Irish Quantum Foundations

Modular differential equations in CFT

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The characters of the $(2,5)$ minimal model in CFT satisfy an ordinary differential equation which is modular. We discuss the corresponding equations in other minimal models and indicate how to generalise them to higher genus Riemann surfaces.

A CFT is soved once all of its N-point functions are known.

Aim: Determine the N-point functions

$$\langle \varphi_1(x_1) \dots \varphi_N(x_N) \rangle$$

of holomorphic fields of RCFT on arbitrary Riemann surfaces.

$g = 0$ is easy

$g = 1$ much is known

For $g > 1$, working with the **analytic** coordinate on $\mathbb{H}^+/\Gamma$ is difficult (problem of classical geometry, e.g. Maskit 1999) - not my concern.

Recent progress to $g = 2$ by sewing tori (Mason, Tuite, Zuevsky)

Suggestion: Use **algebraic** coordinates.

Consider the N-point functions of the **Virasoro field** T on a hyperelliptic genus g Riemann surface

$$X_g : \qquad y^2 = p(x) \,, \qquad n = \deg p = 2g + 1 \,.$$

x varies in $\mathbb{P}^1_{\mathbb{C}}$, y defines a ramified double cover.

For the (μ, ν) minimal model, $\langle TT \ldots \rangle$ can be determined up to a **finite set** of parameters.

Theorem 1. *(M.L. 2012) For the* $(2, 5)$ *minimal model on*

$$X_{g=2}: \quad y^2 = p(x),$$

we have

$$\langle T(x) \rangle = \frac{c}{32}\frac{[p']^2}{p^2}\langle \mathbf{1} \rangle + \frac{A_0 x^3 + A_1 x^2 + A_2 x + A_3}{4p} \, .$$

There are exactly 4 *parameters in*

$$\langle TT \rangle \quad \textit{and} \quad \langle TTT \rangle \, ,$$

given by $\langle \mathbf{1} \rangle$ *and* A_1, A_2, A_3. *All other constants (including* A_0*) are known.*

$\langle \mathbf{1} \rangle$ and A_1, A_2, A_3 depend on the moduli of X_2. Their dependence is governed by ODEs which I have not addressed in previous talks.

What I talk about today is **joint work in progress with W. Nahm**.

In RQFTs, one has a system of differential equations which close up.

Example: $(2,5)$ **minimal model for** $g = 1$

$$\frac{d}{d\tau}\langle \mathbf{1} \rangle = \frac{1}{2\pi i}\oint \langle T(z)\rangle \, dz = \frac{1}{2\pi i}\langle \mathbf{T} \rangle \ . \tag{1}$$

Here the contour integral is along the real period, and $\oint dz = 1$.

$$2\pi i \frac{d}{d\tau}\langle \mathbf{T} \rangle = \oint \langle T(w) T(z)\rangle \, dz = -4 G_2 \langle \mathbf{T} \rangle + \underbrace{\frac{22}{5} G_4 \langle \mathbf{1} \rangle}_{\text{minim. model}} \ . \tag{2}$$

(Here $G_2 \sim \pi^2 E_2$ resp. $G_4 \sim \pi^4 E_4$ (quasi)modular Eisenstein series.)

1st order ODEs (1) & (2) combine to give the **2nd order** ODE (Kaneko, Nagatomo, Sakai 2012)

$$\mathfrak{D}_2 \circ \mathfrak{D}_0 \langle \mathbf{1} \rangle = \frac{11}{3600} E_4 \langle \mathbf{1} \rangle \ , \quad \mathfrak{D}_{2\ell} := \frac{1}{2\pi i}\frac{d}{d\tau} - \frac{\ell}{6} E_2 \quad \text{(Serre derivative).}$$

The two solutions are the well-known Rogers-Ramanujan partition functions.

The two solutions to $\mathfrak{D}_2 \circ \mathfrak{D}_0 \langle \mathbf{1} \rangle = \frac{11}{3600} E_4 \langle \mathbf{1} \rangle$ are the Rogers-Ramanujan partition functions.

$$\langle \mathbf{1} \rangle_1 = q^{\frac{11}{60}} \sum_{n \geq 0} \frac{q^{n^2+n}}{(q)_n} = q^{\frac{11}{60}} + \ldots$$

$$\langle \mathbf{1} \rangle_2 = q^{-\frac{1}{60}} \sum_{n \geq 0} \frac{q^{n^2}}{(q)_n} = q^{-\frac{1}{60}} + \ldots .$$

Here $(q)_n$ is the q-Pochhammer symbol

$$(q)_n = (1-q)(1-q^2)\ldots(1-q^{n-1})(1-q^n) .$$

A different formula for the Teichmüller variation

We define a deformation of the hyperelliptic Riemann surface

$$X_g : \quad y^2 = p(x)\,, n = \deg p\,,$$

with roots X_j, by

$$X_j \mapsto X_j + \xi_j\,, \quad j = 1, \ldots, n,$$

where $\xi_j = dX_j$. Let Φ be a holomorphic field. We have

$$d\langle\Phi\rangle = \sum_{j=1}^{n} \left(\frac{1}{2\pi i} \oint_{\gamma_j} \langle T(z)\Phi\rangle\, dz \right) dX_j\,, \tag{3}$$

where γ_j is a closed path around X_j.

The case $g = 1$ in algebraic coordinates

$$\langle T(x) \rangle = \frac{c}{32} \frac{[p']^2}{p^2} \langle \mathbf{1} \rangle + \frac{A_0 x + A_1}{4p} \,.$$

(Only $A_1 \propto \langle \mathbf{1} \rangle$ is unknown.) Using our new variation formula,

Theorem 2. *(M.L. 2013) Let*

$$p = 4(x - X_1)(x - X_2)(x - X_3).$$

We define a deformation of the Riemann surface by

$$X_j \mapsto X_j + \xi_j \,, \qquad j = 1, 2, 3,$$

where $\xi_i = dX_i$. Assuming $\sum_{i=1}^{3} X_i = 0$, we have

$$d\langle \mathbf{1} \rangle = \frac{c}{8} \langle \mathbf{1} \rangle \left(\frac{X_1 \xi_1}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic} \right) - \frac{1}{8} A_1 \left(\frac{\xi_1}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic} \right).$$

The ODE for $\langle T(x)\rangle$ is given by

$$\Big(d+\Big(\frac{\xi_1 X_1}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\Big)\Big)\frac{\langle\mathbf{1}\rangle^{-1}A_1}{4}$$
$$=\Big(\frac{\xi_1 X_2 X_3}{(X_1-X_2)(X_2-X_3)}+\frac{\xi_1 X_3 X_2}{(X_2-X_3)(X_3-X_1)}+\text{cyclic}\Big)$$
$$-\frac{c^{[1]}}{2c}\Big(\frac{\xi_1}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\Big)$$

In the $(2,5)$ minimal model with

$$p(x)=4(x^3-30G_4x-70G_6)\,,$$

we have

$$c^{[1]}=-51c\,G_4-\left(\frac{\langle\mathbf{1}\rangle^{-1}A_1}{4}\right)^2$$

where $c=-\frac{22}{5}$. Thus we obtain a **system of ODEs that closes up**.

Generalisation to $g > 1$ in algebraic coordinates

Rewrite as

$$d\langle \mathbf{1}\rangle = -\frac{c}{8}\langle \mathbf{1}\rangle \frac{\det \Xi_{3,1}}{\det V_3} - \frac{1}{8}A_1 \frac{\det \Xi_{3,0}}{\det V_3}$$

for the matrices

$$\Xi_{3,0} = \begin{pmatrix} X_1 & X_2 & X_3 \\ 1 & 1 & 1 \\ \xi_1 & \xi_2 & \xi_3 \end{pmatrix}, \quad \Xi_{3,1} = \begin{pmatrix} X_1 & X_2 & X_3 \\ 1 & 1 & 1 \\ \xi_1 X_1 & \xi_2 X_2 & \xi_3 X_3 \end{pmatrix} = dV_3, \quad V_3 := \underbrace{\begin{pmatrix} 1 & X_1 & X_1^2 \\ 1 & X_2 & X_2^2 \\ 1 & X_3 & X_3^2 \end{pmatrix}}_{\text{Vandermonde}}.$$

where $\det \Xi_{3,1} = d \det V_3$. This generalises easily to higher genus. For $g = 2$ and $k = 0, \ldots, 3$, we have to deal with V_5 and

$$\Xi_{5,k} = \begin{pmatrix} X_1^3 & X_2^3 & X_3^3 & X_4^3 & X_5^3 \\ X_1^2 & X_2^2 & X_3^2 & X_4^2 & X_5^2 \\ X_1 & X_2 & X_3^3 & X_4^3 & X_5 \\ 1 & 1 & 1 & 1 & 1 \\ \xi_1 X_1^k & \xi_2 X_2^k & \xi_3 X_3^k & \xi_4 X_4^k & \xi_5 X_5^k \end{pmatrix}.$$

Comparison with analytic result for $g = 1$ & discussion

For $g = 1$, the algebraic result translates into the ODE

$$\left(d - \frac{c\pi i}{24} E_2\, d\tau + \text{const.}\frac{d\lambda}{\lambda}\right)\langle \mathbf{1} \rangle = \frac{1}{2\pi i}\frac{A_1}{4}\, d\tau\ .$$

where $\lambda = (\text{length of real period})^{-1}$.

Translation $z \leftrightarrow x$ is easy:
$\langle \mathbf{1} \rangle_z \sim \langle \mathbf{1} \rangle_x\, \Delta^{\frac{c}{24?}}$, where $\Delta \sim (\det V_3)^2$ is the discriminant.