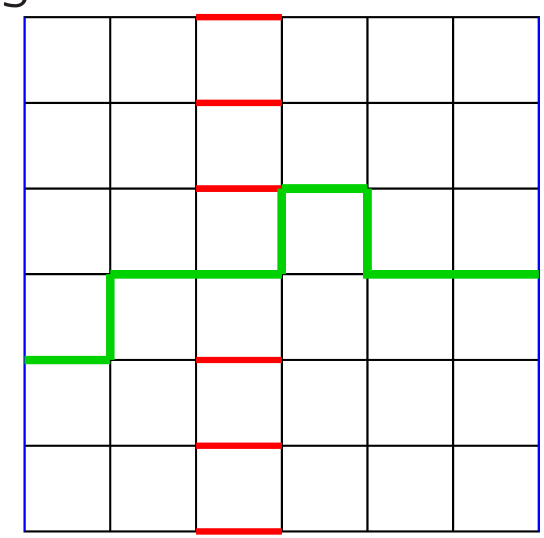
The only dangerous errors are non-local !

They are suppressed by a factor $(\text{noise}/\Delta_{c,f})^L$

Electric noise transfers one Z_2 flux along v-path and flips P_i: Relaxation in *flux* basis or dephasing in *charge* basis.



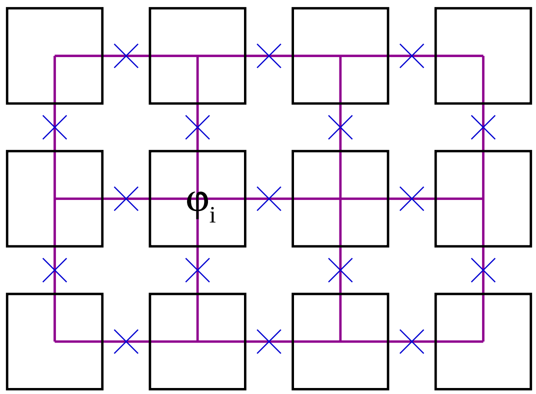
Magnetic noise transfers one Z_2 charge along h-path and flips Q_j: Relaxation in *charge* basis or dephasing in *flux* basis.



Basics of Josephson junction arrays

ϕ_j; local phase of Cooper pair condensate

$\hat{n}_j = \frac{\partial}{i\partial\phi_j}$: number of Cooper pairs on island j



$$\Delta\phi_j \Delta n_j \simeq 2\pi$$

$$A_{ij} = \frac{2\pi}{\Phi_0}\int_i^j \vec{A}_{ij}.d\vec{r}$$

$$H = -E_\mathrm{J}\sum_{\langle ij\rangle}\cos(\phi_i - \phi_j - A_{ij}) + \frac{E_\mathrm{C}}{2}\sum_{ij}(C_{ij}^{-1})\hat{n}_i\hat{n}_j$$

E_J: Josephson coupling energy

E_C: Charging energy

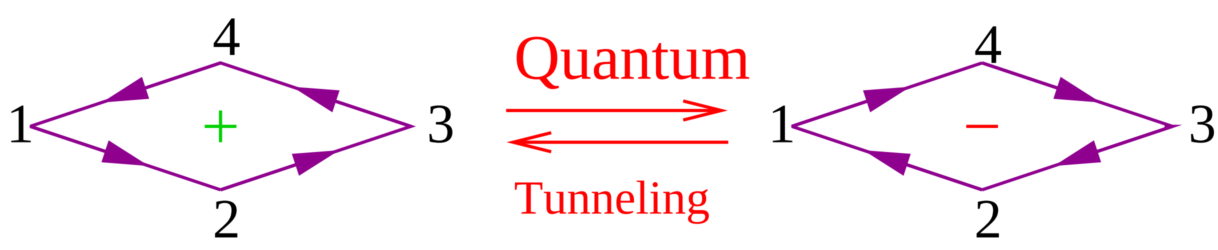
A rhombus with half a flux quantum

Define $\theta_{ij} = \phi_i - \phi_j - A_{ij}$, then:

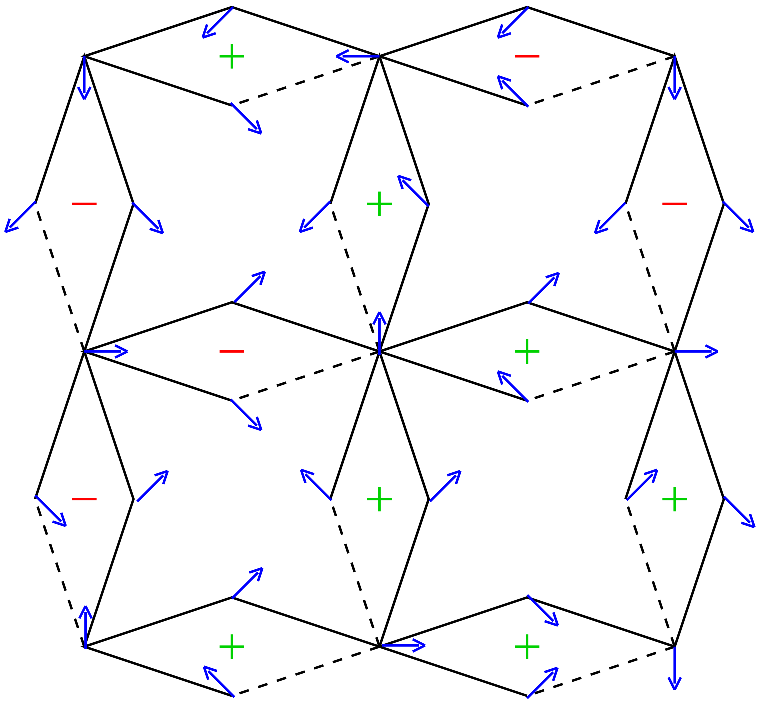
$$\theta_{12} + \theta_{23} + \theta_{34} + \theta_{41} \equiv \pi, \ \text{mod } 2\pi$$

$\rightarrow$ Get two-fold degenerate classical ground-state, with $\theta_{ij} = \pm\frac{\pi}{4}$

$\rightarrow$ Quantum fluctuations ($E_C \neq 0$) of phases lift this degeneracy

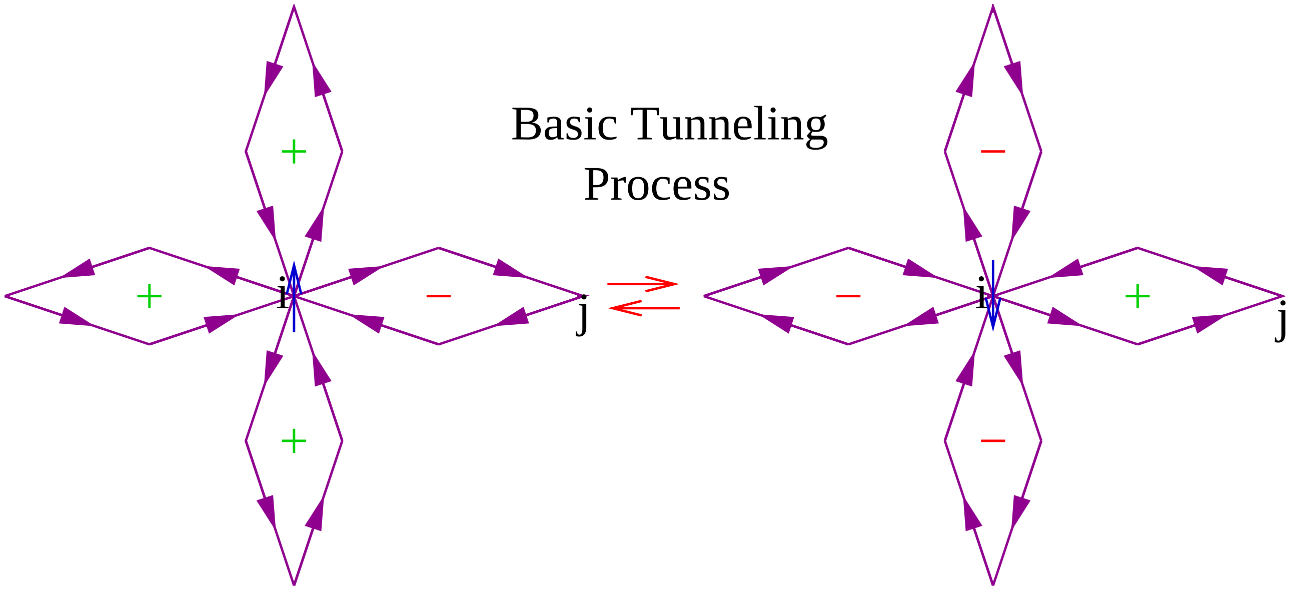


Local constraint on the classical ground-states



Enforces $B_{\Box} = 1$ for classical ground-states.
Physical origin of Δ_f.

Effect of quantum fluctuations of phase variables



Basic tunneling process acts as: $\prod_j^{(i)} \sigma_{ij}^x$

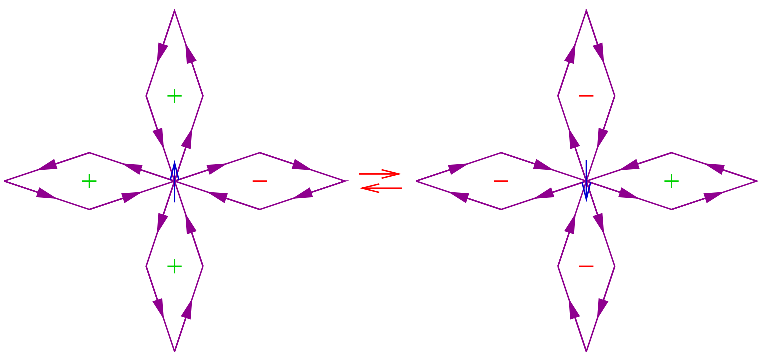
Tunnel rate: $\Delta_\text{C} \simeq E_\text{J}^{3/4} E_\text{C}^{1/4} \exp(-4 S_0)$

where: $S_0 = 1.61 (E_\text{J}/E_\text{C})^{1/2}$, (Ioffe and Feigel'man, 2002)

Localization of Cooper pairs, and charge $4e$ condensate

Physical interpretation of local flip operator U_j:

$$U_j|\phi_j\rangle = |\phi_j + \pi\rangle$$



$$U_j|n_j\rangle = (-1)^{n_j}|n_j\rangle$$

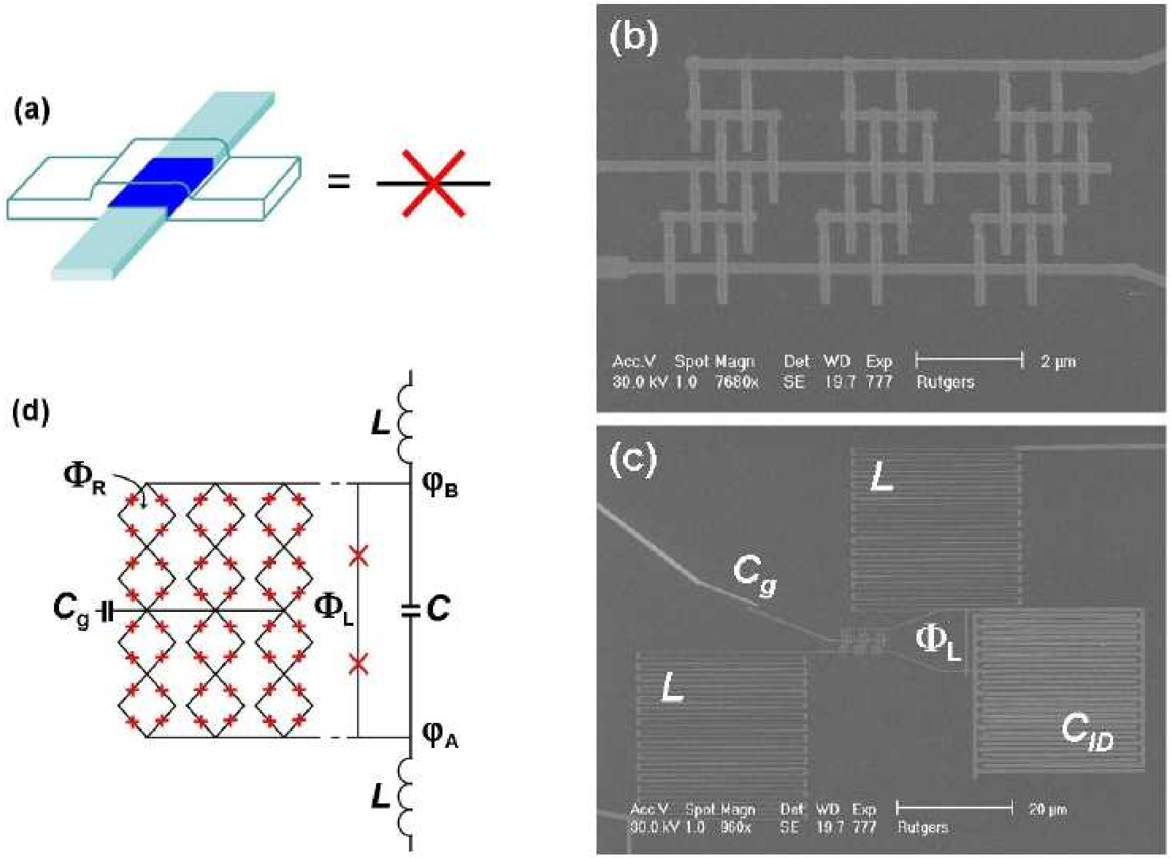
The parity U_j of n_j is conserved and single Cooper pairs are localized in Aharonov-Bohm cages.

$$\langle \exp(i\phi_j) \rangle = 0$$

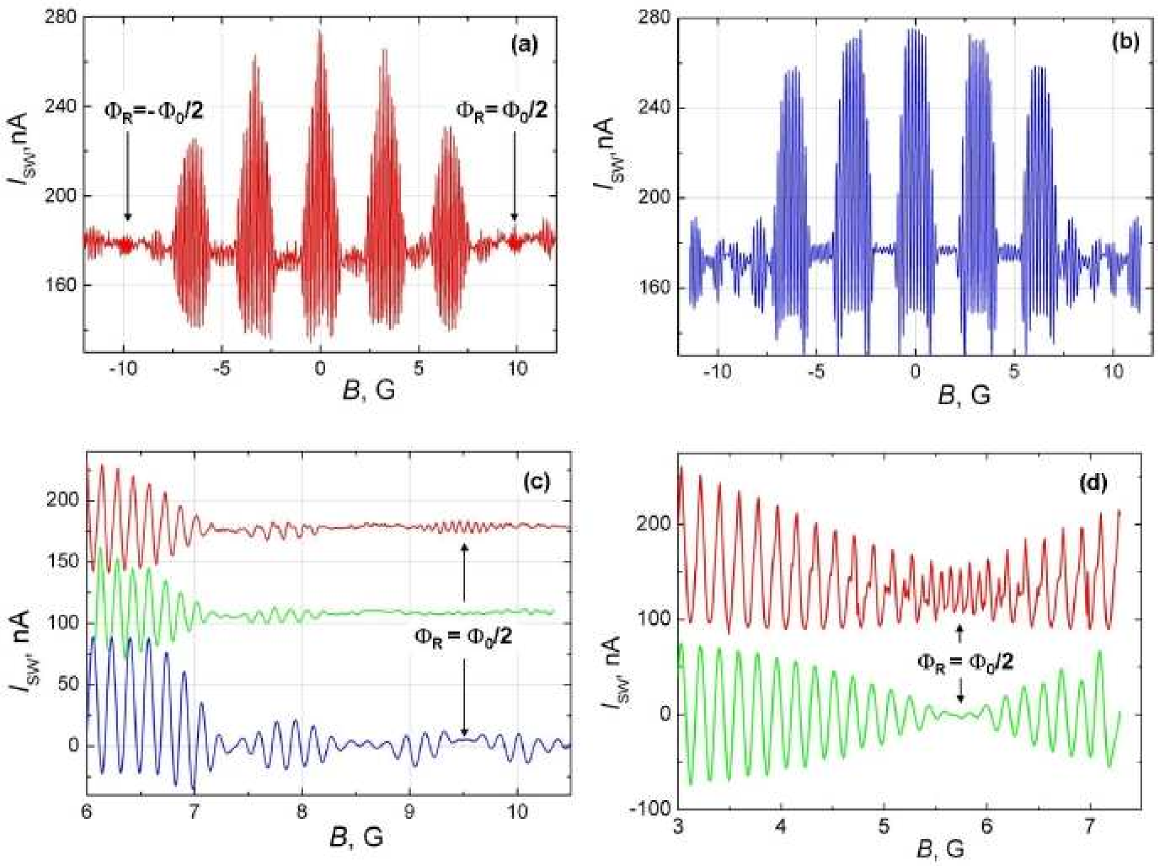
$$\langle \exp(i2\phi_j) \rangle \neq 0$$

No 2e condensate if $\Delta_C \neq 0$, but $4e$ condensate!

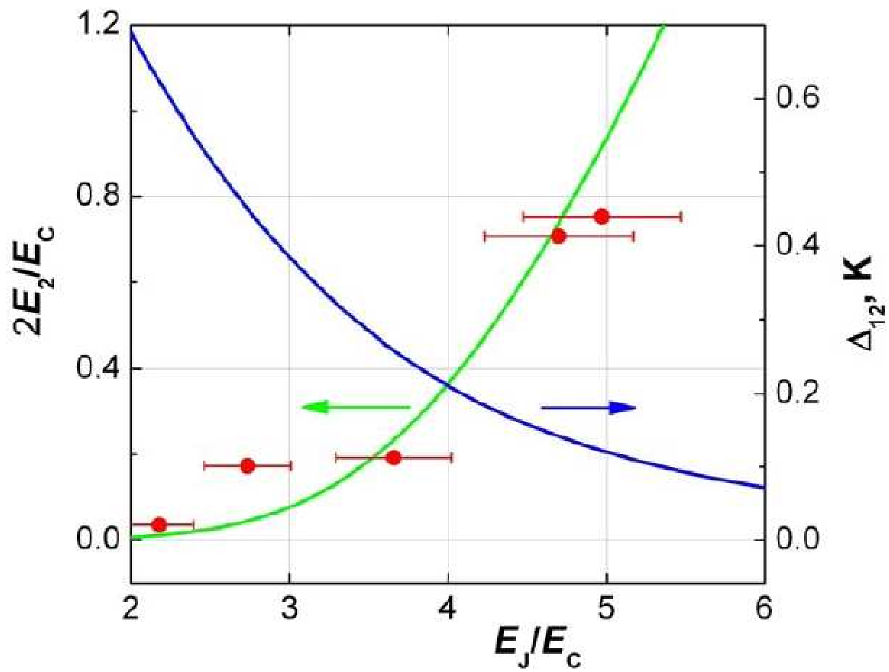
Experimental realization: M. Gershenson et al. (2007)



Evidence for finite Δ_C and charge $4e$ condensate



Phase stiffness of charge $4e$ condenstate



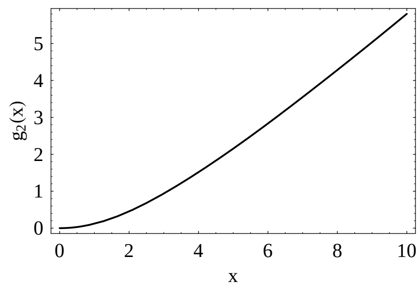
Computational issues I: hierarchical approximation

Series composition of Z_2 junctions

$$V(\phi) = -E_2 \cos(2\phi)$$

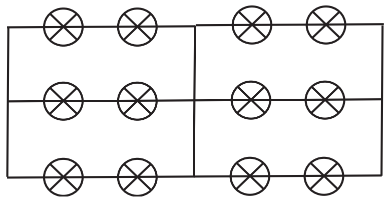
$$E'_2 = \left[1 - \frac{7}{256}\left(\frac{E_2}{E_C}\right)^2\right]\frac{1}{8}\frac{E_2^2}{E_C}$$

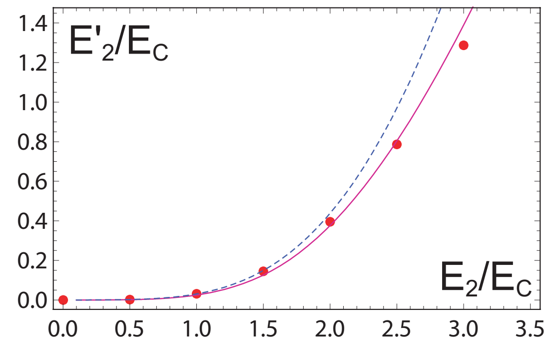
$$E'_C = \left[1 - \frac{1}{16}\left(\frac{E_2}{E_C}\right)^2\right] 2E_C$$



Parallel composition

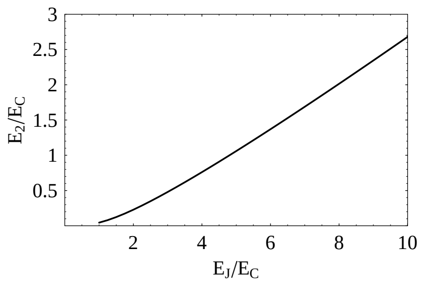
$$E'_2 = K E_2$$
$$E'_C = K^{-1} E_C$$



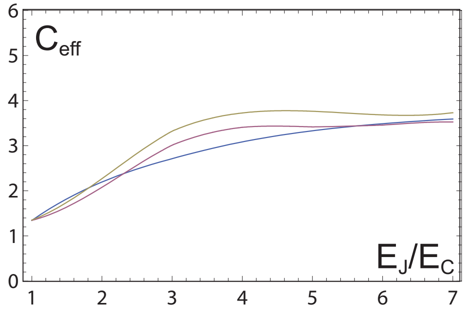


Computational issues II: single rhombus as Z_2 junction

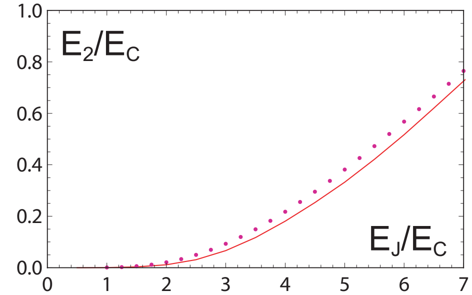
Effective E_2

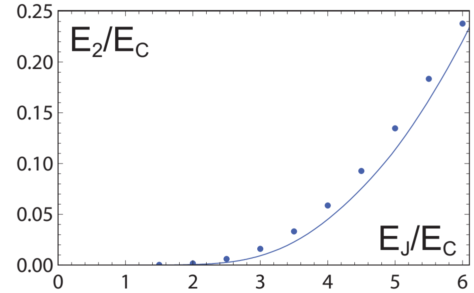


Effective capacitance



Test of coarse graining: 2 and 3 rhombi in series





Decoherence induced by finite frequency fluctuations

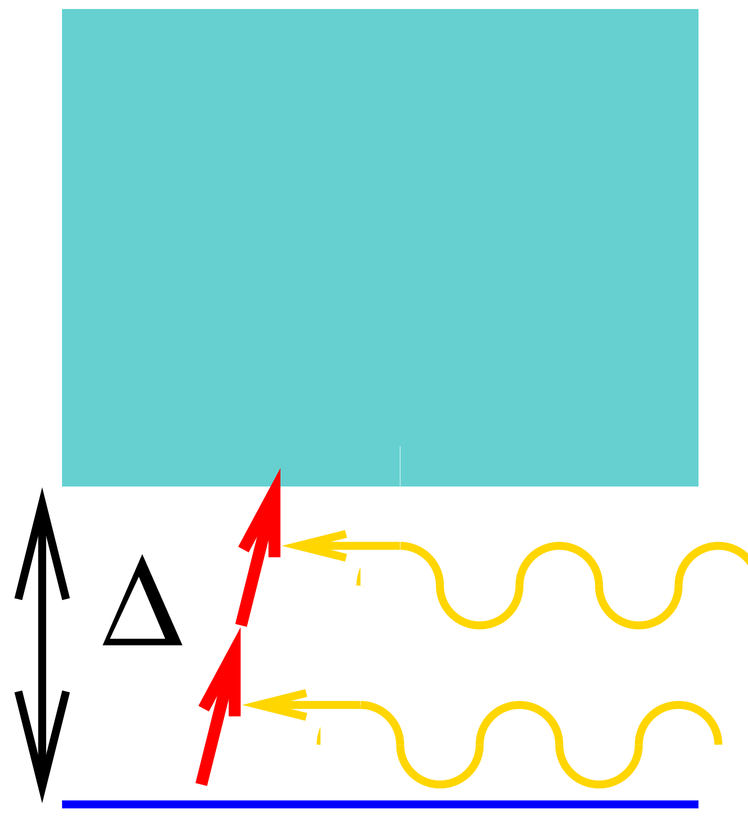
So far, we have considered only *virtual* transition to excited states.

But the bath may provide some energy: problem of *real* transitions.

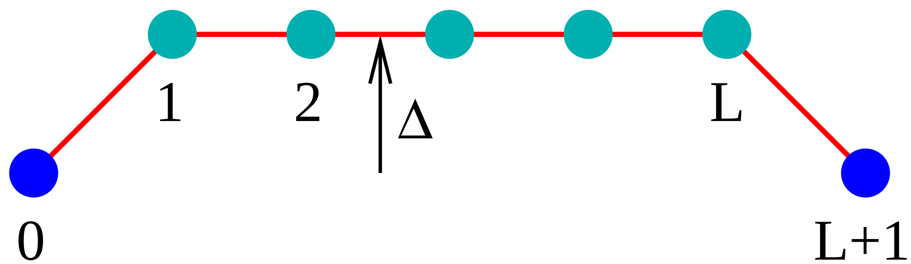
Spectral width of bath: D

$$D_{\text{eff}} = \text{Min}(k_B T, D)$$

$$n = \Delta / D_{\text{eff}}$$



Toy model



$$H = H_{\mathsf{syst}} + H_{\mathsf{bath}} + H_{\mathsf{c}}$$

$$H_{\mathsf{syst}} = \Delta \sum_{j=1}^{L} |j\rangle\langle j|$$

$$H_{\mathsf{c}} = - \sum_{j=0}^{L} |j\rangle\langle j+1| \otimes X_{j+1/2}$$

Tree approximation

Density of states at generation p:

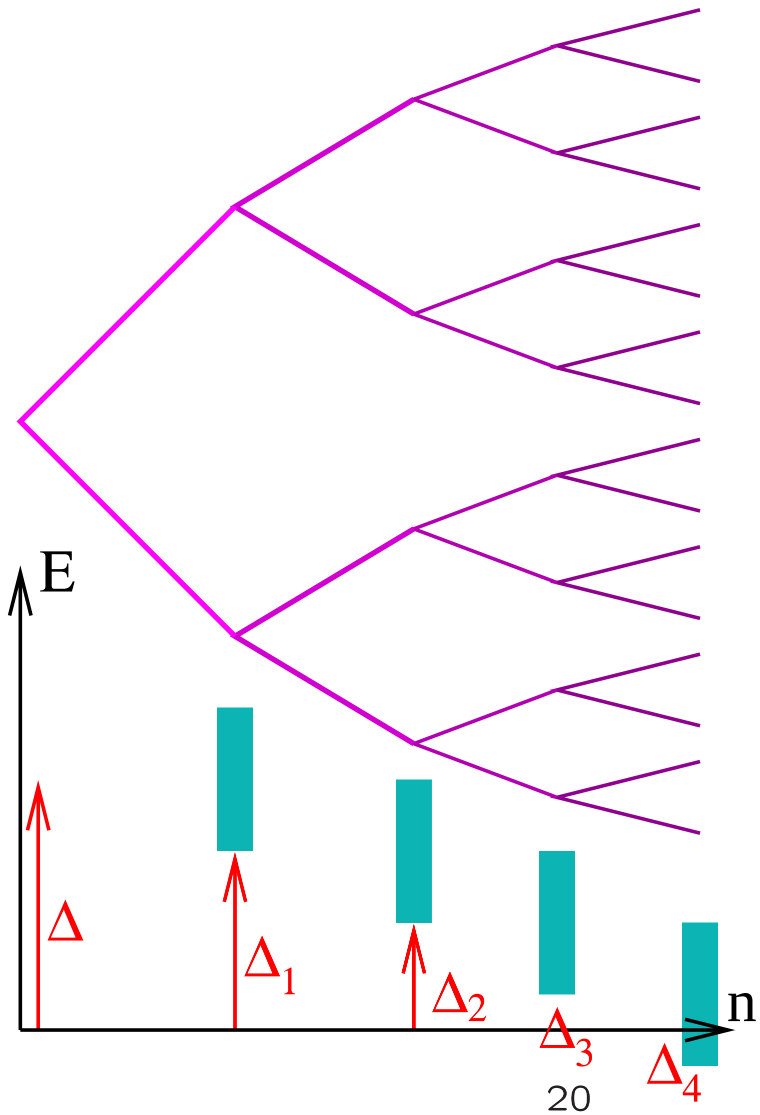
$$\rho_p(\omega) = \alpha_p(\omega - \Delta_p)^{r_p},$$

restricted to $\Delta_p < \omega < \Delta_p + D_p$.

$$R(z) = \langle \mathrm{in}|(z-H)^{-1}|\mathrm{in}\rangle$$

$$R(z) = \frac{1}{z - \Sigma_1(z)}$$

$$\Sigma_p(z) = \alpha_p W_p^2 \int \frac{\rho_p(\omega)\, d\omega}{z - \omega - \Sigma_{p+1}(z)}$$



Weak coupling analysis

Assume $z \simeq 0$, and

$$\begin{aligned} z - \Re\Sigma_1(z) &\leq \Delta_1 \\ \ldots &\leq \ldots \\ z - \Re\Sigma_{n-1}(z) &\leq \Delta_{n-1} \\ z - \Re\Sigma_n(z) &\geq \Delta_n \end{aligned}$$

Then, imaginary parts satisfy:

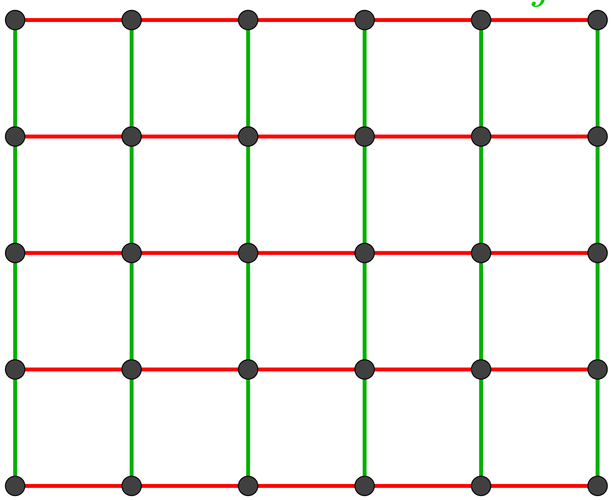
$$\begin{aligned} -\Im\Sigma_1(z) &\simeq c_1\alpha_1 W_1^2 D_1^{r_1-1}(-\Im\Sigma_2) \\ \ldots &\simeq \ldots \\ -\Im\Sigma_{n-1}(z) &\simeq c_{n-1}\alpha_{n-1}W_{n-1}^2 D_{n-1}^{r_{n-1}-1}(-\Im\Sigma_n) \\ -\Im\Sigma_n(z) &= \alpha_n W_n^2(z+|\Delta_n|)^{r_n} \end{aligned}$$

$$-\Im\Sigma_1(z) \simeq \left(\frac{W_1 \ldots W_n}{D_1 \ldots D_{n-1}}\right)^2 \left(c_1\alpha_1 D_1^{r_1+1} \ldots c_{n-1}\alpha_{n-1}D_{n-1}^{r_{n-1}+1}\right)\alpha_n(z+|\Delta_n|)^{r_n}$$

$\rightarrow$ Master equation, with rates appearing at order $2n$ in perturbative expansion.

$\rightarrow$ No use to make systems of size L with $L > n$.

Assume $[H, P_i] = [H, Q_j] = 0$



$$P_i^2 = 1, \quad [P_i, P_j] = 0$$
$$Q_i^2 = 1, \quad [Q_i, Q_j] = 0$$

$$\{P_{\text{row}}, Q_{\text{column}}\} = 0$$

Can diagonalize *simultaneously*:

$$P_1, P_2, ..., P_M, Q_1Q_2, ..., Q_1Q_N$$

Gives only two-dimensional irreducible representations!

1)Start with $|\uparrow\rangle$, such that:

$$P_i|\uparrow\rangle = \alpha_i|\uparrow\rangle$$
$$Q_1Q_j|\uparrow\rangle = \beta_j|\uparrow\rangle$$

2)Define $|\downarrow\rangle$ as:

$$|\downarrow\rangle = Q_1|\uparrow\rangle$$

3)$|\downarrow\rangle$ satisfies:

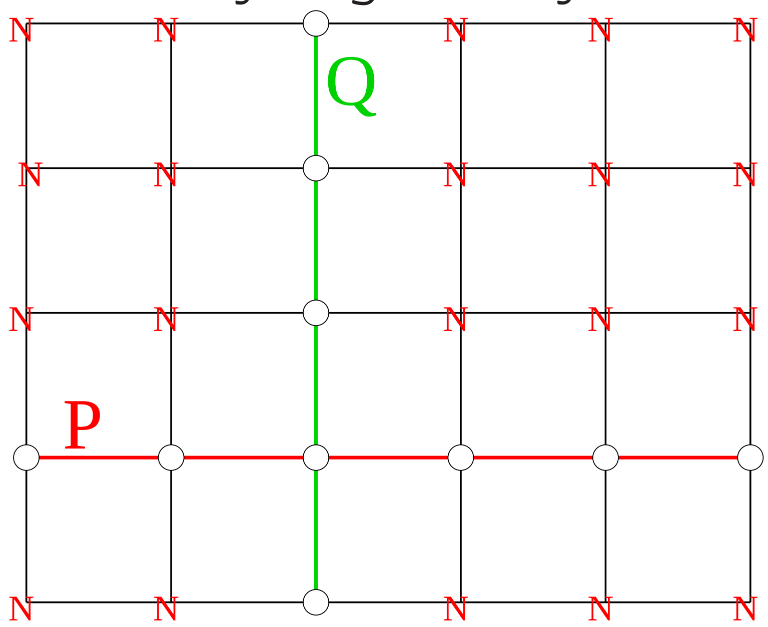
$$P_i|\downarrow\rangle = -\alpha_i|\downarrow\rangle$$
$$Q_1Q_j|\downarrow\rangle = \beta_j|\downarrow\rangle$$

4)Furthermore:

$$Q_j|\uparrow\rangle = \beta_j|\downarrow\rangle$$

Doublets exist as long as one can find at least ONE pair P_i, Q_j, commuting with H.

Example of a static disorder configuration which does NOT lift any degeneracy!

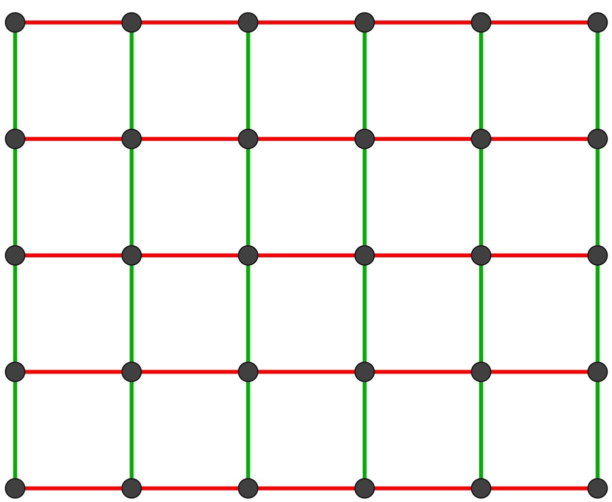


Local noise breaks degeneracies only at high orders (M, or N) in perturbation theory!

Example: X-Z Ising model

Douçot, Feigel'man, Ioffe, Ioselevich, P. R. B. **71**, (2005)

$$H = -J_x \sum_{\langle ij \rangle}^{(h)} \sigma_i^x \sigma_j^x - J_z \sum_{\langle ij \rangle}^{(v)} \sigma_i^z \sigma_j^z$$



Conservation laws

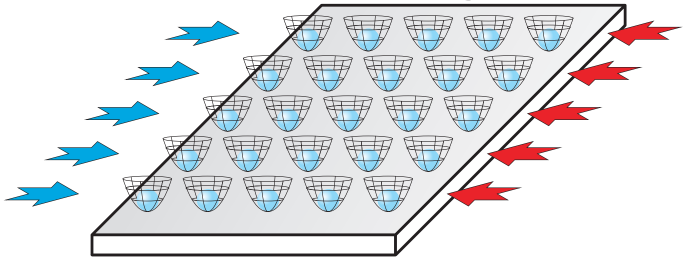
$$P_{\text{row}} = \prod_{r \in \text{row}} \sigma_r^z$$
$$Q_{\text{column}} = \prod_{r \in \text{column}} \sigma_r^x$$

$$P_i^2 = 1, \quad [P_i, P_j] = 0$$
$$Q_i^2 = 1, \quad [Q_i, Q_j] = 0$$

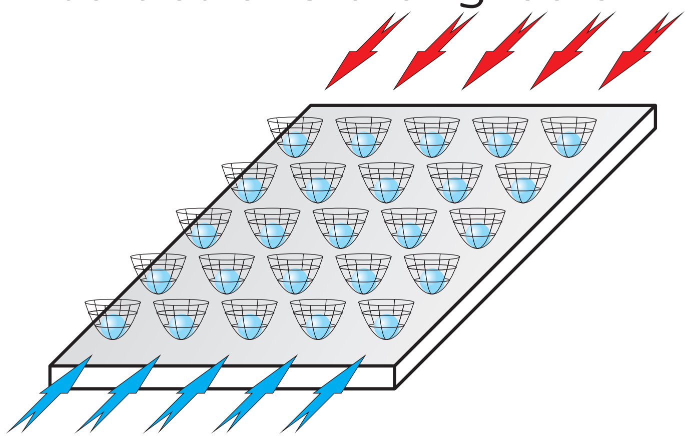
$$\{P_{\text{row}}, Q_{\text{column}}\} = 0$$

Implementation with trapped ions

Interactions along rows



Interactions along columns



with T. Coudreau, P. Milman, and L. Ioffe

Phonon-mediated long-ranged spin coupling

The Sørensen-Mølmer process

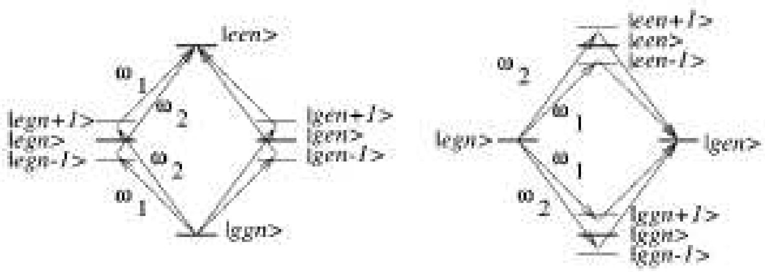


FIG. 2. Level scheme for a pair of ions sharing an oscillator degree of freedom. Left: By application of laser light with frequencies $\omega_{eg} \pm \delta$, where δ is somewhat smaller than the vibrational frequency ν, we identify four transition paths between the states $|gg\rangle|n\rangle$ and $|ee\rangle|n\rangle$, which interfere as described in the text. Right: Four similar transition paths are identified between states $|eg\rangle|n\rangle$ and $|ge\rangle|n\rangle$, yielding the same effective coupling among these states as between the states in the left panel.

Effective interaction

$$J_{\text{eff}} = (\eta\Omega)^2/|\nu - \delta|$$

Ω: Light intensity (Rabi frequency)

η: Photon energy/recoil energy

ν: Phonon frequency

δ: detuning of the main transition

Constraints for pratical implementation

Wish to maximize J_{eff}, because energy gap has to be larger than main source of noise, likely to be due to laser frequency noise, typically $\delta f \sim 500\text{Hz}$

Weak coupling: $\eta\Omega < |\nu - \delta|$, so $J_{\text{eff}} < \eta\Omega$, but:

One has to couple only to one phonon mode: $\eta\Omega < \Delta\nu$

In one dimension: $\Delta\nu = (\sqrt{3} - 1)\nu$

In two dimensions: $(5 \times 5$ array): $\Delta\nu \simeq 0.1\nu$

Increasing ν decreases the distance between ions

Optimal size seems to be $N \leq 3$ (1D) or $N \leq 5$ (2D)

Long range interactions help, because they induce larger gaps!

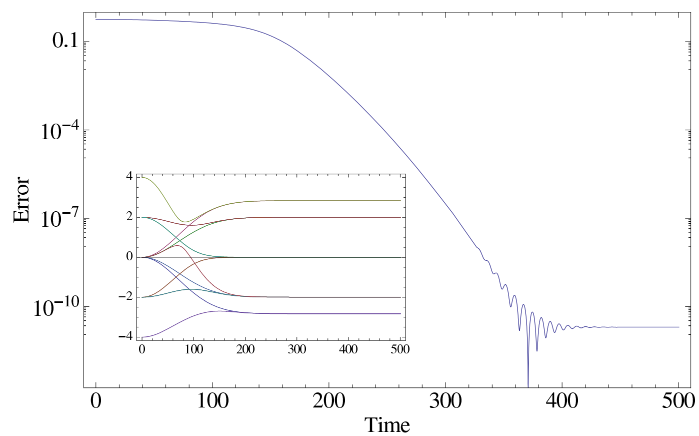
	2×2	3×3	4×4	5×5
SRI	0.84	0.58	0.32	0.20
LRI	0.84	0.96	0.92	0.80

Estimates for decoherence time

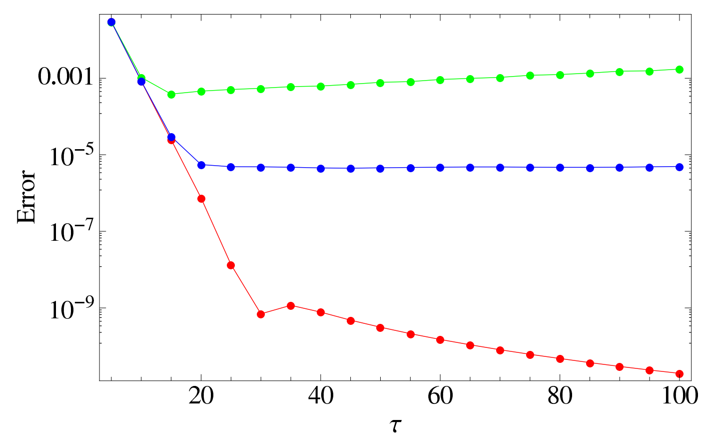
	4 ions	9 ions	5×5 ions
Γ_{eff} (Hz)	$1.5 \cdot 10^{-3}$	$7.5 \cdot 10^{-5}$	$1.9 \cdot 10^{-11}$
$\tau(s)$	$6.6 \cdot 10^{2}$	$1.3 \cdot 10^{4}$	$5.3 \cdot 10^{10}$

Initialization of the protected qubit

Switching off a local field:



Effect of a static noise:



Conclusions

1) Kitaev's Z_2 lattice model implemented in the low energy sector of some Josephson junction arrays.

2) These arrays are composed of fully frustrated rhombi.

3) Topological protection arises in the phase where quantum phase fluctuations destroy the $2e$ condensate, while preserving the $4e$ condensate.

4) Experimental evidence for this phase: observation of enhanced immunity against static flux fluctuations, evidence of a finite Δ_{c}.

5) Protection still works in the presence of dynamical fluctuations, up to order $n = \Delta / D_{\text{eff}}$.

6) Alternative implementations of protection by non-local symmetries.

Appendix: Lattice gauge theories with a finite gauge group G

(A. Kitaev, quant-ph/9707021)

If G is a permutation group S_n for n large enough, can generate universal quantum computation!

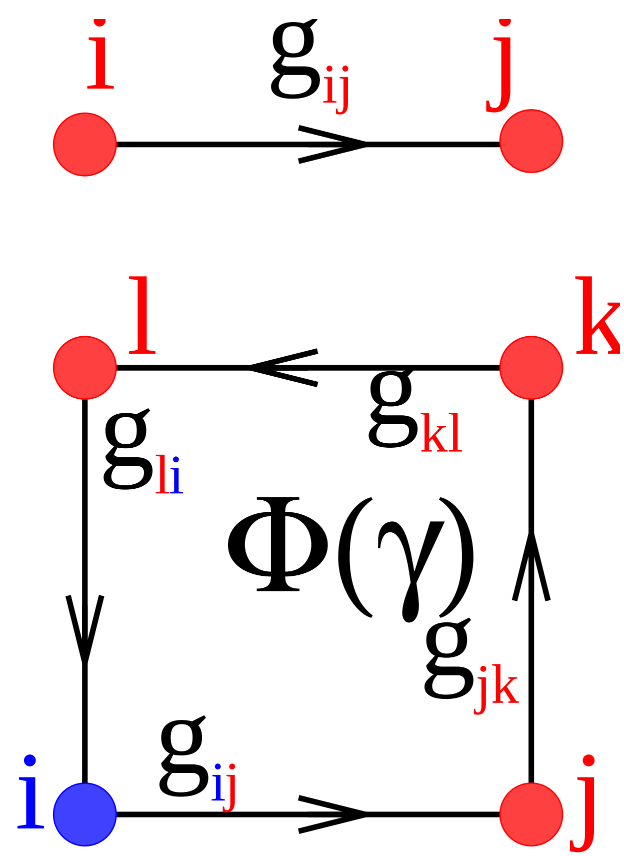
Mochon, Phys. Rev. A **67**, 022315 (2003) and **69**, 032306 (2004)

Basics of lattice gauge theory

Link $ij \longrightarrow g_{ij} \in G$, G finite group
Path $\gamma \longrightarrow \Phi(\gamma) = g_{ij}g_{jk}g_{kl}g_{li}$
Local gauge transformation:
start from $h_j \in G$

$$\begin{aligned} g_{ij} &\longrightarrow g'_{ij} = h_i g_{ij} {h_j}^{-1} \\ \Phi(\gamma) &\longrightarrow \Phi'(\gamma) = h_i \Phi(\gamma) {h_i}^{-1} \end{aligned}$$

States of localized flux $\Longleftrightarrow$ conjugacy classes in G
States of several localized fluxes: a choice of a common origin for defining fluxes is crucial.

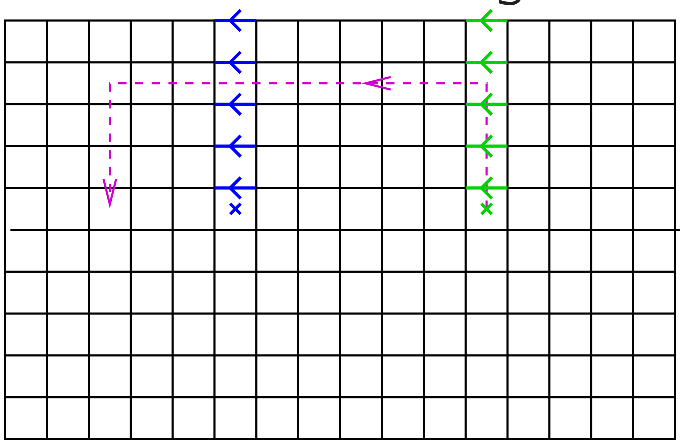


Non-trivial holonomy of fluxons: pair exchange

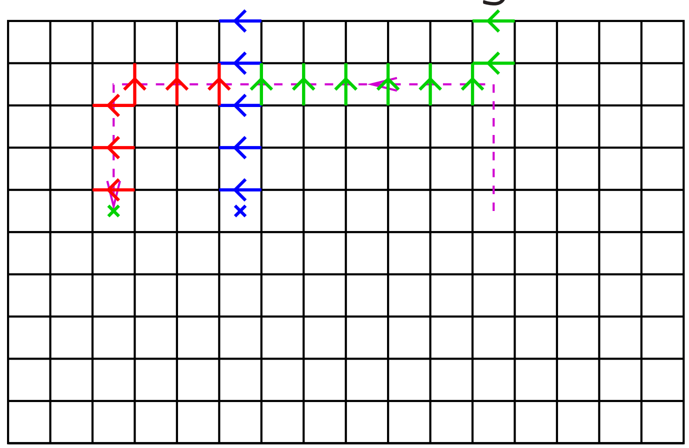
Bais, Nucl. Phys. **B 170**, 32, (1980)

Lo and Preskill, Phys. Rev. **D 48**, 4821, (1993))

Before exchange:



After exchange:

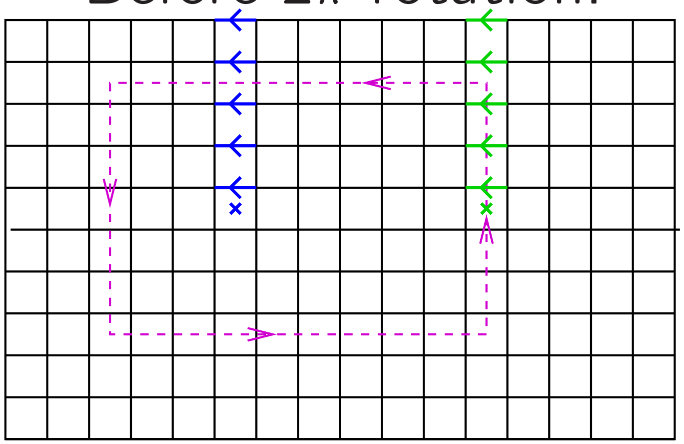


$$g \longrightarrow \tilde{g} = h^{-1} g h$$

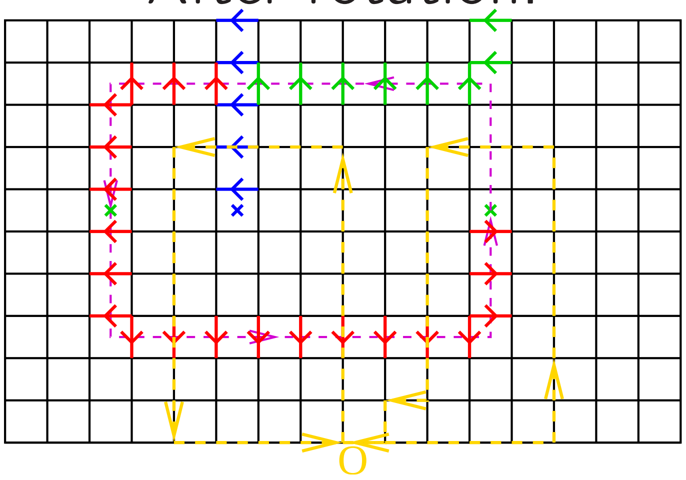
$$h \longrightarrow h$$

Non-trivial holonomy of fluxons: 2π rotation around a fixed flux

Before 2π rotation:



After rotation:



$$g \longrightarrow \tilde{g} = h^{-1} g h$$
$$h \longrightarrow \tilde{h} = (h^{-1} g h)^{-1} h (h^{-1} g h)$$

Universal computation with anyons

Mochon, Phys. Rev. A **67**, 022315 (2003) and **69**, 032306 (2004)

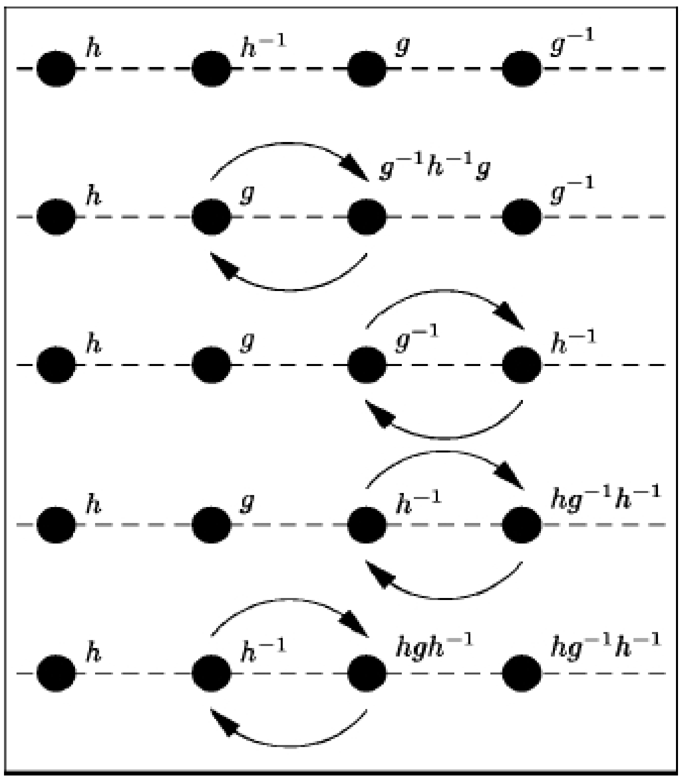


FIG. 2. Conjugating a pair of anyons.

0) G "sufficiently" non-Abelian (non-solvable)

1) We can braid or exchange any two excitations

2) We can fuse a pair of anyons and detect whether there is a particle left behind or whether they had vacuum quantum numbers.

3)We can produce a pair of anyons in a state that is chosen at random from the two-particle subspace that has vacuum quantum numbers.

4) We have ancilla pairs $|g\rangle \otimes |g^{-1}\rangle$ for any $g \in G$, where the individaul anyons have trivial electric charge.