# Higher Genus Ward Identities

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# Introduction

1. Vertex Operator Algebras (VOAs).
2. Genus One Zhu Theory and Ward Identities.
3. The Partition and Correlation Functions for a VOA on a Genus Two Riemann Surface.
4. Genus Two Zhu Theory and Ward Identities.

# Vertex Operator Algebras (VOAs) - Chiral CFT

**A Vector Space** $V$**.** $\mathbb{Z}$-graded $V = \oplus_{n\geq 0} V_n$ with $\dim V_n < \infty$.

**Vacuum Vector.** A distinguished element $\mathbf{1} \in V_0$.

**Vertex Operators.** For each $u \in V$ there exists a vertex operator

$$Y(u,z) = \sum_{n\in\mathbb{Z}} u(n) z^{-n-1}, \quad \text{(not physics modes!)}$$

a formal Laurent series in $z$ with modes $u(n) \in \mathrm{End}(V)$.

**Creativity.** $Y(u,z)\mathbf{1} = u + O(z)$ i.e. $u(n)\mathbf{1} = \delta_{n,-1}u$, for all $n \geq -1$.

**Locality.** For each $u, v \in V$ and sufficiently large integer $N$

$$(x-y)^N \left[Y(u,x), Y(v,y)\right] = 0.$$

**Conformal Virasoro Vector.** A distinguished vector $\omega \in V_2$ with

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}, \quad \text{(physics modes here!)}$$

$$[L(m), L(n)] = (m-n)L(m+n) + C\frac{m^3-m}{12}\delta_{m,-n}\,\mathrm{Id}_V\,.$$

$L(0)$ gives $\mathbb{Z}$-grading: $L(0)u = nu$ for $u \in V_n$ for weight $\mathrm{wt}(u) = n$.

**Translation.** $L(-1)\mathbf{1} = 0$ and

$$[L(-1), Y(u,z)] = \frac{\partial}{\partial z}Y(u,z) \equiv \sum_{n\in\mathbb{Z}}(-n-1)u(n)z^{-n-2}.$$

**Some consequences** (amongst many!):

$$o(u) : V_k \to V_k \quad \text{for } o(u) \equiv u(\mathrm{wt}(u)-1) \text{ (physics zero mode)}$$

$$[u(m), Y(v,z)] = \sum_{i\geq 0}\binom{m}{i} Y(u(i)v, z) z^{m-i}, \quad \text{(a finite sum).}$$

# The Li-Zamolodchikov metric

We can define an invariant symmetric bilinear form $\langle\, ,\rangle$ on $V$ where

$$\langle Y(u,z)a,b\rangle = \langle a, Y^{\dagger}(u,z)b\rangle,$$

for adjoint

$$Y^{\dagger}(u,z) = \sum_{n\in\mathbb{Z}} u^{\dagger}(n) z^{-n-1} = Y\left(e^{zL(1)}\left(-z^{-2}\right)^{L(0)} u, z^{-1}\right).$$

For quasiprimary $u$ (for which $L(1)u = 1$) of weight $\mathrm{wt}(u)$

$$u^{\dagger}(n) = (-1)^{\mathrm{wt}(u)} u(-n - 2\,\mathrm{wt}(u) - 2).$$

Thus $L^{\dagger}(n) = L(-n)$ and $\langle a,b\rangle = 0$ if $\mathrm{wt}(a) \neq \mathrm{wt}(b)$.

We consider $V$ for which $\langle\, ,\rangle$ is unique (with normalization $\langle \mathbf{1},\mathbf{1}\rangle = 1$) and invertible and call such a form the Li-Zamolodchikov or Li-Z metric.

# The Heisenberg Algebra or Bosonic String

Consider $a \in V_1$ whose modes obey

$$[a(m), a(n)] = m\delta_{m,-n} \operatorname{Id}_V \quad \text{i.e.} \quad [a(m), Y(a,z)] = \operatorname{Id}_V m z^{m-1}.$$

$Y(a,z)$ generates a VOA with vector space $V$ with Fock basis

$$a(-1)^{r_1} a(-2)^{r_2} \ldots a(-n)^{r_n} \mathbf{1}, \quad r_i \geq 0$$

with $a = a(-1)\mathbf{1}$ and $a(n)\mathbf{1} = 0$ for all $n \geq 0$.
The Virasoro vector is $\omega = \frac{1}{2} a(-1)^2 \mathbf{1}$ for central charge $C = 1$ with

$$L(0) = \frac{1}{2} a(0)^2 + \sum_{m>0} a(-m) a(m).$$

Each Fock vector $a(-1)^{r_1} \ldots a(-n)^{r_n} \mathbf{1} \in V_k$ has $\mathbb{Z}$-grade

$$k = 1.r_1 + 2.r_2 + \ldots + n.r_n, \qquad \text{an integer partition of } k$$

Since $a(n)^\dagger = -a(-n)$, Fock vectors form a Li-Z metric orthogonal basis.

# The Genus One Partition and Correlation Functions

Define the genus one partition function (for formal parameter $q$ later identified as a modular parameter).

$$Z_V^{(1)}(q) = \mathrm{Tr}_V \left(q^{L(0)-C/24}\right) = \sum_{k\geq 0} \dim V_n \, q^{n-C/24}.$$

Formal 1-point and $n$-point correlation functions are defined by

$$Z_V^{(1)}(u,q) = \mathrm{Tr}_V \left(o(u)q^{L(0)-C/24}\right),$$
$$Z_V^{(1)}(u_1,z_1;\ldots;u_n,z_n;q) = Z_V^{(1)}(Y[u_1,z_1]\ldots Y[u_n,z_n]\mathbf{1},q),$$

for "square-bracket" operators

$$Y[u,z] = \sum_{n\in\mathbb{Z}} u[n]z^{-n-1} = Y(e^{zL(0)}u, e^z - 1).$$

These satisfy a VOA isomorphic to $V$ for Virasoro vector $\widetilde{\omega} = \omega - \frac{C}{24}\mathbf{1}$.

# Genus One Zhu Recursion Theory

A deep and effective theory for understanding how modular and elliptic functions arise for VOAs. Based on locality, Zhu recursion relates $n$-pt functions to $n-1$-pt functions. Thus

$$\begin{aligned} Z_V^{(1)}(u,x;v,y;q) &= \mathrm{Tr}_V\left(o(u)o(v)q^{L(0)-C/24}\right) \\ &\quad + \sum_{m\geq 0}\frac{1}{m!}\frac{\partial^m}{\partial y^m}P_1(x-y)Z_V^{(1)}(u[m]v,q), \end{aligned}$$

for elliptic and modular functions

$$\begin{aligned} P_1(z) &= -\sum_{n\in\mathbb{Z},n\neq 0}\frac{e^{nz}}{1-q^n} = \frac{1}{z}-\sum_{n\geq 1}E_{2n}(q)z^{2n-1}, \\ E_{2n}(q) &= -\frac{B_{2n}}{(2n)!}+\frac{2}{(2n-1)!}\sum_{r\geq 0}\frac{r^{2n-1}q^r}{1-q^r}, \end{aligned}$$

for Eisenstein series of even weight $2n$ and Bernoulli nos $B_{2n}$.
In particular, $P_2(z) = -\frac{\partial}{\partial z}P_1(z) = \wp(z)+E_2(q)$, for the Weierstrass function $\wp(z)$ for $z\in\mathbb{C}$ and $q=e^{2\pi i\tau}$ for $\tau\in\mathbb{H}_1$.

# Some Ward and Heisenberg Identities

For the Virasoro vector $\widetilde{\omega} = \omega - \frac{C}{24}\mathbf{1}$ we find

$$Z^{(1)}(\widetilde{\omega}, q) = \mathrm{Tr}_V((L(0) - \frac{C}{24})q^{L(0)-C/24}) = q\frac{\partial}{\partial q}Z^{(1)}(q) = \frac{1}{2\pi i}\frac{\partial}{\partial \tau}Z^{(1)}(q).$$

For $n$ primary vectors $u_1, \ldots, u_n$, Zhu reduction gives the Ward identity

$$Z_V^{(1)}(\tilde{\omega}, x; u_1, z_1; \ldots; u_n, z_n; q) = q\frac{\partial}{\partial q}Z_V^{(1)}(u_1, z_1; \ldots; u_n, z_n; q)$$
$$+ \sum_{1\leq i\leq n}\left[P_1(x - z_i)\frac{\partial}{\partial z_i} + \mathrm{wt}[u_i]\frac{\partial}{\partial z_i}P_1(x - z_i)\right] Z_V^{(1)}(u_1, z_1; \ldots; u_n, z_n; q)$$

For the Heisenberg VOA $M$, Zhu reduction gives all $n$-pt functions e.g.

$$Z_M^{(1)}(a, x; a, y; q) = P_2(x - y)Z_M^{(1)}(q).$$

Then $Z_M^{(1)}(\widetilde{\omega}, q) = \frac{1}{2}\lim_{x\to y}\left(Z_M^{(1)}(a, x; a, y; q) - \frac{Z_M^{(1)}(q)}{(x-y)^2}\right) = \frac{1}{2}E_2(q)Z_M^{(1)}(q).$

$\Rightarrow$ ODE: $q\frac{\partial}{\partial q}Z_M^{(1)}(q) = \frac{1}{2}E_2(q)Z_M^{(1)}(q)$ i.e. $Z_M^{(1)}(q) = 1/\eta(q).$

# Heisenberg Modules

Zhu reduction also applies to $n$-pt correlation functions for a VOA module. For a Heisenberg module $N_\alpha$ (i.e. $a(n)w = \alpha\delta_{n0}w$ for all $w \in N_\alpha$ and $n \geq 0$ for some $\alpha \in \mathbb{C}$) one finds directly that

$$Z_\alpha^{(1)}(q) = \mathrm{Tr}_{N_\alpha}(q^{L(0)-1/24}) = \frac{q^{\alpha^2/2}}{\eta(q)}.$$

The Virasoro 1-pt function is again

$$Z_\alpha^{(1)}(\widetilde{\omega}, q) = q\frac{\partial}{\partial q} Z_\alpha^{(1)}(q).$$

On the other hand, Zhu reduction gives

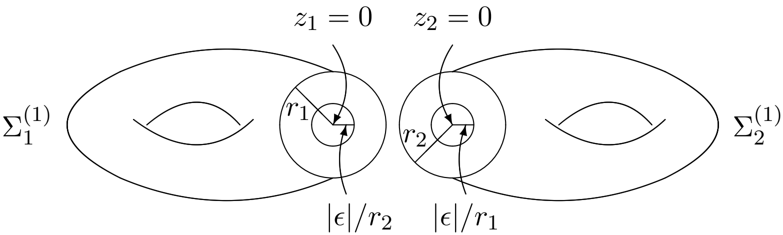
$$Z_\alpha^{(1)}(a, x; a, y; q) = \left(\frac{\alpha^2}{2} + P_2(x-y)\right) Z_\alpha^{(1)}(q),$$

so that

$$Z_\alpha^{(1)}(\widetilde{\omega}, q) = \left(\frac{\alpha^2}{2} + \frac{1}{2}E_2(q)\right) Z_\alpha^{(1)}(q).$$

Comparing implies $Z_\alpha^{(1)}(\widetilde{\omega}, q)$ satisfies an ODE with solution $q^{\alpha^2/2}/\eta(q)$.

# Genus Two Riemann Surface from Two Sewn Tori




Consider two tori $\Sigma_a^{(1)}$ with modular parameters $q_a = e^{2\pi i \tau_a}$ for $a = 1, 2$. Identify annular regions shown via the sewing relation

$$z_1 z_2 = \epsilon, \quad \epsilon \in \mathbb{C}.$$

Defines a genus two Riemann surface $\Sigma^{(2)}$ parameterized by

$$\mathcal{D}^\epsilon = \{(\tau_1, \tau_2, \epsilon) \in \mathbb{H}_1 \times \mathbb{H}_1 \times \mathbb{C} : |\epsilon| < \frac{1}{4} D(\tau_1) D(\tau_2)\},$$

(where $D(\tau) = 2\pi \min_{(m,n)\neq(0,0)} |m + n\tau|$).

# Structures on $\Sigma^{(2)}$

For standard homology basis $a_i, b_j$ with $i, j = 1, 2$ on a genus $2$ Riemann surface consider the normalized differential of the second kind. This is a symmetric meromorphic form $\omega^{(2)}(x, y)$ obeying

$$\omega^{(2)}(x, y) \sim \frac{dxdy}{(x-y)^2},$$

for local coordinates $x \sim y$ with $\oint_{a_i} \omega^{(2)}(x, \cdot) = 0$.
We can use $\omega^{(2)}$ to find a normalized basis of holomorphic 1-forms $\nu_i$ and the period matrix $\Omega_{ij}$:

$$\begin{aligned} \nu_i(x) &= \oint_{b_i} \omega^{(2)}(x, \cdot), \quad \oint_{a_j} \nu_i = \delta_{ij}, \\ \Omega_{ij} &= \frac{1}{2\pi i} \oint_{b_i} \nu_i. \end{aligned}$$

# $\omega^{(2)}$ on the Sewn Surface $\Sigma^{(2)}$

$\omega^{(2)}$ can be determined from the genus one normalized differential

$$\omega^{(1)} = P_2(x-y)dxdy, \quad P_2(z) = \frac{1}{z^2} + \sum_{n\geq 1}(2n-1)E_{2n}(q)z^{2n-2},$$

for each sewn torus [Yamada, Mason-T].
We define an infinite matrix indexed by $k, l \geq 1$

$$A(k,l,\tau,\epsilon) = \epsilon^{(k+l)/2}\tfrac{1}{\sqrt{kl}}(-1)^{k+1}\tfrac{(k+l-1)!}{(k-1)!(l-1)!}E_{k+l}(q)$$

$$= \begin{bmatrix} \epsilon E_2(q) & 0 & \sqrt{3}\epsilon^2 E_4(q) & 0 & \cdots \\ 0 & -3\epsilon^2 E_4(q) & 0 & -5\sqrt{2}\epsilon^3 E_6(q) & \cdots \\ \sqrt{3}\epsilon^2 E_4(q) & 0 & 10\epsilon^3 E_6(q) & 0 & \cdots \\ 0 & -5\sqrt{2}\epsilon^3 E_6(q) & 0 & -35\epsilon^4 E_8(q) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

# Convergence of $(I - A_1A_2)^{-1}$ and $\det(I - A_1A_2)$

Let $A_1$ and $A_2$ denote the matrix for modular parameter $\tau_1$ and $\tau_2$ resp. and consider the infinite matrix $I - A_1A_2$ where $I$ is the infinite identity matrix. We define $(I - A_1A_2)^{-1}$ and $\det(I - A_1A_2)$ by

$$
\begin{aligned}
(I - A_1A_2)^{-1} &= \sum_{n\geq 0}(A_1A_2)^n, \\
\log\det(I - A_1A_2) &= \operatorname{Tr}\log(I - A_1A_2) = -\sum_{n\geq 1}\frac{1}{n}\operatorname{Tr}((A_1A_2)^n).
\end{aligned}
$$

**Theorem (Mason-T)**

- $(I - A_1A_2)^{-1}$ *is convergent on the sewing domain* $\mathcal{D}^\epsilon$.
- $\det(I - A_1A_2)$ *is non-vanishing and holomorphic on* $\mathcal{D}^\epsilon$.

# The Genus Two Period Matrix

**Theorem (Mason-T)**

*$\Omega(\tau_1, \tau_2, \epsilon)$ is holomorphic on $\mathcal{D}^\epsilon$ and is given by*

$$\begin{aligned} 2\pi i \Omega_{11} &= 2\pi i \tau_1 + \epsilon \left( A_2 (I - A_1 A_2)^{-1} \right)(1,1), \\ 2\pi i \Omega_{22} &= 2\pi i \tau_2 + \epsilon \left( (A_1 (I - A_2 A_1)^{-1} \right)(1,1), \\ 2\pi i \Omega_{12} &= -\epsilon \left( (I - A_1 A_2)^{-1} (1,1). \end{aligned}$$

*Here $(1,1)$ refers to the $(1,1)$-entry of a matrix.*

There are related closed formulas for the normalized differential $\omega^{(2)}(x,y)$ and the holomorphic 1-forms $\nu_1(x)$ and $\nu_2(x)$.

# The Genus Two Partition Function

We define the genus two partition function in terms of 1-point functions $Z_V^{(1)}(u, \tau_a)$ for all $u \in V$.

$$Z_V^{(2)}(\tau_1, \tau_2, \epsilon) = \sum_{n \geq 0} \epsilon^n \sum_{u \in V_{[n]}} Z_V^{(1)}(u, q_1) Z_V^{(1)}(\overline{u}, q_2),$$

for genus one 1-pt functions $Z_V^{(1)}(u, q_1)$ etc and summing over any $V$-basis $\{u\}$ with LiZ dual basis $\{\overline{u}\}$.

**Theorem (Mason-T)**

*For the rank Heisenberg VOA $M$ we have*

- $Z_M^{(2)}(\tau_1, \tau_2, \epsilon) = \dfrac{1}{\eta(q_1)\eta(q_2)}(\det(I - A_1 A_2))^{-1/2}$.
- $Z_M^{(2)}(\tau_1, \tau_2, \epsilon)$ *is holomorphic on* $\mathcal{D}^\epsilon$.
- $Z_M^{(2)}$ *is automorphic wrt* $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z}) \subset Sp(4, \mathbb{Z})$ *(with an automorphy factor* $\det(C\Omega + D)^{-1/2}$ *and multiplier system).*

Let $N_{\alpha_1}, N_{\alpha_2}$ be a pair of Heisenberg modules. Define

$$Z^{(2)}_{\alpha_1,\alpha_2}(\tau_1,\tau_2,\epsilon) = \sum_{n\geq 0}\epsilon^n \sum_{u\in M_{[n]}} Z^{(1)}_{\alpha_1}(u,q_1) Z^{(1)}_{\alpha_2}(\overline{u},q_2),$$

for genus one 1-pt functions $Z^{(1)}_{\alpha_1}(u,q_1)$ etc.

Find a natural generalization of the genus one result $Z^{(1)}_{\alpha}(q) = q^{\alpha^2/2} Z^{(1)}_M(q)$:

**Theorem (Mason-T)**

$Z^{(2)}_{\alpha_1,\alpha_2}(\tau_1,\tau_2,\epsilon)$ *is holomorphic on the sewing domain* $\mathcal{D}^{\epsilon}$ *and is given by*

$$Z^{(2)}_{\alpha_1,\alpha_2}(\tau_1,\tau_2,\epsilon) = e^{i\pi\alpha.\Omega.\alpha} Z^{(2)}_M(\tau_1,\tau_2,\epsilon),$$

*where* $\alpha.\Omega.\alpha = \alpha_1\Omega_{11}\alpha_1 + \alpha_2\Omega_{22}\alpha_2 + 2\alpha_1\Omega_{12}\alpha_2$.

# Genus Two $n$-Point Correlation Functions

We define a genus two $n$-pt function for $z_1, \ldots, z_n \in \Sigma_1^{(1)}$ by

$$Z_V^{(2)}(u_1, z_1; \ldots; u_n, z_n) = \sum_u \epsilon^n Z_V^{(1)}(u_1, z_1; \ldots; u_n, z_n; u, 0; \tau_1) Z_V^{(1)}(\overline{u}, \tau_2),$$

(suppressing $\tau_1, \tau_2, \epsilon$) and similarly for other insertions on $\Sigma_1^{(1)}, \Sigma_2^{(1)}$.

**Theorem (Mason-T)**

*All Heisenberg VOA $M$ genus two $n$-pt functions are explicitly known e.g.*

$$Z_M^{(2)}(a, x; a, y) dx dy = \omega^{(2)}(x, y) Z_M^{(2)}.$$

This implies the Heisenberg genus two Virasoro 1-point function is

$$Z_M^{(2)}(\widetilde{\omega}, x) dx^2 = \frac{1}{12} s^{(2)}(x) Z_M^{(2)},$$

for projective connection: $s^{(2)}(x) = 6 \lim_{x \to y} \left( \omega^{(2)}(x, y) - \frac{dx dy}{(x-y)^2} \right)$.

This is the genus 2 analogue of $Z_M^{(1)}(\widetilde{\omega}, q) = \frac{1}{2} E_2(q) Z_M^{(1)}(q)$.

# Genus Two Zhu Theory - See Tom Gilroy's Talk

We have developed a genus two Zhu theory from which Ward identities follow [Gilroy-T]. The genus 2 analogue of $Z_V^{(1)}(\omega, q) = q\frac{\partial}{\partial q} Z_V^{(1)}(q)$ is

**Theorem (Gilroy-T)**

$$Z_V^{(2)}(\widetilde{\omega}, x)dx^2 = \mathbb{D}_x Z_V^{(2)},$$

*where* $\mathbb{D}_x = \mathbb{A}(x)q_1\frac{\partial}{\partial q_1} + \mathbb{B}(x)q_2\frac{\partial}{\partial q_2} + \mathbb{C}(x)\epsilon\frac{\partial}{\partial \epsilon}$ *for specific local 2-forms* $\mathbb{A}, \mathbb{B}$ *and* $\mathbb{C}$*.*

Thus for the Heisenberg VOA $M$ we find $Z_M^{(2)}$ satisfies the PDE:

$$\mathbb{D}_x Z_M^{(2)} = \frac{1}{12} s^{(2)}(x) Z_M^{(2)},$$

the genus 2 analogue of $q\frac{\partial}{\partial q} Z_M^{(1)}(q) = \frac{1}{2}E_2(q) Z_M^{(1)}(q)$.

# Heisenberg Modules

**Theorem (Mason-T)**

*All $n$-pt functions for a Heisenberg module pair $N_{\alpha_1}, N_{\alpha_2}$ are known e.g.*

$$Z^{(2)}_{\alpha_1,\alpha_2}(a,x;a,y)dxdy = \left[\frac{1}{2}\nu_\alpha(x)\nu_\alpha(y) + \omega^{(2)}(x,y)\right] Z^{(2)}_{\alpha_1,\alpha_2},$$

*for 1-form $\nu_\alpha(x) = \alpha_1\nu_1(x) + \alpha_2\nu_2(x)$,*

Thus $Z^{(2)}_{\alpha_1,\alpha_2} = e^{i\pi\alpha.\Omega.\alpha} Z^{(2)}_M$ satisfies the PDE:

$$\mathbb{D}_x Z^{(2)}_{\alpha_1,\alpha_2} = \left[\frac{1}{2}\nu_\alpha(x)^2 + \frac{1}{12}s^{(2)}(x)\right] Z^{(2)}_{\alpha_1,\alpha_2}.$$

It follows that for $i,j = 1,2$

$$\mathbb{D}_x \Omega_{ij} = \nu_i(x)\nu_j(x) \Rightarrow \mathbb{D}_x f(\Omega) = \frac{1}{2\pi i}\sum_{i\leq j}\nu_i(x)\nu_j(x)\frac{\partial}{\partial\Omega_{ij}} f(\Omega).$$

for any differentiable function $f(\Omega)$.

# Outlook

- What is the geometrical meaning of the coefficient functions and the modular derivative operator $\mathbb{D}_x$ in the genus two Zhu reduction formula?
- Prove convergence of partition and $n$-pt functions as solutions to genus two modular differential equations.
- Generalize Zhu reduction, $\mathbb{D}_x$ operator etc to higher genus.
- Schottky problem?