Fractional Quantum Hall Effect of Lattice Bosons

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Plus some new stuff …

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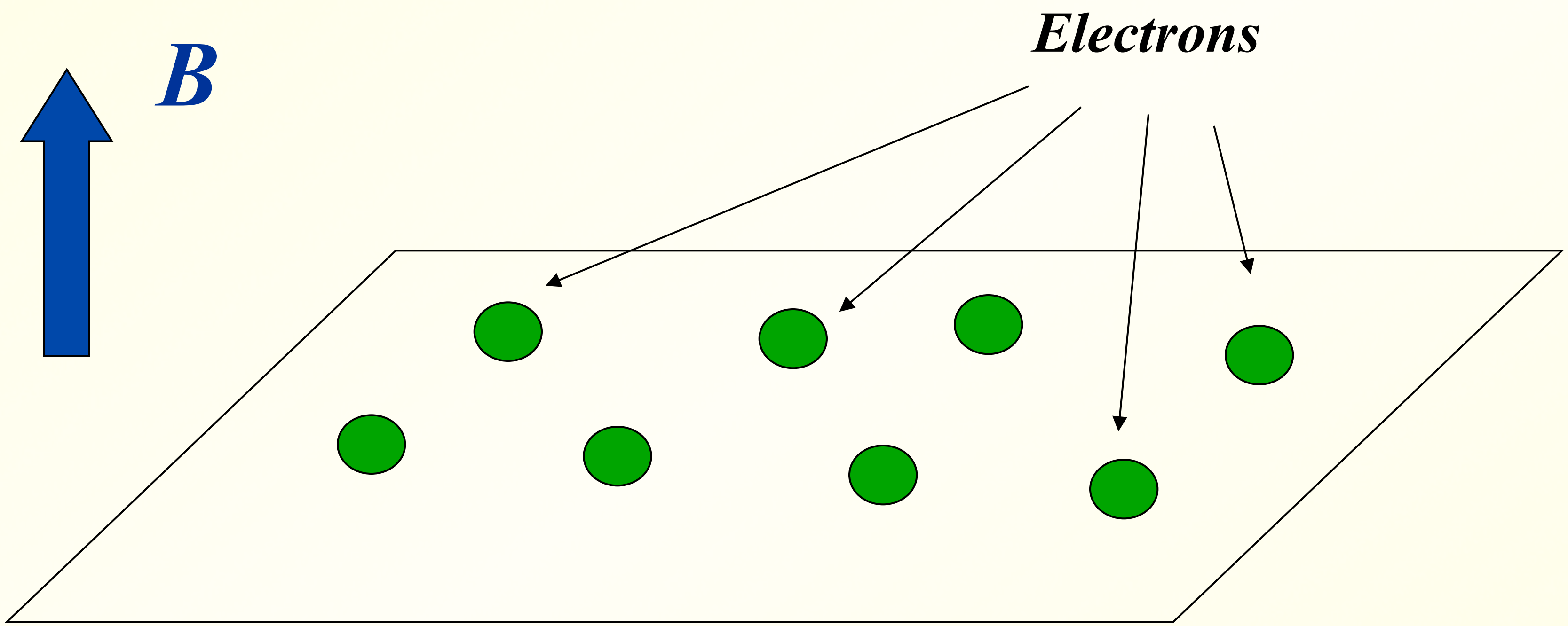
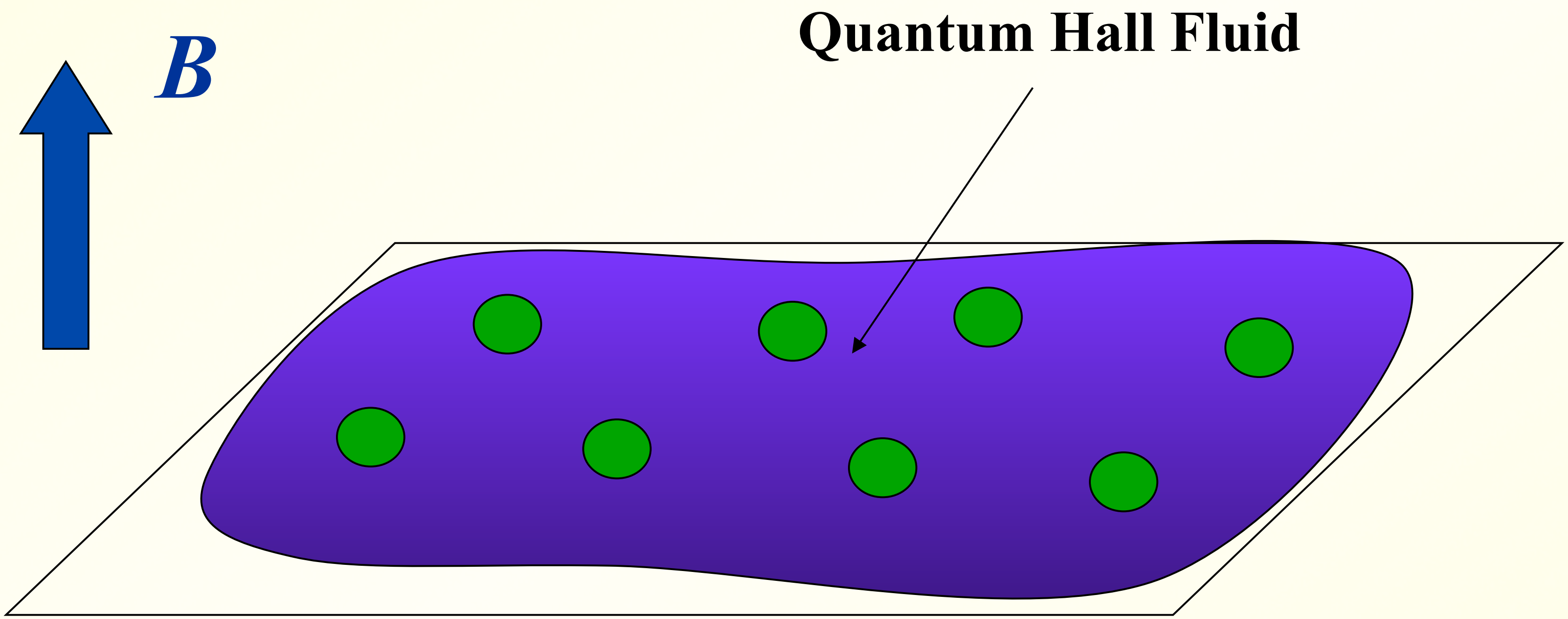
Support: NIST/NRC; Marie Curie IIF

Fractional Quantum Hall Effect



A two dimensional gas of interacting electrons in a strong magnetic field ***B***.

Fractional Quantum Hall Effect

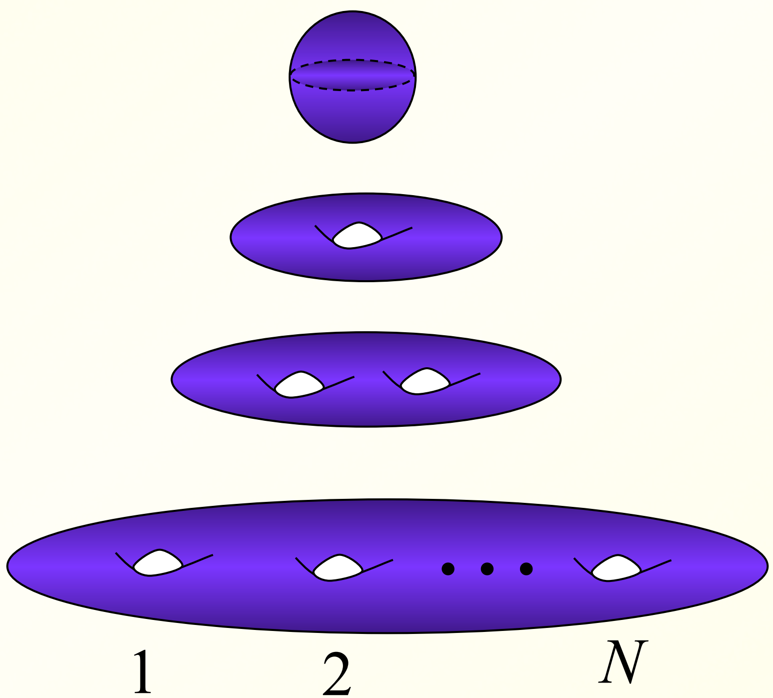


An **incompressible quantum liquid** can form when the Landau level filling fraction $\nu = n_e(h/eB)$ is a rational fraction.

Topological Degeneracy *(X.G. Wen)*

A theoretical curiosity: FQH states on **topologically nontrivial surfaces** have degenerate ground states which **can only be distinguished by global measurements**.

For the $\nu = 1/3$ state:



Degeneracy

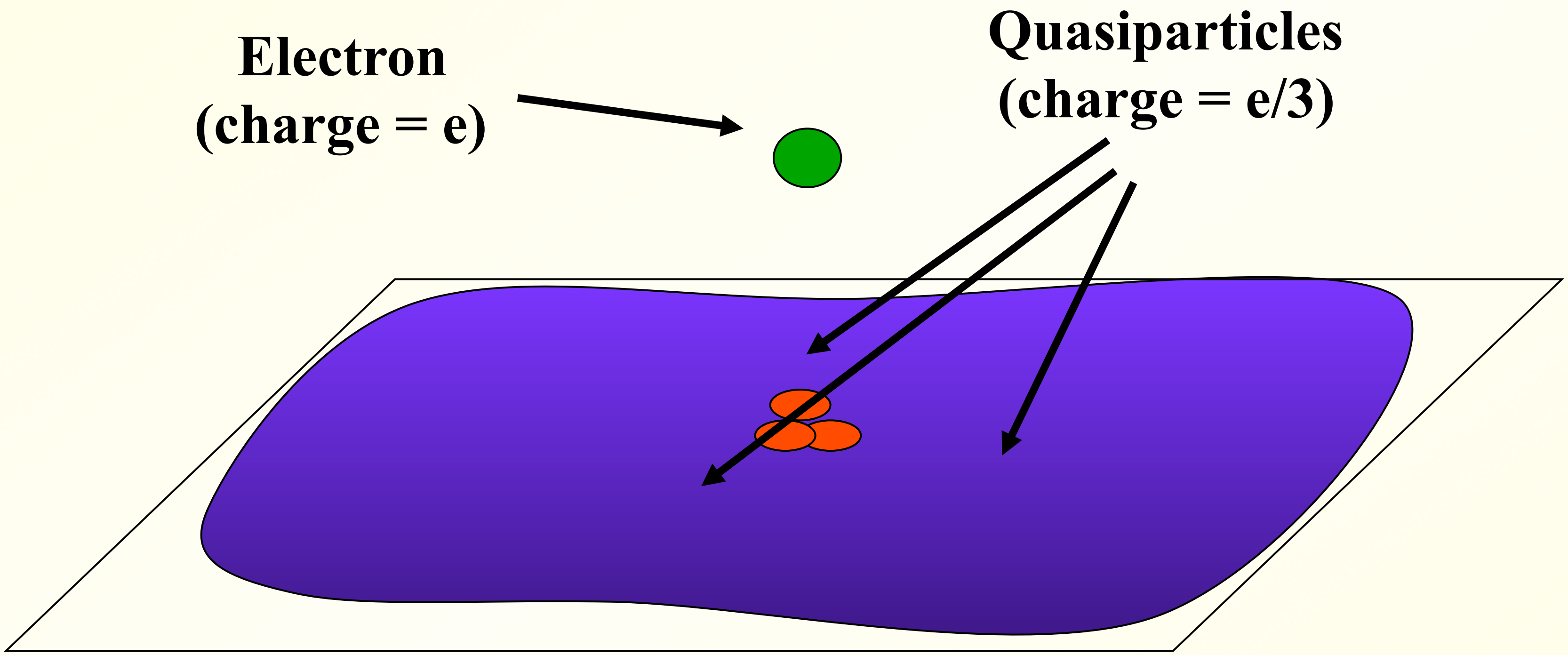
1

3

9

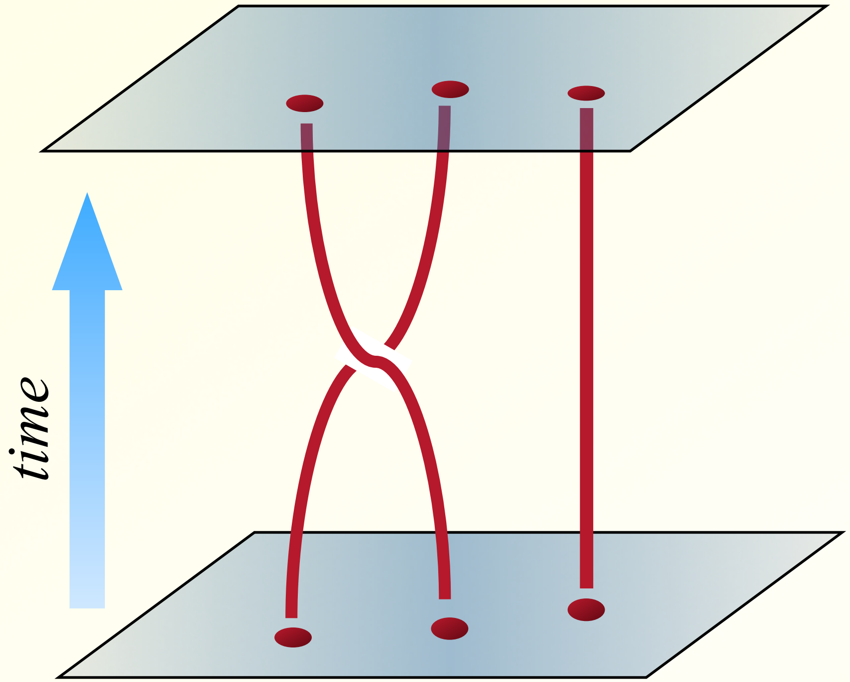
3^N

Fractional Quantum Hall Effect



When an electron is added to a FQH state it can be **fractionalized** --- i.e., it can break apart into **fractionally charged quasiparticles.**

Fractional Statistics: Abelian Anyons



$$|\psi_f\rangle = e^{i\phi}|\psi_i\rangle$$

'any' phase

$$|\psi_i\rangle$$

$e^{i\phi} = +1$ Bosons

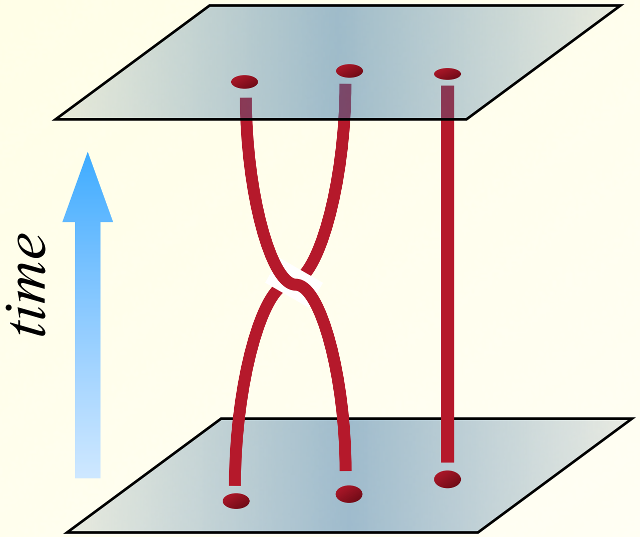
$e^{i\phi} = -1$ Fermions

$\nu = 1/3$ quasiparticles → $\phi = \pi/3$

→ Fractional Abelian Statistics!

Non-Abelian Anyons

Moore, Read, '91



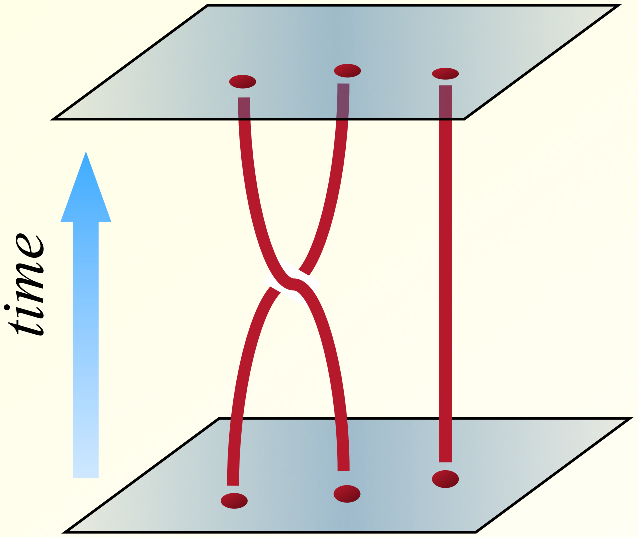
$$|\psi_f\rangle = \tilde{\alpha}|\psi_0\rangle + \tilde{\beta}|\psi_1\rangle$$

$$|\psi_i\rangle = \alpha|\psi_0\rangle + \beta|\psi_1\rangle$$

orthogonal degenerate states

Non-Abelian Anyons

Moore, Read, '91



$$|\psi_f\rangle = \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Unitary matrix represents topological braid operation.

$$|\psi_i\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

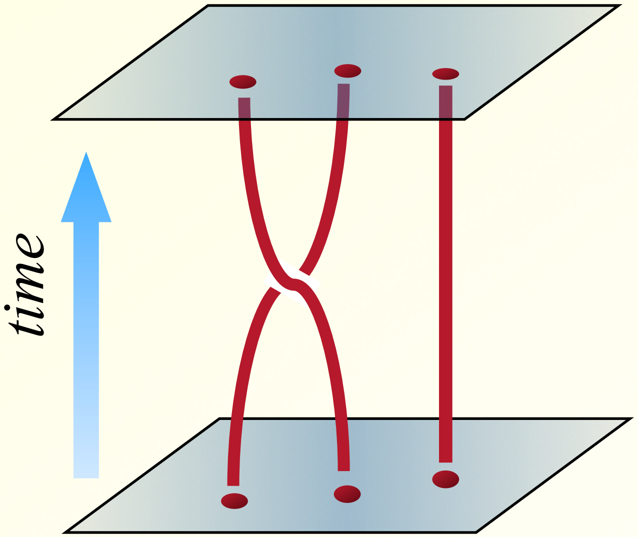
orthogonal degenerate states

Matrices form a **non-Abelian** representation of the **braid group**.

→ Non-Abelian Statistics

Non-Abelian Anyons

Moore, Read, ‘91



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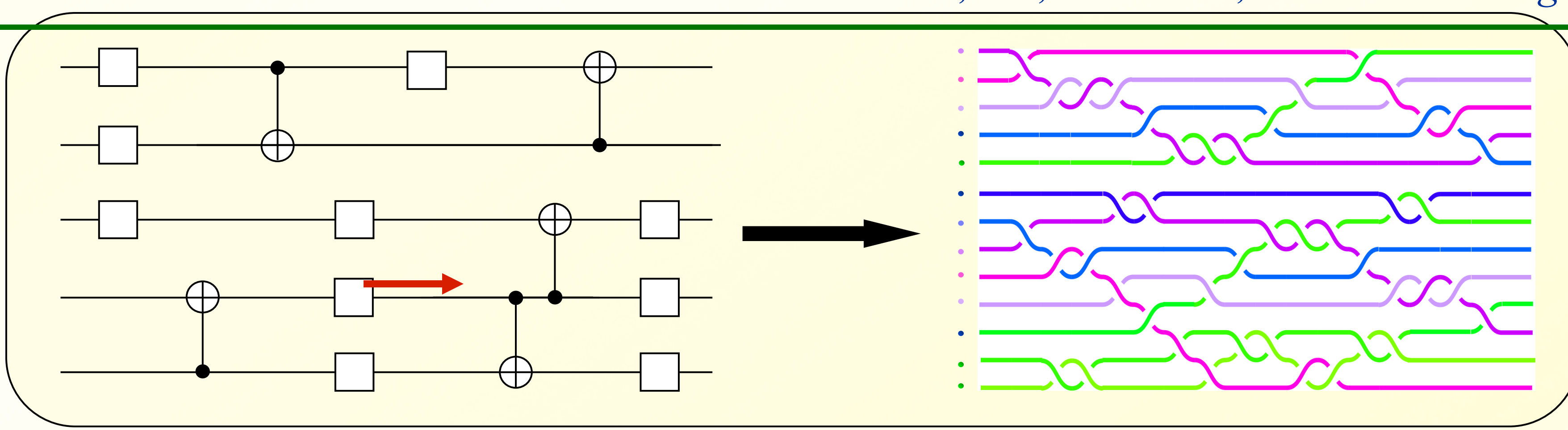
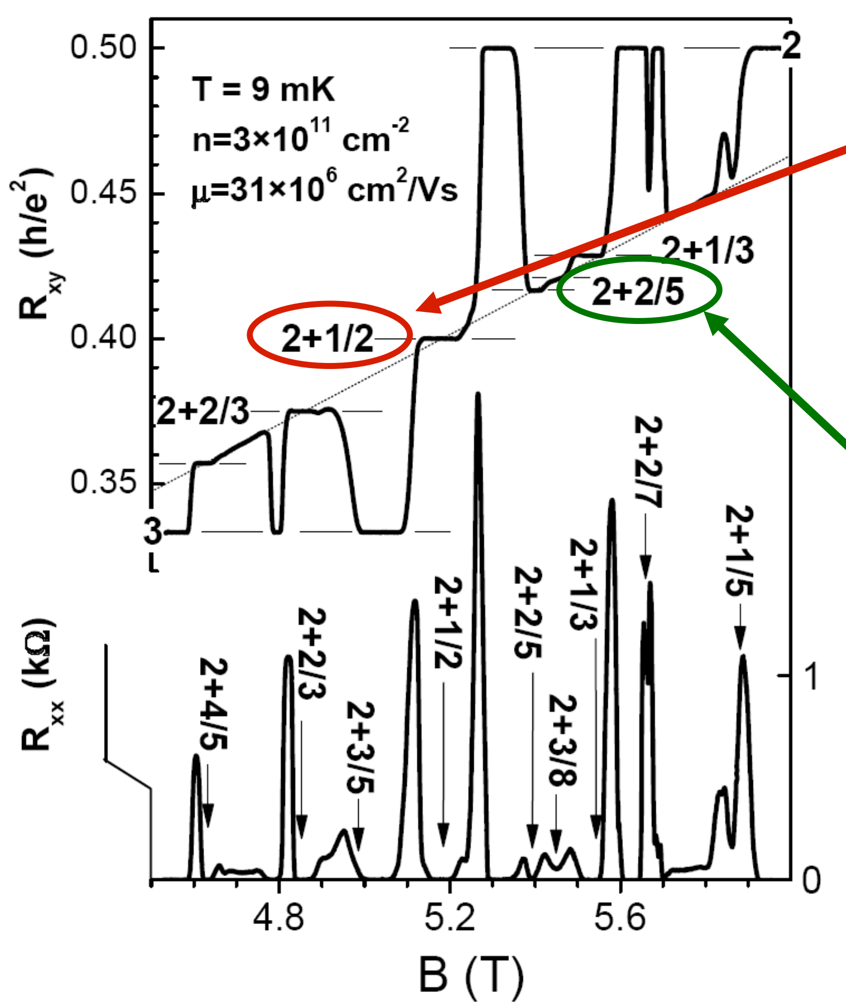
Topological Quantum Computer

Kitaev, ‘97; Freedman, Larsen and Wang ‘00



Non-Abelian FQH States



J.S. Xia et al., '04

$\nu = 5/2$: Believed to be the Moore-Read "Pfaffian" state.

Moore and Read, '91; Morf, '98

Not sufficiently rich non-Abelian statistics for pure topological quantum computing.

$\nu = 12/5$: Possibly a $k = 3$ Read-Rezayi "Parafermion" state.

Read and Rezayi, '99; Rezayi and Read, '06

→ *Fibonacci Anyons*

Good for quantum computation!

But these states are very delicate – hard to stabilize and manipulate.

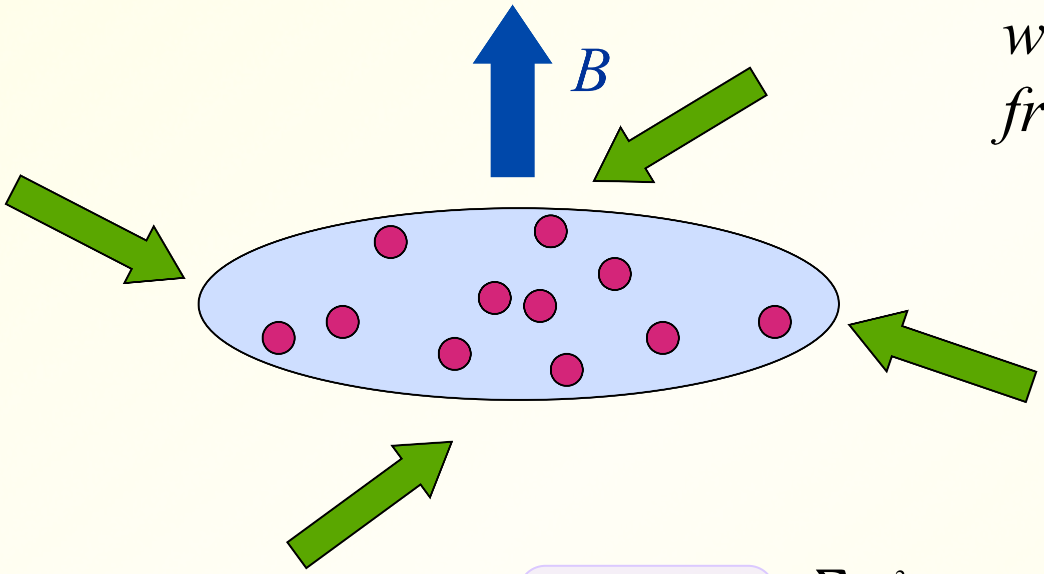
2D Bosons under Magnetic Field

Ultracold atomic gasses (usually bosons)

Synthetic magnetic field (rotation, laser-induces fields)

Contact interaction → *At sufficiently small $\nu = N / N_\phi$ we expect Bosonic versions of fractional quantum Hall states.*

Cooper et al., '01



$z = x + iy$

$\nu = \frac{1}{2}$ $\quad \psi_L(\{z_i\}) = \prod_{i<j}(z_i - z_j)^2 e^{-\sum_i |z_i|^2/4}$ *Laughlin State*

$\nu = 1$ $\quad \psi_{MR}(\{z_i\}) = Pf\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)\, e^{-\sum_i |z_i|^2/4}$ *Moore-Read State*

2D Bosons under Magnetic Field – Lattice

Ultracold atomic gasses (usually bosons)

Synthetic magnetic field (rotation, laser-induces fields)

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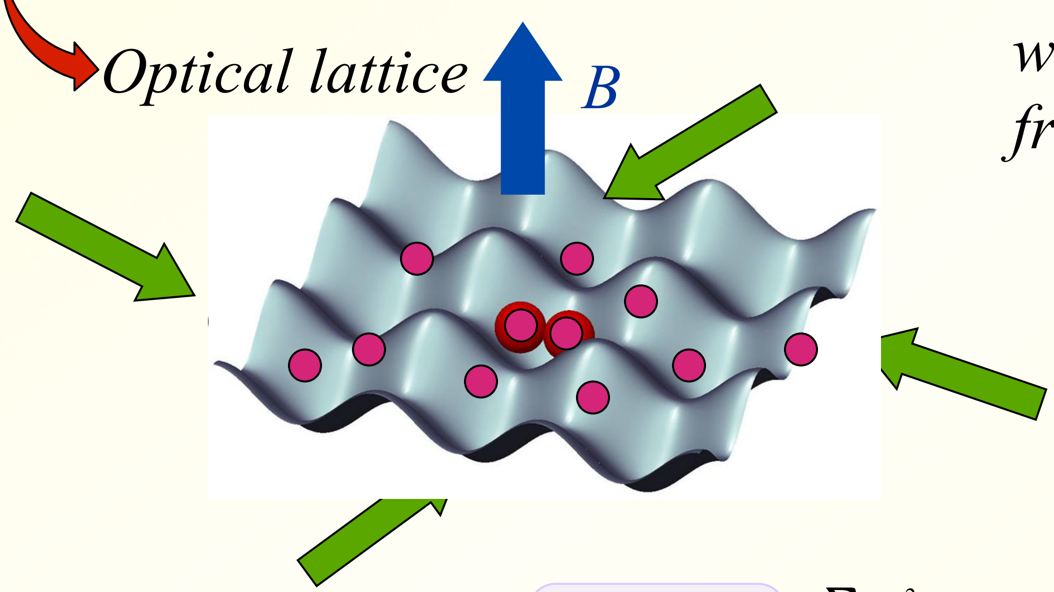
Cooper et al., '01

Optical lattice



How do the quantum Hall states modify in the presence of a lattice?

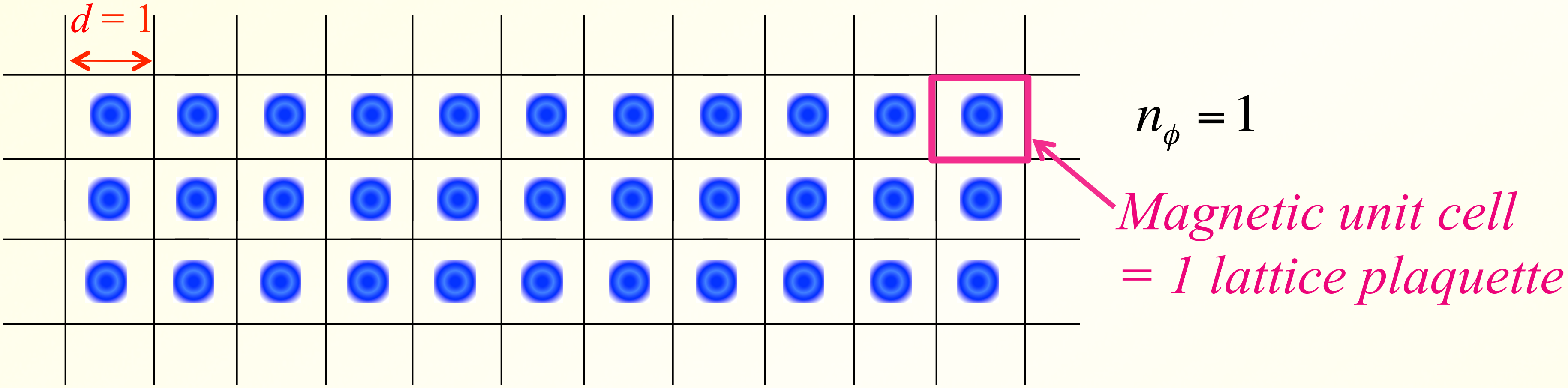
$$\nu = \frac{1}{2} \qquad \psi_L(\{z_i\}) = \prod_{i<j}(z_i - z_j)^2 e^{-\sum_i |z_i|^2/4}$$

Laughlin State

$$\nu = 1 \qquad \psi_{MR}(\{z_i\}) = Pf\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)\, e^{-\sum_i |z_i|^2/4}$$

Moore-Read State

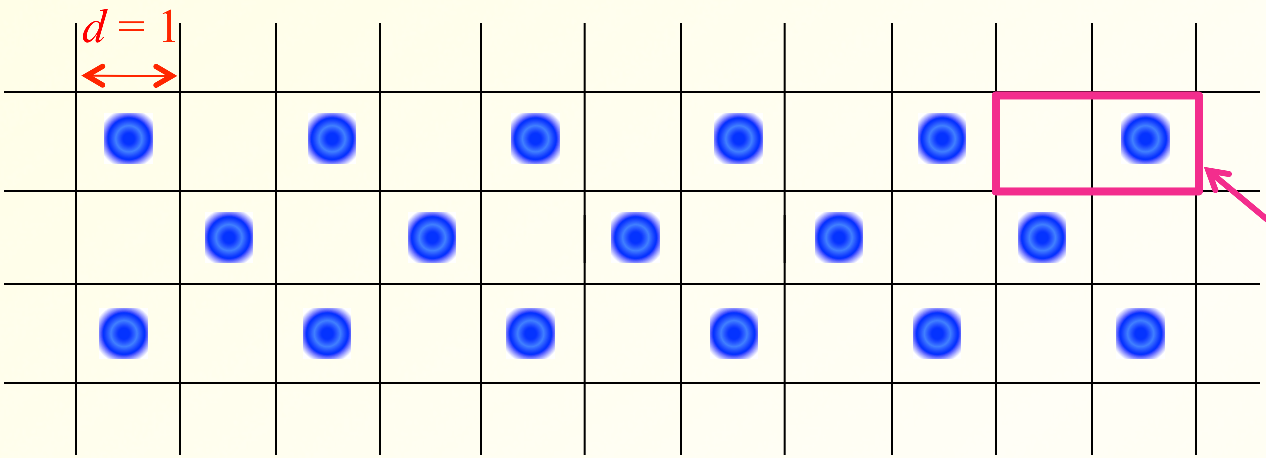
2D Bosons under Magnetic Field – Lattice



Flux density $n_\phi = \dfrac{Bd^2}{h/e}$

$0 \le n_\phi < 1$

2D Bosons under Magnetic Field – Lattice



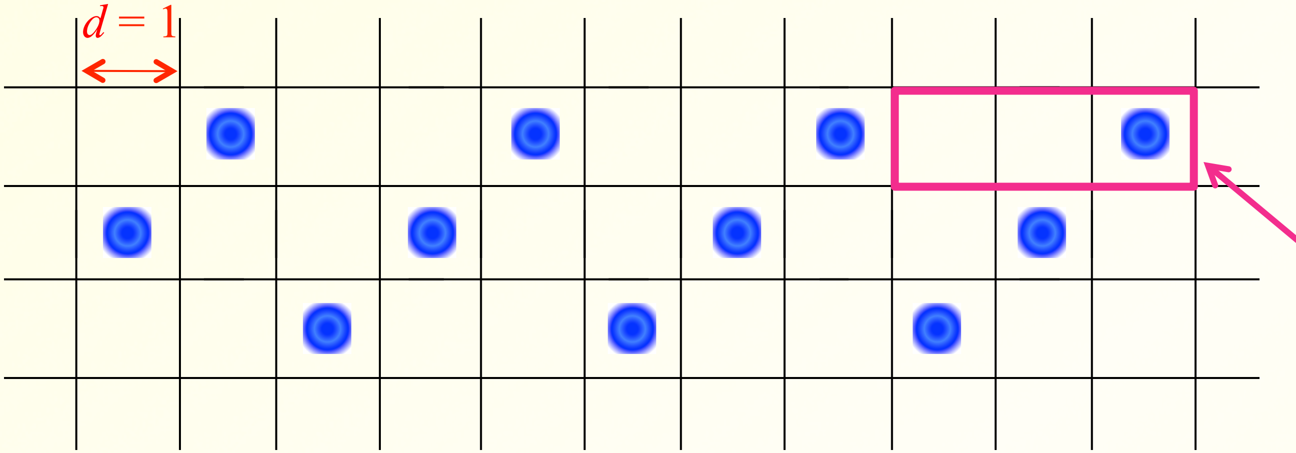
$n_\phi = \frac{1}{2}$

Magnetic unit cell
= 2 lattice plaquettes

Flux density $n_\phi = \frac{Bd^2}{h/e}$

$0 \le n_\phi < 1$

2D Bosons under Magnetic Field – Lattice



$$n_\phi = \frac{1}{3}$$

Magnetic unit cell = 3 lattice plaquettes

Flux density $n_\phi = \dfrac{Bd^2}{h/e}$

$0 \le n_\phi < 1$

$n_\phi = \dfrac{p}{q}$ → *p flux quanta per q plaquettes*

$n_\phi << 1$ → *Effectively the continuum limit*

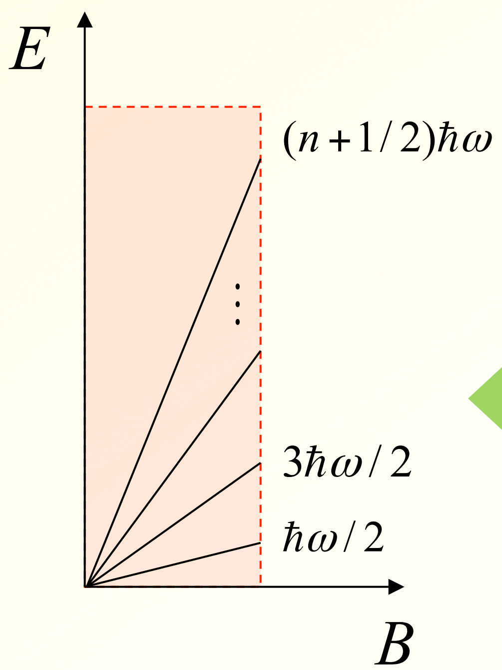
Sorensen et al., PRL '05
Hafezi et al., PRA '07

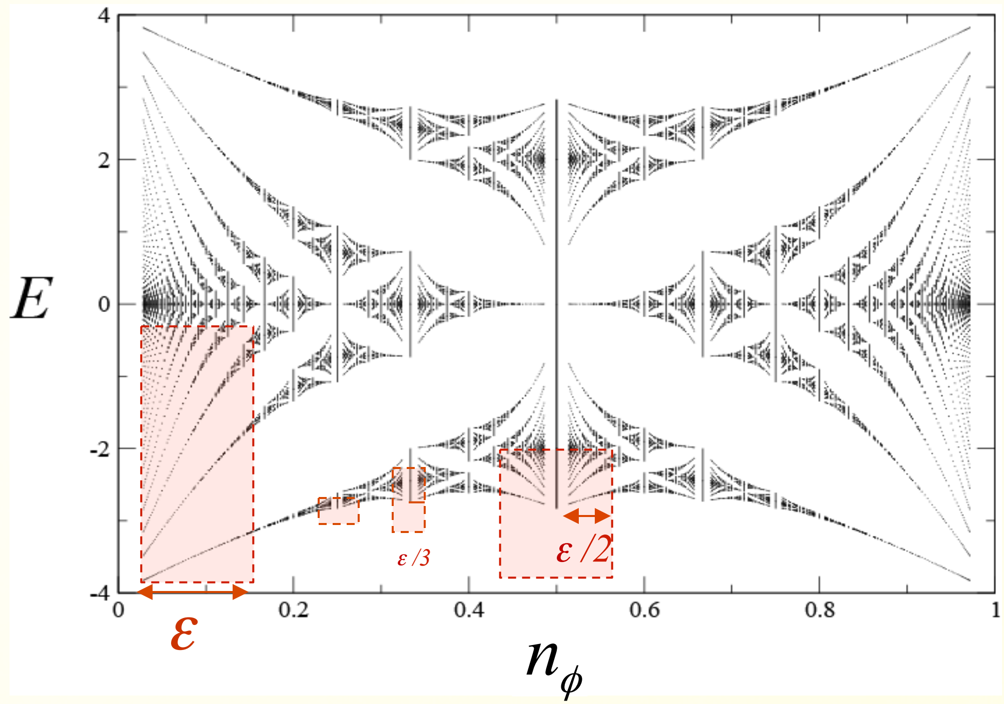
$n_\phi \sim p/q$ → *Map the lattice to a multi-layer model in the continuum limit.*

Palmer and Jacksch, PRL '06
Moller and Cooper, PRL '09
Powell et al., PRL '10

Charged Particles in a Magnetic Field – Single Particle Picture

Continuum *Lattice*





Landau Levels

$E_n = (n+1/2)\hbar\omega$ $\omega = qB/m$

Hofstadter Butterfly

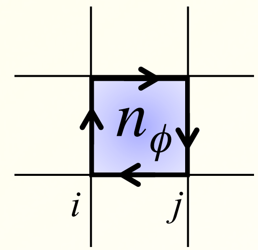
Hofstadter, '76

Map the lattice near rational n_ϕ to a model in the continuum!

Non-interacting System: $n_\phi = \varepsilon << 1$

Hofstadter, '76
Palmer and Jacksch, '06

$$H = -J \sum_{\langle ij \rangle} \left(e^{i\theta_{ij}} c_i^\dagger c_j + h.c. \right)$$



$$\theta_{ij} = \frac{2\pi}{h/e} \int_i^j \vec{A} \cdot d\vec{l}$$

$$\sum_{plaquette} \theta_{ij} = 2\pi n_\phi$$

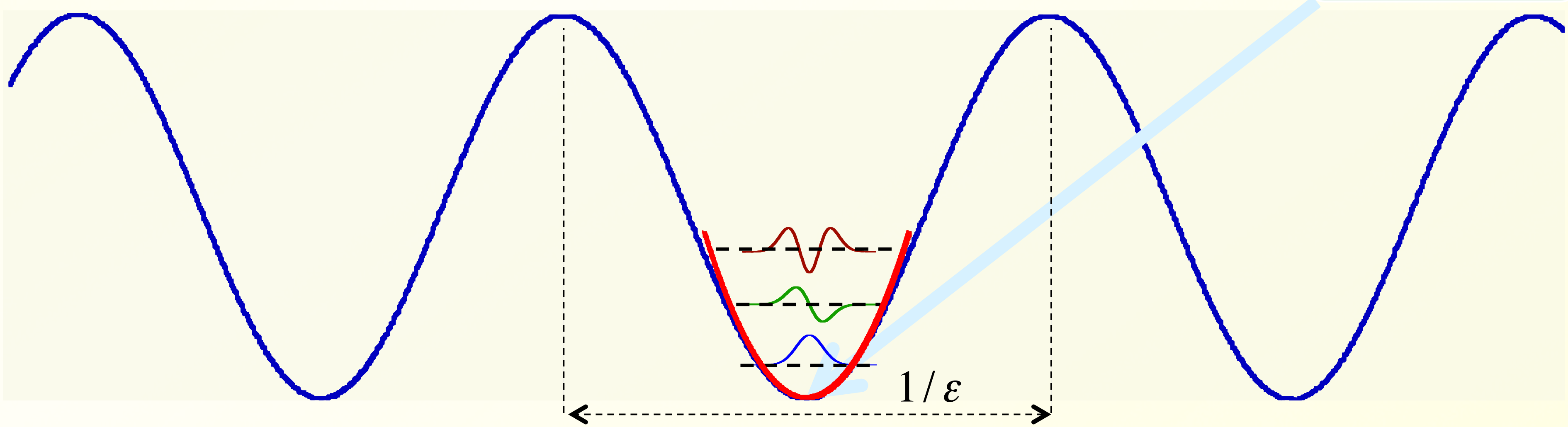
Landau Gauge
$\vec{A} = (0, -Bx, 0)$ → $\psi_k(x,y) \sim \phi(x)\, e^{iky}$

$$-\phi(x+1) - \phi(x-1) - 2Cos\,(2\pi n_\phi\, x - k)\phi(x) = E/J\, \phi(x)$$

$$l_0 = \frac{1}{\sqrt{2\pi\varepsilon}}$$

$n_\phi = \varepsilon << 1$ → $-\partial_x^2 \phi(x) + 2\pi\varepsilon (x - x_k)^2 \phi(x) = \tilde{E} \phi(x)$

$$x_k = \frac{k + 2m\pi}{2\pi\varepsilon}$$



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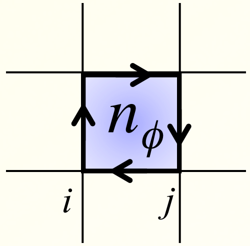


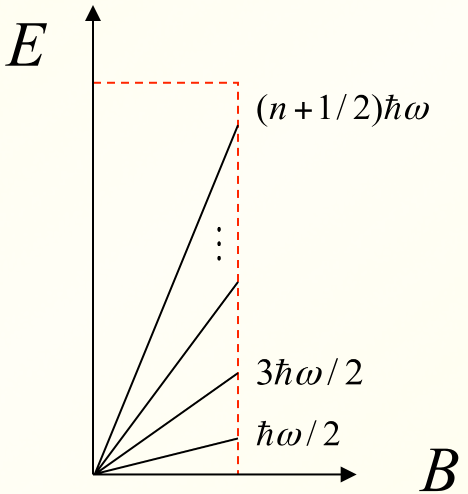
$$\theta_{ij} = \frac{2\pi}{h/e}\int_i^j \vec{A}\cdot d\vec{l}$$

$$\sum_{plaquette}\theta_{ij} = 2\pi n_\phi$$

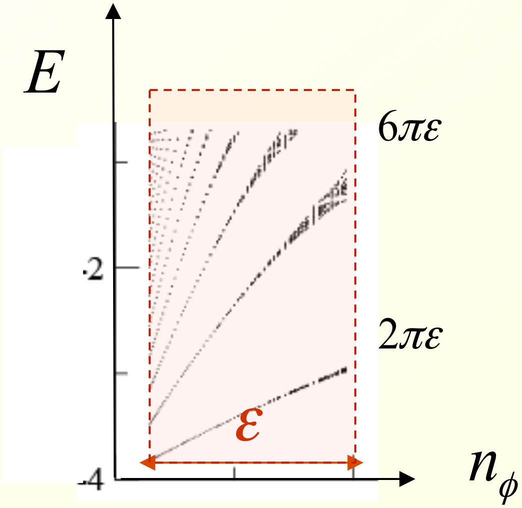
Landau Gauge $\vec{A} = (0,-Bx,0)$ → $$H = -\partial_x^2 + 2\pi\varepsilon(x-x_k)^2$$ $$x_k = \frac{k}{2\pi\varepsilon}$$

Ground State → $\psi_k(x,y) \sim e^{-\pi\varepsilon(x-x_k)^2}e^{iky}$ → *Lowest Landau Level*

Energy Spectrum → $E_n = 4\pi\varepsilon J(n+1/2)$



$\hbar\omega \equiv 4J\pi\varepsilon$



Non-interacting System: $n_\phi = 1/2 + \varepsilon$

Hofstadter, '76
Palmer and Jacksch, '06

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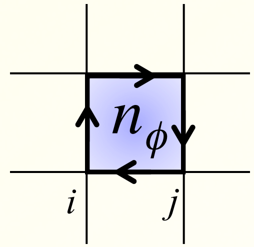

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Landau Gauge
$\vec{A} = (0, -Bx, 0)$

→ $\psi_k(x,y) \sim \phi(x)\, e^{iky}$

$$-\phi(x+1) - \phi(x-1) - 2Cos\,(2\pi n_\phi x - k)\phi(x) = E/J\,\phi(x)$$

$$-2Cos\,(\pi x + 2\pi\varepsilon x - k)$$

$$-2(-1)^x Cos\,(2\pi\varepsilon x - k)$$

Non-interacting System: $n_\phi = 1/2 + \varepsilon$

Hofstadter, '76
Palmer and Jacksch, '06

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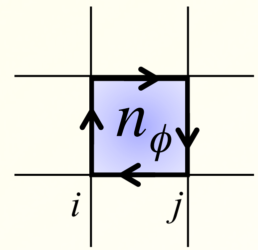

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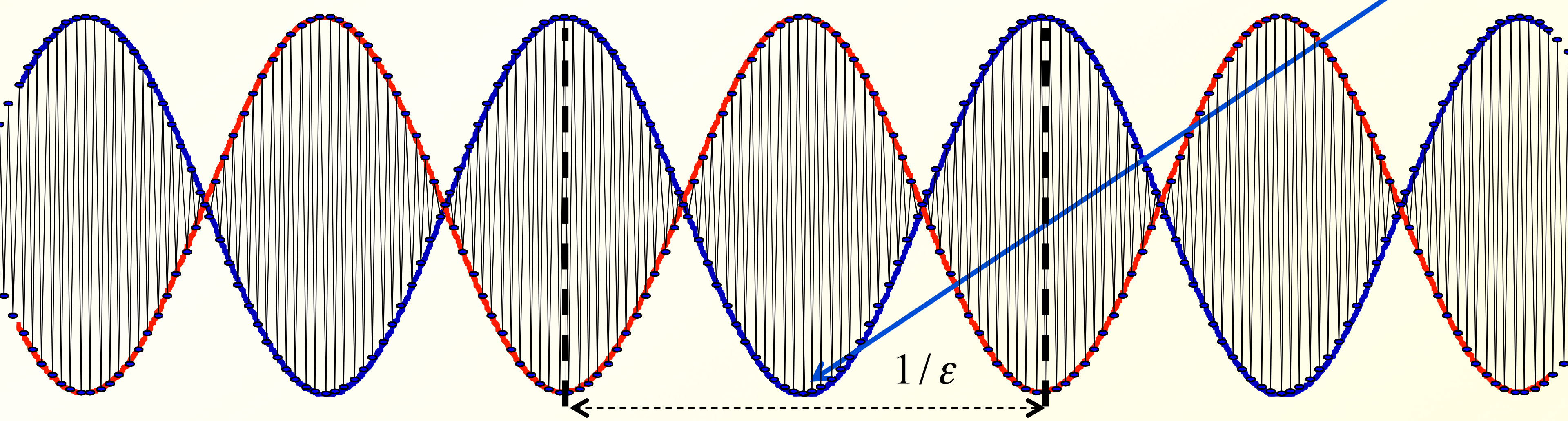
$\varepsilon << 1$ → $$-\partial_x^2\phi(x) + 2\pi\varepsilon(-1)^x(x - x_k)^2\phi(x) = \tilde{E}\phi(x)$$

$$x_k = \frac{k + 2m\pi}{2\pi\varepsilon}$$



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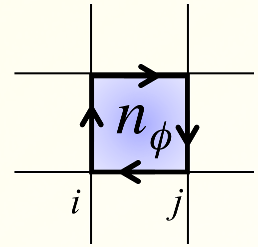

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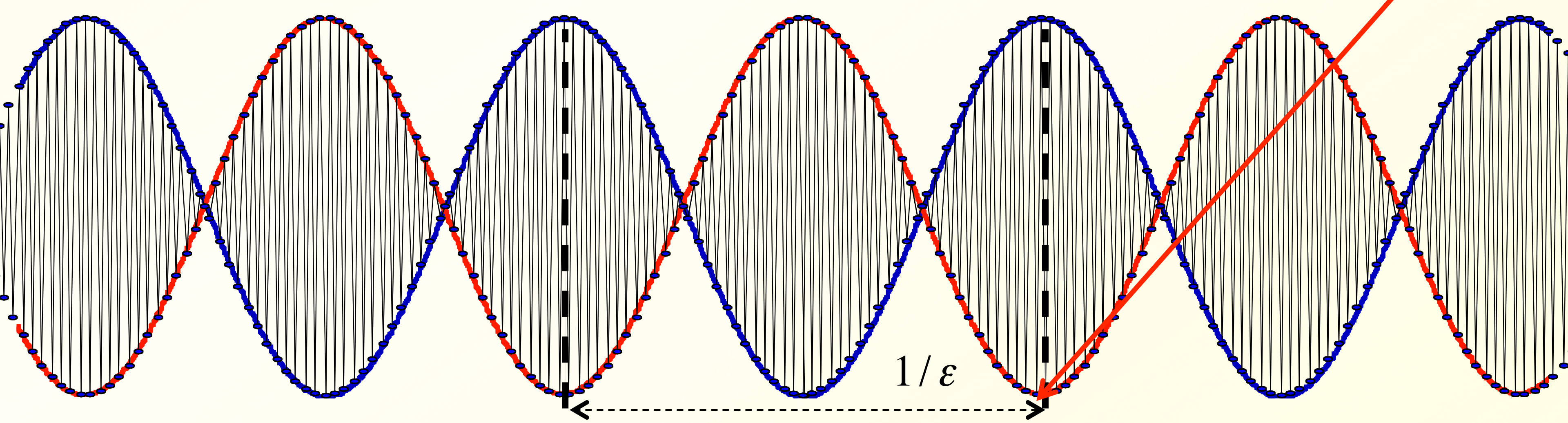
$\varepsilon << 1$ → $-\partial_x^2\phi(x) + 2\pi\varepsilon(-1)^{x+1}(x - x_{k+\pi})^2\phi(x) = \tilde{E}\phi(x)$ $\quad x_{k+\pi} = \dfrac{k + 2m\pi + \pi}{2\pi\varepsilon}$



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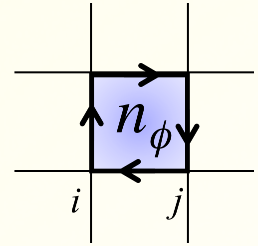

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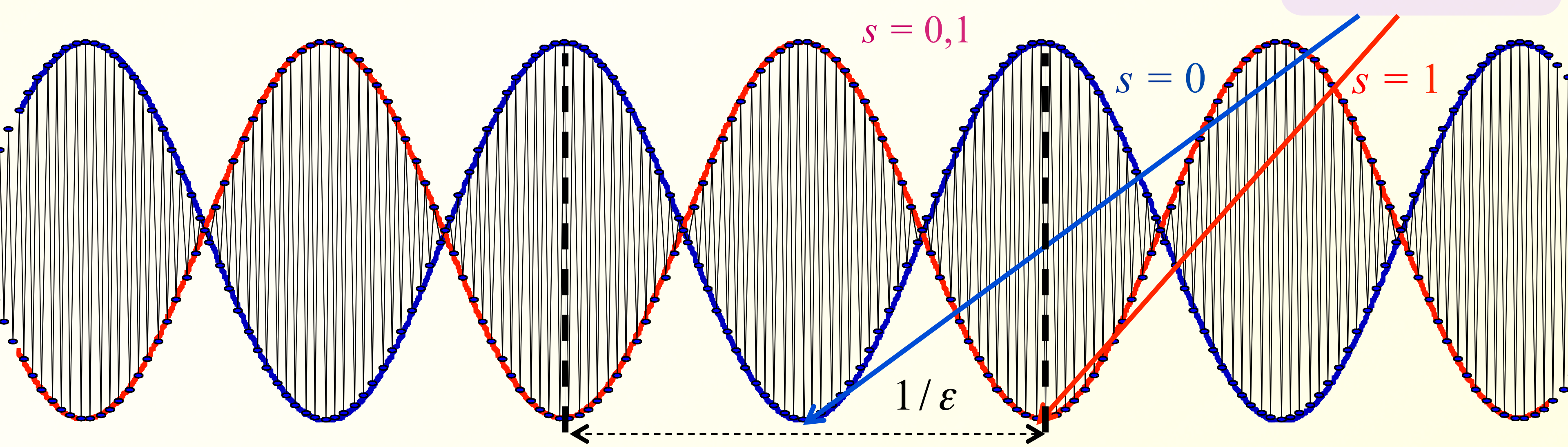
$$x_{k+s\pi} = \frac{k + 2m\pi + s\pi}{2\pi\varepsilon}$$



Non-interacting System: $n_\phi = 1/2 + \varepsilon$

Palmer and Jacksch, '06

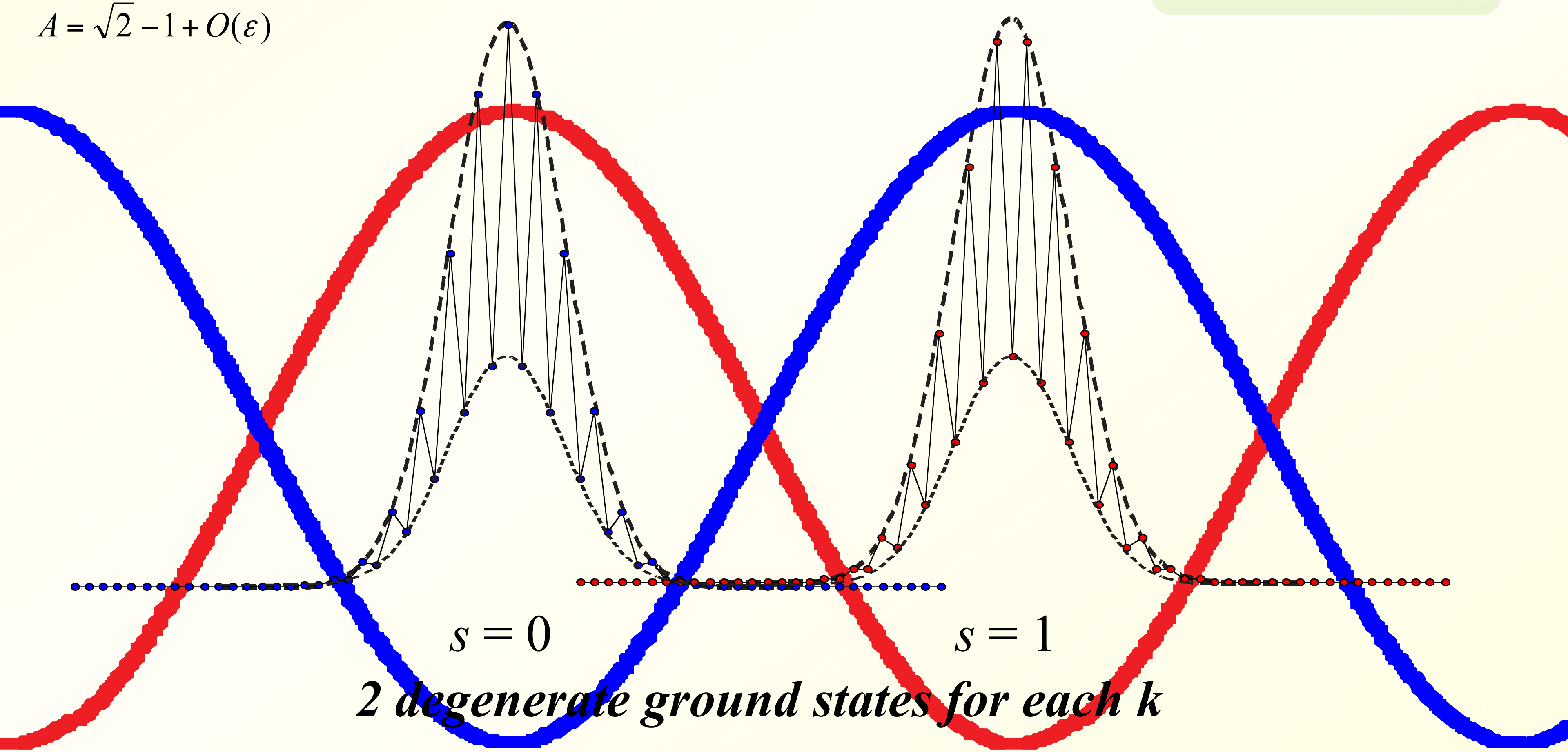
Ground State Ansatz → $\psi_{k,s}(x,y) \sim F_s(x)\, e^{-\pi\varepsilon(x - x_{k-s\pi})^2} e^{iky}$

$F_s(x) = (1 + (-1)^{s+x} A)$

$s = 0,\ 1$ → *band index*

$x_{k-s\pi} = \dfrac{k - s\pi}{2\pi\varepsilon}$

$A = \sqrt{2} - 1 + O(\varepsilon)$



2 degenerate ground states for each k

Non-interacting System: $n_\phi = 1/2 + \varepsilon$

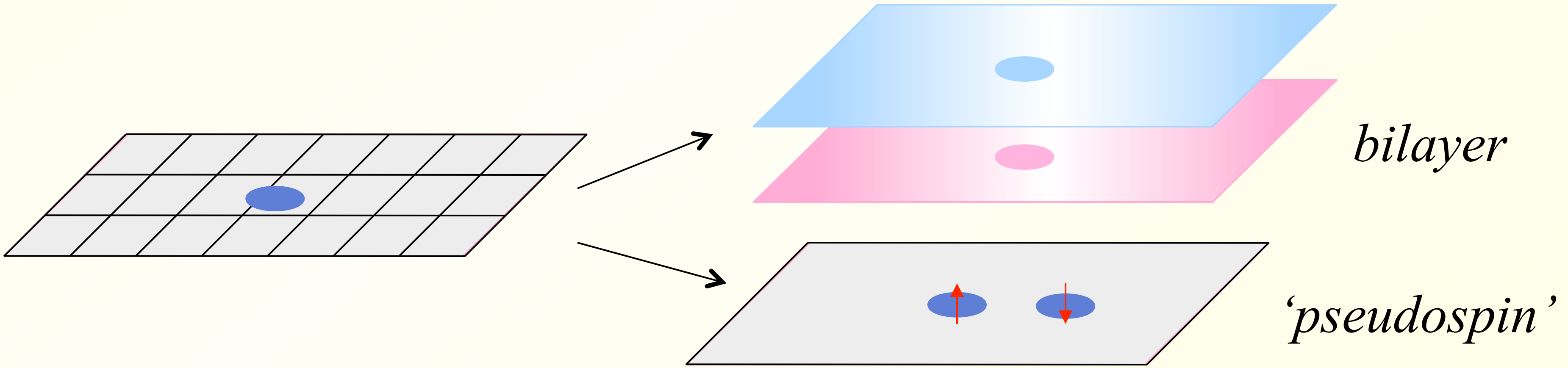
Palmer and Jacksch, '06

Ground State Ansatz → $\psi_{k,s}(x,y) \sim F_s(x) e^{-\pi\varepsilon(x - x_{k-s\pi})^2} e^{iky}$

$F_s(x) = (1 + (-1)^{s+x} A)$ $s = 0, 1$ → *band index* → *2-fold degeneracy*

$A = \sqrt{2} - 1 + O(\varepsilon)$

Similar to the continuum case ($n_\phi = \varepsilon$) but now with a form factor that depends on band index.



By introducing the band index we can map the energy spectrum near n_ϕ = ½ to Landau levels with each level being two-fold degenerate.

Interacting System

Haldane, '83
Cooper, '08

Interaction $$\hat{U} = \frac{1}{2}\sum_{k_1k_2k_3k_4} U_{k_1k_2k_3k_4} c^\dagger_{k_1} c^\dagger_{k2} c_{k_3} c_{k_4}$$

$$U_{k_1k_2k_3k_4} = \int dr_1 dr_2 U(r_1 - r_2)\psi^*_{k_1}(r_1)\psi^*_{k_2}(r_2)\psi_{k_3}(r_2)\psi_{k_4}(r_1)$$

$\psi_{k_i}(r_\alpha)$ ← *single-particle basis states at the lowest Landau level*

We carry out exact diagonalization of the potential for finite size systems.

Contact Interaction $$\hat{U} = U\sum_{i<j}\delta(r_i - r_j)$$

Continuum Limit

$$n_\phi = \varepsilon << 1$$

Single Particle Ground State $$\psi_k(x,y) \sim e^{-\pi\varepsilon(x-x_k)^2} e^{iky}$$

→ $$U_{k_1k_2k_3k_4} = U\sqrt{\varepsilon}\, e^{-\sum_{i<j}(k_i-k_j)^2/(16\pi\varepsilon)} \delta_{k_1+k_2,k_3+k_4}$$ → $\forall i,j \;\; k_i \approx k_j$

Interacting Case Near $n_\phi = 1/2$

Single Particle Ground State

$$\psi_k(x,y) \sim e^{-\pi\varepsilon(x-x_k)^2} e^{iky}$$

Continuum

$$n_\phi = \varepsilon << 1$$

$$\rightarrow \quad U_{k_1k_2k_3k_4} = U\sqrt{\varepsilon}\, e^{-\sum_{i<j}(k_i-k_j)^2/(16\pi\varepsilon)} \delta_{k_1+k_2,k_3+k_4}$$

Single Particle Ground State

$$\psi_{k,s}(x,y) \sim F_s(x) e^{-\pi\varepsilon(x-x_{k-s\pi})^2} e^{iky}$$

$$n_\varphi = \frac{1}{2} + \varepsilon$$

$$F_s(x) = (1+(-1)^{s+x}A)$$

$$\rightarrow \quad U_{\substack{k_1k_2k_3k_4 \\ s_1s_2s_3s_4}} = U\sqrt{\varepsilon}\, G_{s_1s_2s_3s_4}\, e^{-\sum_{i<j}(k_i-k_j-\pi(s_i-s_j))^2/(16\pi\varepsilon)} \delta_{k_1+k_2,k_3+k_4}$$

Similar to the continuum case but now with matrix elements that depend on pseudospin.

$$G_{s_1s_2s_3s_4} = \frac{1}{2}\sum_{x=0}^{1} F_{s_1}(x)F_{s_2}(x)F_{s_3}(x)F_{s_4}(x)$$

Interacting Case at $n_\phi = 1/2 + \varepsilon$

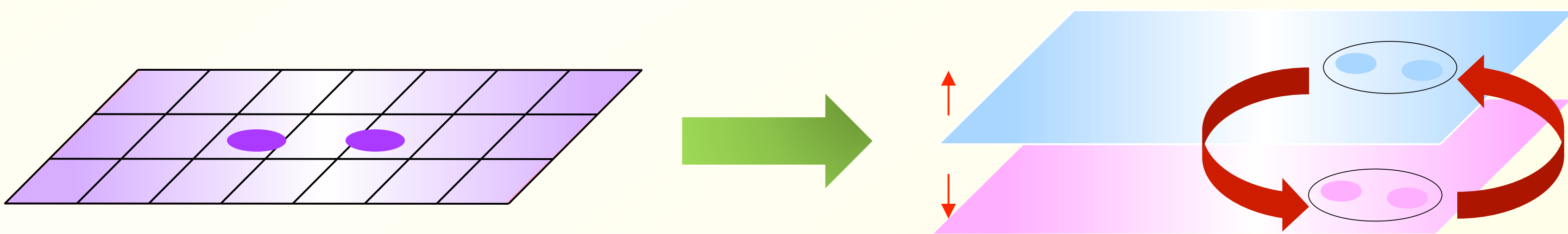
Pseudospin → $\uparrow\uparrow \quad \uparrow\downarrow \quad \downarrow\uparrow \quad \downarrow\downarrow$

$G_{s_1s_2s_3s_4}$

$$U_{\substack{k_1k_2k_3k_4\\ s_1s_2s_3s_4}} = 2U\sqrt{\varepsilon}\begin{pmatrix}1 & & & -\pi\varepsilon\\ & 1 & 1 & \\ & 1 & 1 & \\ -\pi\varepsilon & & & 1\end{pmatrix} e^{-\sum_{i<j}(k_i-k_j-\pi(s_i-s_j))^2/(16\pi\varepsilon)} \; \delta_{k_1+k_2,k_3+k_4}$$

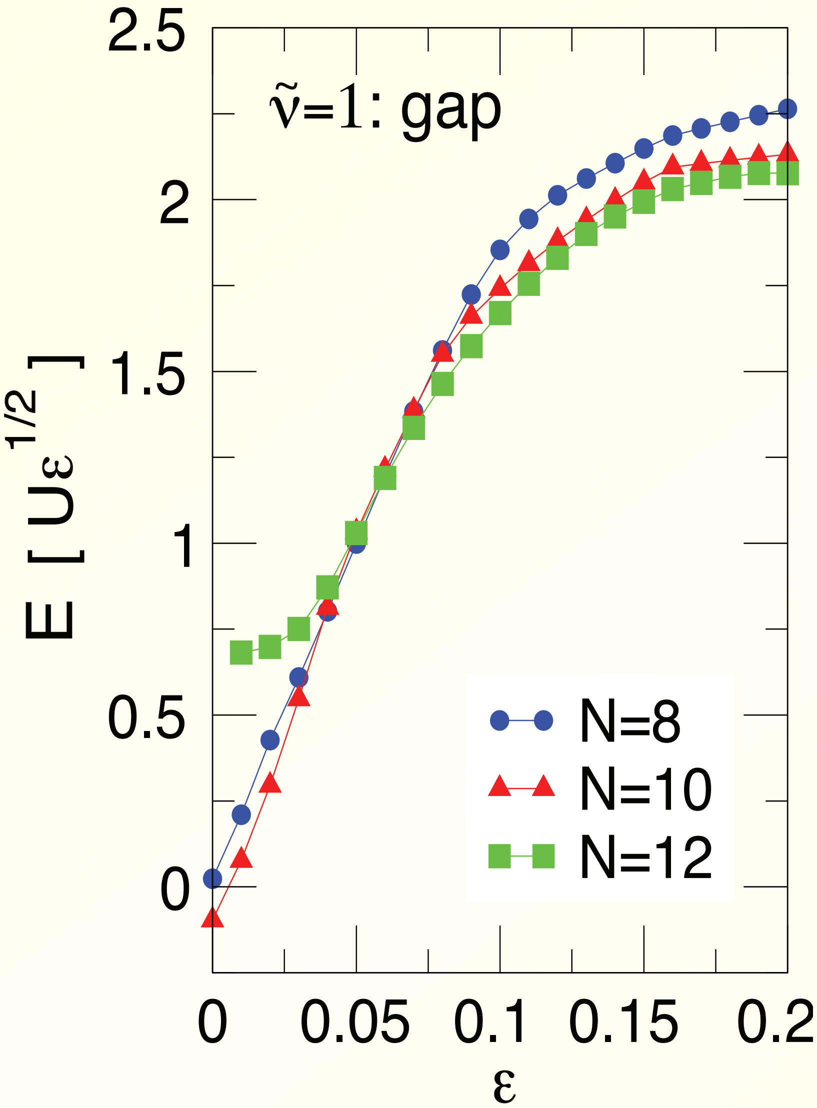
Non-pseudospin conserving terms → $k_1 + k_2 = k_3 + k_4 + 2\pi$ *umklapp scattering*

Pairs of particles can flip their pseudospin/tunnel between layers.



Can a pairing process be observed in an incompressible state?

Incompressible States



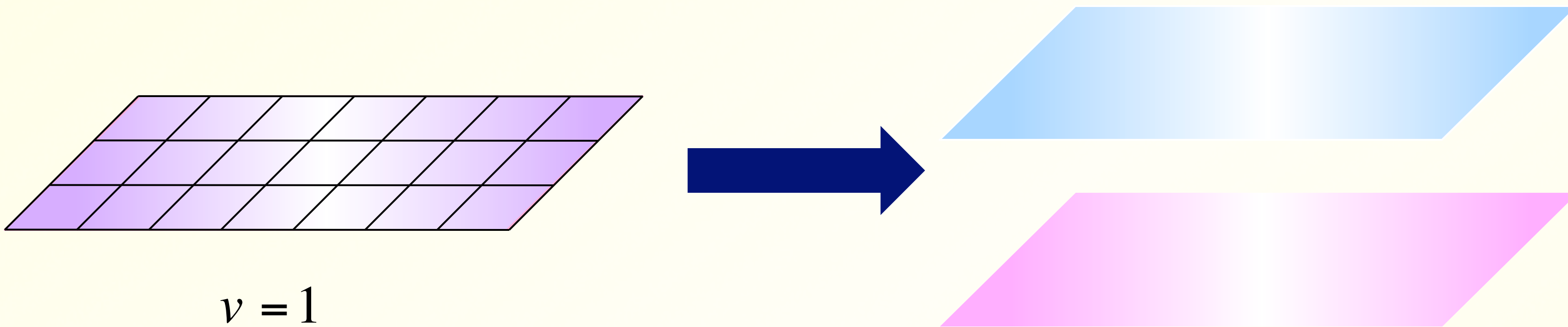
There is an incompressible state at $\nu = 1$ that becomes more robust as ε increases.

Gap closes as ε vanishes!

→ *The ε-dependent off-diagonal matrix elements seem to stabilize the incompressible state.*

Trial Wavefunction

Near flux density n_ϕ = ½: Effectively a bilayer system $\quad \nu = \frac{1}{2} + \frac{1}{2}$



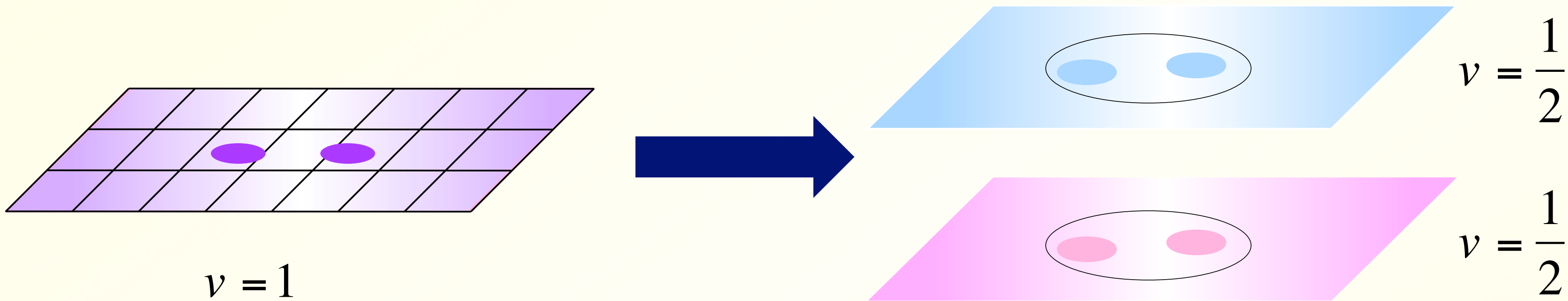
Continuum $\quad \nu = \frac{1}{2} \quad \Psi_L(\{z_i\}) = \prod_{i<j} (z_i - z_j)^2$ $\quad z = x + iy$

$n_\phi = \frac{1}{2} + \varepsilon \quad \Psi(\{z_i\})_{trial} = \prod_{i<j} (z_i^\uparrow - z_j^\uparrow)^2 \prod_{i<j} (z_i^\downarrow - z_j^\downarrow)^2$

- *Does not prevent particles of opposite pseudospin from approaching one another --- not energetically favorable.*
- *The overlap is not good either.*

Trial Wavefunction

Near flux density n_ϕ = ½: Effectively a bilayer system $\nu = \frac{1}{2} + \frac{1}{2}$



Continuum $\Psi_{MR}(\{z_i\}) = Pf\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)$ $\nu = 1$

$n_\phi = \frac{1}{2} + \varepsilon$ $\Psi(\{z_i\})_{trial} = \prod_{i<j}(z_i^\uparrow - z_j^\uparrow)Pf\left(\frac{1}{z_i^\uparrow - z_j^\uparrow}\right)\prod_{i<j}(z_i^\downarrow - z_j^\downarrow)Pf\left(\frac{1}{z_i^\downarrow - z_j^\downarrow}\right)$

- *Prevents particles with opposite pseudospin from approaching one another*
- *The pfaffian factors permit the particles with the same pseudospin to pair up, which is consistent with our pairing conjecture.*

How Good is the Trial Wave function?

$$\Psi(\{z_i\})_{trial} = \prod_{i<j}(z_i^{\uparrow} - z_j^{\uparrow}) Pf\left(\frac{1}{z_i^{\uparrow} - z_j^{\uparrow}}\right)\prod_{i<j}(z_i^{\downarrow} - z_j^{\downarrow}) Pf\left(\frac{1}{z_i^{\downarrow} - z_j^{\downarrow}}\right)\prod_{i\neq j}(z_i^{\uparrow} - z_j^{\downarrow})$$

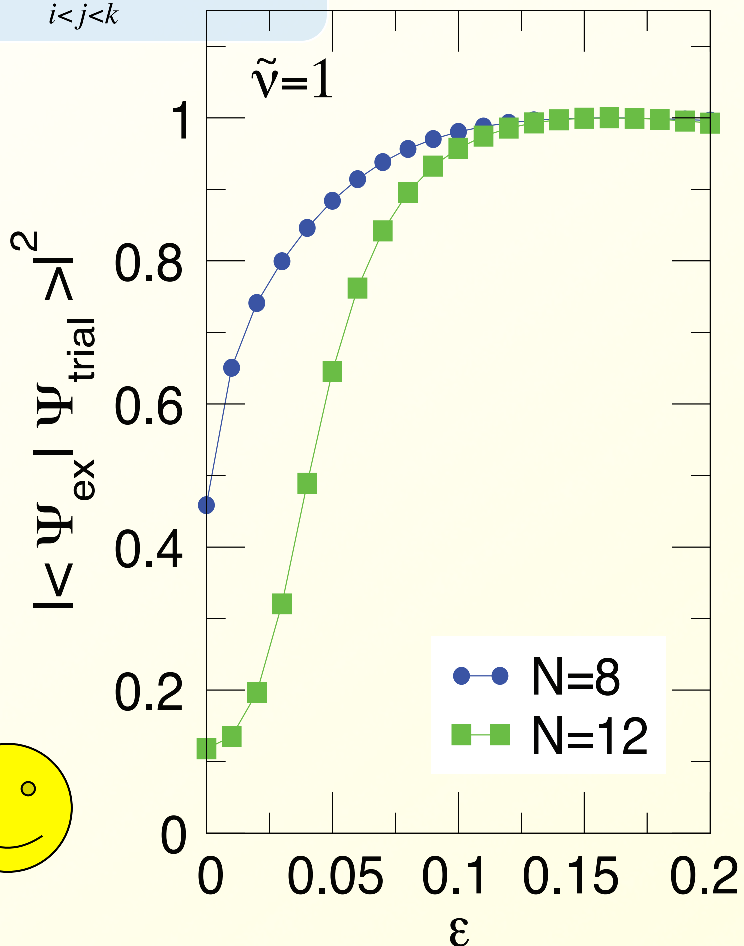
exact groundstate!

$$H_{model} = \sum_{i<j<k} \delta(z_i^{\uparrow} - z_j^{\uparrow})\,\delta(z_j^{\uparrow} - z_k^{\uparrow}) + \sum_{i<j<k} \delta(z_i^{\downarrow} - z_j^{\downarrow})\,\delta(z_j^{\downarrow} - z_k^{\downarrow}) + \sum_{i<j<k} \delta(z_i^{\uparrow} - z_j^{\downarrow})$$

$$H_{real} \sim \sum_{k_i s_i} G_{s_i}\, e^{-\sum_{i<j}(k_i - k_j - \pi(s_i - s_j))^2/(16\pi\varepsilon)}\, \delta_{k_1+k_2,k_3+k_4}\, c^{\dagger}_{k_1 s_1} c^{\dagger}_{k_2 s_2} c_{k_3 s_3} c_{k_4 s_4}$$

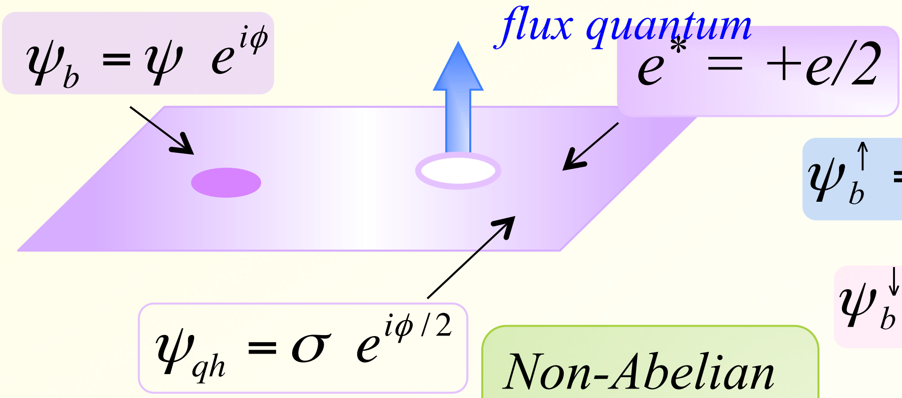
$$G = \begin{pmatrix} & \uparrow\uparrow & \uparrow\downarrow & \downarrow\uparrow & \downarrow\downarrow \\ & 1 & & & -\pi\varepsilon \\ & & 1 & 1 & \\ & & 1 & 1 & \\ & -\pi\varepsilon & & & 1 \end{pmatrix}$$

For $\varepsilon = 0.16$, the overlap is 0.99999!



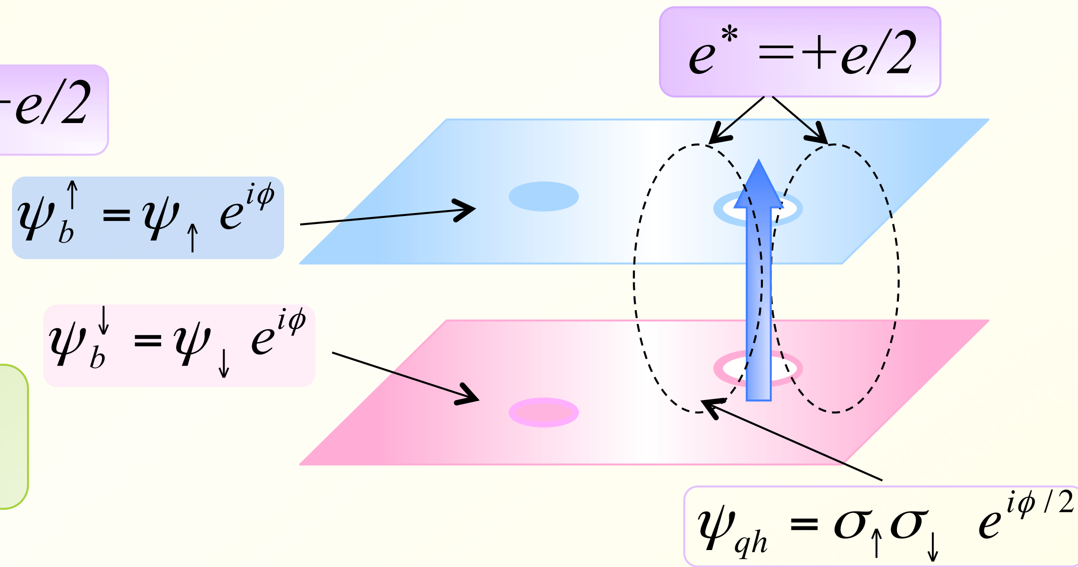
Abelian or Not?

Moore-Read State



Non-Abelian excitations

*Two **coupled** Moore-Read States*



Ising CFT: $\{1, \sigma, \psi\}$

2 ***coupled*** copies of Ising CFT: $\{1, \sigma^{\uparrow}, \psi^{\uparrow}\} \times \{1, \sigma^{\downarrow}, \psi^{\downarrow}\}$

$\downarrow$

$U(1)_4$

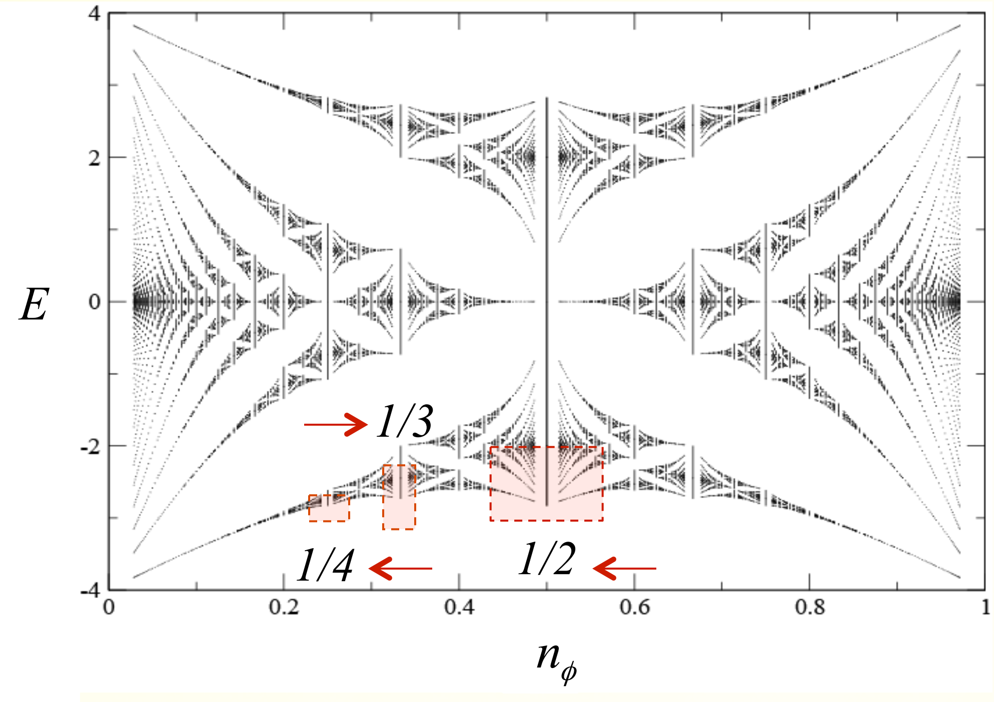
$\leftarrow$ *The combined excitations are effectively Abelian.*

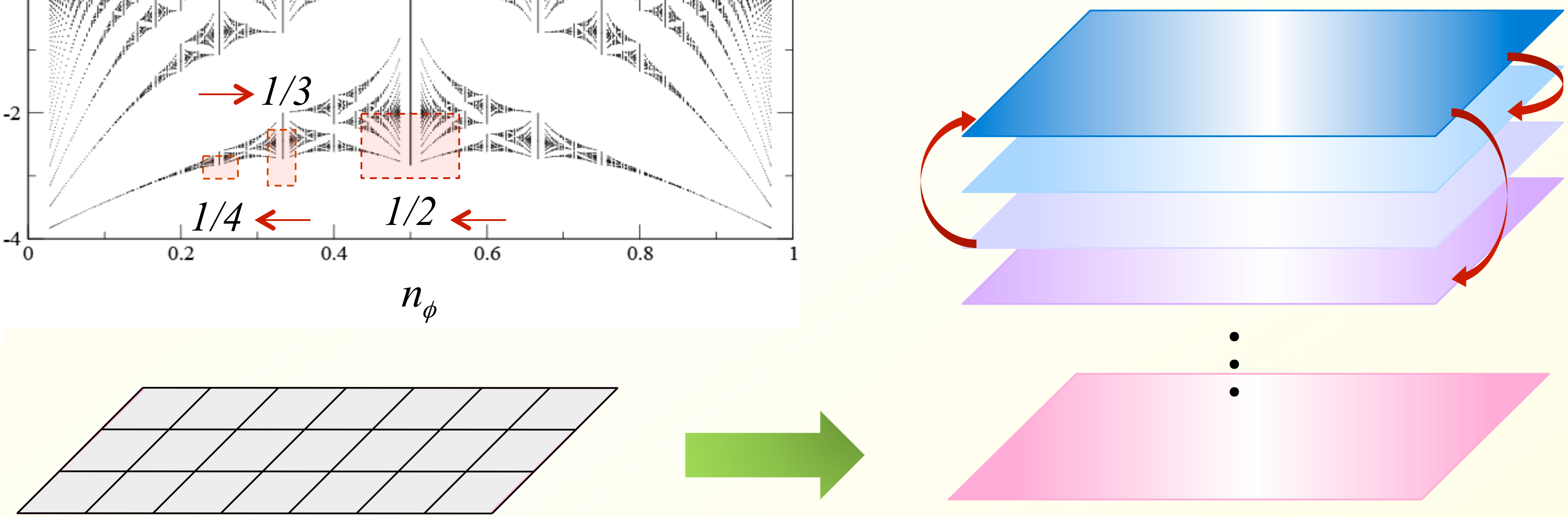
Free Chiral Bose Field: ϕ

$$\Psi_g(\{z_i\}) = \langle \psi_b(z_1)\psi_b(z_2)\ldots\psi_b(z_N)\rangle$$

$$\Psi_e(\{z_i\},\{w_i\}) = \langle \psi_b(z_1)\psi_b(z_2)\ldots\psi_b(z_N)\psi_{qh}(w_1)\psi_{qh}(w_2)\ldots\psi_{qh}(w_M)\rangle$$

Generalization to $n_\phi = p/q + \varepsilon$





Lattice near $n_\phi = p/q + \varepsilon$

q-layer continuum system

Potentially more interesting states but probably harder to realize…

Summary and Outlook

- *Near* $n_\phi = ½$ → *two-fold degeneracy due to pseudospin.*
- *Interaction potential suggests pairing of particles with the same pseudospin.*
- *At* $\nu = 1$ *pairing terms stabilize the groundstate.*
- *Trial wave function for the groundstate of* $\nu = 1$ *has excellent overlap with ED result and the excitation spectrum matches the prediction for coupled Moore Read states.*

$$\Psi(\{z_i\})_{trial} = \prod_{i<j}(z_i^{\uparrow} - z_j^{\uparrow}) Pf\left(\frac{1}{z_i^{\uparrow} - z_j^{\uparrow}}\right) \prod_{i<j}(z_i^{\downarrow} - z_j^{\downarrow}) Pf\left(\frac{1}{z_i^{\downarrow} - z_j^{\downarrow}}\right) \prod_{i \neq j}(z_i^{\uparrow} - z_j^{\downarrow})$$

- *Pairing terms might be important for other filling fractions, flux densities, other types of interactions, fermions, etc.*

- *'Dislocations'* → *modifying topology?* → *Non-Abelian States?*

Barkeshli and Qi, '12