A tour of G_2 manifolds

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No physics

I don't know any physics, so I won't be able to connect this mathematical story to physics much.

Why G_2 metrics?

A G_2-manifold is a 7-dimensional manifold with a metric of Euclidean signature which admits a parallel spinor, and is not (even locally) a product metric.

Why a parallel spinor?

A parallel spinor is an essential ingredient in some physical models in order to allow $N = 1$ SUSY to survive in an effective field theory after compactifying 7-dimensions.

To be explained

1926	Élie Cartan	holonomy group
1955	Marcel Berger	attempted classification
1987	Robert Bryant	G_2 manifolds exist (perhaps not compact)
1989	Bryant & Salamon	complete G_2 manifolds exist (thin necks)
1992	Bryant & Altschuler	try to melt metrics "close to G_2" into G_2.
1992	Dominic Joyce	compact G_2 manifolds exist.
2012	Karigiannis & Lotay	Conical singularities exist

Why are they called G_2?

To first order, any Riemannian metric is flat (curvature is 2nd order). Pick a point, and look in local coordinates at the linear transformations (i.e. transformations at first order) preserving the metric and the point. These are just the rotations.

Why are they called G_2?

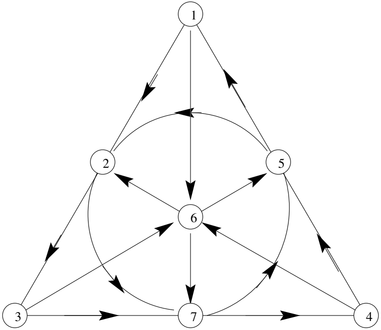
But if you also want to preserve a spinor, you get a smaller group. In 6-dimensions, you get SU(3), and a metric with parallel spinor is a CY metric. In 7-dimensions, you get G_2, a 14-dimensional exceptional simple Lie group.

Why are they called G_2?

A metric on a 7-dimensional manifold is a G_2-metric if it has a parallel spinor, and is not locally a product metric. No explicit compact examples are known.

What is G_2?

For each vertex in this drawing,



take a variable. Following arrows along a straight edge or around the circle, if you hit three variables a, b, c then let $ab = c$. Also add in all relations $a^2 = -1$. This defines a noncommutative, nonassociative real algebra, called the *octonions* or *octaves*. The group of symmetries of this algebra is G_2.

First examples

Bryant proved that G_2-metrics exist, but just in "little pieces", maybe not compact or maybe not even complete. Write the problem as a huge system of partial differential equations, and prove the existence of local solutions using big machinery.

Complete examples

Bryant and Salamon constructed explicit examples; thin necks which look like smoothed out cones over S^3, S^4 or $\mathbb{CP}^2$. Method: look for symmetries, and reduce to ODEs. Need some reason why there might be such a spinor, topologically for example, and that gives some insight into why S^3, S^4 and $\mathbb{CP}^2$.

Compact ones exist

Joyce: any G_2-metric has a finite dimensional smooth deformation space.
Joyce: take a flat 7-dimensional torus. Quotient by a finite group action. Quotient space is not smooth. Deform away the singularities, cleverly. At some moment during the deformation, you can perturb the metric to get a G_2-metric.
Choose different finite group actions on 7-dimensional tori: hundreds of different examples, with different topology.

Melting

Heat flows have been powerful in recent mathematics. Make some PDE which looks like a nonlinear heat equation, and understand how its singularities look.

Melting

There are many flows we can apply to metrics and spinors to try to flow both into a metric with parallel spinor. Several flows have short time existence of solutions, but we know nothing about long time behaviour. Bad news: all explicit examples have been shown develop singularities in the flow. Solitons for the singularities are not classified. A few explicit solitons are known.

Singularities

"String theory": compactification of M-theory on a G_2-manifold gives chiral parity, unless perhaps the G_2-manifold has a conical singularity.

The singularities that led mathematicians away from this subject are now seen as good.

Singularities

Karigiannis & Lotay: finite dimensional smooth moduli space of conical singularity G_2-metrics on any compact space. Can sometimes remove these singularities and paste in something smooth (string theory?).

Singularities

Link on a point singularity is a “nearly Kähler” 6-manifold. A few examples known; not classified.

Things inside

Infinitesimal picture: the algebra $ab = c$ we constructed is not associative, but contains many *associative* 3-planes. Big picture: any G_2-metric will have a family of 3-dimensional absolutely minimal submanifolds, called *associative*, with these tangent planes. These are apparently analogues of the Riemann surfaces in string theory compactified on a CY manifold.

The same story happens with 4-dimensional objects, called *coassociative submanifolds*.

Things inside

"String theory": when you compactify M-theory on a G_2-manifold, to get nonabelian YM fields in the effective theory, you need associative singularities. Compact G_2-objects with such singularities are *not* known to exist.
Examples of associatives and coassociatives are known on some compact G_2 manifolds.

Open problem

What does M-theory suggest we should prove about G_2-metrics?