Genus 2 Recursion for Correlation Functions of Vertex Operator Algebras

Thomas Patrick Gilroy
Supervised by Dr. Michael P. Tuite

National University of Ireland Galway

May 4, 2013

Introduction

- A brief recap of Michael's talk.
- Outline of the method.
- General formula for genus 2 1-point Correlation Function.
- The Heisenberg Case and the genus 2 Ward Identity.

A Brief Recap

A *Vertex Operator Algebra* (VOA) V is a $\mathbb{Z}$-graded vector space over $\mathbb{C}$ $V = \oplus_{n\geq 0} V_n$ with $\dim V_n < \infty$.

The *genus* 1 *partition function* for V is

$$Z_V^{(1)}(\tau) = \mathrm{Tr}_V \left(q^{L(0)-C/24} \right)$$

where $\tau \in \mathbb{H}$ and $q = e^{2\pi i \tau}$. 1-point and n-point *correlation functions* for V are defined by

$$Z_V^{(1)}(u,\tau) = \mathrm{Tr}_V \left(o(u) q^{L(0)-C/24} \right),$$
$$Z_V^{(1)}(u_1,z_1;\ldots;u_n,z_n;\tau) = Z_V^{(1)}(Y[u_1,z_1]\ldots Y[u_n,z_n]\mathbf{1},\tau),$$

respectively.

A Brief Recap

Genus 1 Zhu Recursion Theory relates n-point functions to $(n-1)$-point functions. For a 2-point function, we have

$$Z_V^{(1)}(u,x;v,y;\tau) = \mathrm{Tr}_V\left(o(u)o(v)q^{L(0)-C/24}\right) + \sum_{m\geq 0}\frac{1}{m!}\frac{\partial^m}{\partial y^m}P_1(x-y,\tau)Z_V^{(1)}(u[m]v,\tau).$$

The aim of my thesis is to develop an analogue of Zhu Theory at genus 2.

The Approach

If we have a genus 2 surface formed from two sewn tori, the genus 2 partition function is defined by

$$Z_V^{(2)}(\tau_1, \tau_2, \epsilon) = \sum_{n \geq 0} \epsilon^n \sum_{u \in V[n]} Z_V^{(1)}(u, \tau_1) Z_V^{(1)}(\overline{u}, \tau_2)$$

and the genus 2 1-point function (left insertion) is defined by

$$Z_V^{(2)}(v, x; \tau_1, \tau_2, \epsilon) = \sum_{n \geq 0} \epsilon^n \sum_{u \in V[n]} Z_V^{(1)}(Y[v, x]u, \tau_1) Z_V^{(1)}(\overline{u}, \tau_2)$$

where $\overline{u}$ is dual to u w.r.t the Li-Z metric.

The Approach

We can apply Zhu recursion to the left factor of the 1-point function, giving terms of the form

$$X(m) = \sum_{n\geq 0} \epsilon^n \sum_{u \in V[n]} Z_V^{(1)}(v[m]u, \tau_1) Z_V^{(1)}(\overline{u}, \tau_2),$$

where $m \geq 0$. If v is *quasiprimary*, we may move the mode $v[m]$ from left to right using the dagger, so

$$X(m) = \sum_{n\geq 0} \epsilon^n \sum_{u \in V[n]} Z_V^{(1)}(u, \tau_1) Z_V^{(1)}(v^\dagger[m]\overline{u}, \tau_2)$$

$$= (-1)^{\mathrm{wt}[v]} \epsilon^{m+1-\mathrm{wt}[v]} \sum_{n\geq 0} \epsilon^n \sum_{u \in V[n]} Z_V^{(1)}(u, \tau_1) Z_V^{(1)}(v[2\mathrm{wt}[v] - m - 2]\overline{u}, \tau_2)$$

and we can apply Zhu recursion to the right factor if $2\mathrm{wt}[v] - m - 2 \leq -1$.

The Result

If we let

$$\mathbb{X}(m) = \frac{\epsilon^{-\frac{m}{2}}}{\sqrt{m}} \sum_{n\geq 0} \epsilon^n \sum_{u\in V[n]} Z_V^{(1)}(v[m]u, \tau_1) Z_V^{(1)}(\overline{u}, \tau_2)$$

and

$$\mathbb{Y}(r) = \frac{\epsilon^{-\frac{r}{2}}}{\sqrt{r}} \sum_{n\geq 0} \epsilon^n \sum_{u\in V[n]} Z_V^{(1)}(u, \tau_1) Z_V^{(1)}(v[r]\overline{u}, \tau_2),$$

we eventually find the 1-point recursion formula...

The Result

$$
\begin{aligned}
&Z_V^{(2)}(v, x; \tau_1, \tau_2, \epsilon) = \\
&O_1 + \widetilde{\mathbb{P}}(x)\left(\mathbb{1} - \widetilde{A}_2\widetilde{A}_1\right)^{-1}\widetilde{A}_2\Theta\mathbb{O}_1 \\
&+ (-1)^{\mathrm{wt}[v]}\widetilde{\mathbb{P}}(x)\left(\mathbb{1} - \widetilde{A}_2\widetilde{A}_1\right)^{-1}\Theta\mathbb{O}_2 \\
&+ \mathbb{P}(x)\mathbb{X}_0 + \widetilde{\mathbb{P}}(x)\left(\mathbb{1} - \widetilde{A}_2\widetilde{A}_1\right)^{-1}\widetilde{A}_2 A_1\mathbb{X}_0 \\
&+ (-1)^{\mathrm{wt}[v]}\widetilde{\mathbb{P}}(x)\left(\mathbb{1} - \widetilde{A}_2\widetilde{A}_1\right)^{-1} A_2\mathbb{Y}_0,
\end{aligned}
$$

assuming $\left(\mathbb{1} - \widetilde{A}_2\widetilde{A}_1\right)^{-1}$ exists, where...

The Result

$$\alpha = 2\mathrm{wt}[v] - 2,$$

$$\Delta(m,n) = \sqrt{\frac{n}{m}}\delta_{m-n,\alpha}, \qquad \Theta(m,n) = \sqrt{\frac{n}{m}}\delta_{n-m,\alpha},$$

$$A(m,n) = \frac{\epsilon^{\frac{m+n}{2}}}{\sqrt{mn}}(-1)^{m+1}\frac{(m+n-1)!}{(m-1)!(n-1)!}E_{m+n}(\tau),$$

$$(\mathbb{P}(x))(m) = \epsilon^{m/2}\sqrt{m}\frac{(-1)^{m+1}}{m!}\frac{\partial^m}{\partial x^m}P_1(x,\tau_1),$$

$$O_1 = \sum_{n\geq 0}\left(\epsilon^n \sum_{u\in V[n]} \mathrm{Tr}_V\left(o(v)o(u)q_1^{L(0)-c/24}\right) Z_V^{(1)}(\overline{u},\tau_2)\right),$$

$$\mathbb{O}_1(m) = \frac{\epsilon^{1/2}}{\sqrt{m}}(-1)^{\mathrm{wt}[v]}\delta_{m,\alpha+1}O_1,$$

$$\mathbb{X}_0 = \mathbb{X} - \Delta\Theta\mathbb{X}, \qquad \mathbb{Y}_0 = \mathbb{Y} - \Delta\Theta\mathbb{Y},$$

$$\widetilde{A} = A\Delta, \qquad \widetilde{\mathbb{P}}(x) = \mathbb{P}(x)\Delta.$$

The Heisenberg Case

Let N_{β_1}, N_{β_2} be a pair of Heisenberg modules. Define

$$Z^{(2)}_{\beta_1,\beta_2}(a, x; \tau_1, \tau_2, \epsilon) = \sum_{n\geq 0} \epsilon^n \sum_{u\in M_{[n]}} Z^{(1)}_{\beta_1}(Y[a, x]u, q_1) Z^{(1)}_{\beta_2}(\overline{u}, q_2).$$

We have that

$$O_1 = \beta_1 Z^{(1)}_V(\tau_1, \tau_2, \epsilon),$$
$$O_2 = \beta_2 Z^{(1)}_V(\tau_1, \tau_2, \epsilon)$$

and since $\alpha = 0$, we have $\mathbb{X}_0 = 0 = \mathbb{Y}_0$ and $\Delta = \mathbb{1} = \Theta$.

The Heisenberg Case

Thus we have

$$
\begin{aligned}
&Z^{(2)}_{\beta_1,\beta_2}(a,x;\tau_1,\tau_2,\epsilon)dx = \\
&\beta_1 \left(1 - \epsilon^{1/2} \sum_{m\geq 1} \left(\mathbb{P}(x) \left(\mathbb{1} - A_2 A_1 \right)^{-1} A_2 \right)(m,1) \right) Z^{(2)}_{\beta_1,\beta_2}(\tau_1,\tau_2,\epsilon)dx \\
&+ \beta_2 \left(\epsilon^{1/2} \sum_{m\geq 1} \left(\mathbb{P}(x) \left(\mathbb{1} - A_2 A_1 \right)^{-1} \right)(m,1) \right) Z^{(2)}_{\beta_1,\beta_2}(\tau_1,\tau_2,\epsilon)dx \\
&= \left(\beta_1 \nu_1(x) + \beta_2 \nu_2(x) \right) Z^{(2)}_{\beta_1,\beta_2}(\tau_1,\tau_2,\epsilon)
\end{aligned}
$$

for holomorphic 1-forms ν_1, ν_2. This agrees with calculations by Mason and Tuite.

The Virasoro Case and Ward Identity

For the conformal vector $\widetilde{\omega} \in V_{[2]}$, we have

$$O_1 = q_1 \partial_{q_1} Z_V^{(2)}(\tau_1, \tau_2, \epsilon),$$
$$O_2 = q_2 \partial_{q_2} Z_V^{(2)}(\tau_1, \tau_2, \epsilon)$$

.

Since $\alpha = 2$, we have that $\mathbb{X}_0 = \mathbb{Y}_0$ with

$$\mathbb{X}(1) = \epsilon^{1/2} \partial_\epsilon Z_V^{(2)}(\tau_1, \tau_2, \epsilon) = \mathbb{Y}(1)$$

and $\mathbb{X}(m) = 0 = \mathbb{Y}(m)$ for $m > 1$.

The Virasoro Case and Ward Identity

This allows us to find $\mathbb{A}(x), \mathbb{B}(x)$ and $\mathbb{C}(x)$ such that

$$Z_V^{(2)}(\widetilde{\omega}, x; \tau_1, \tau_2, \epsilon) = \left(\mathbb{A}(x)q_1\partial_{q_1} + \mathbb{B}(x)q_2\partial_{q_2} + \mathbb{C}(x)\epsilon\partial_\epsilon\right) Z_V^{(2)}(\tau_1, \tau_2, \epsilon)$$
$$= \mathbb{D}_x Z_V^{(2)}(\tau_1, \tau_2, \epsilon).$$

If $v \in V$ is a primary vector, we have the *Ward Identity*

$$\begin{aligned} & Z_V^{(2)}(\widetilde{\omega}, x; v, y; \tau_1, \tau_2, \epsilon) \\ & = \mathbb{D}_x Z_V^{(2)}(v, y; \tau_1, \tau_2, \epsilon) \\ & + \mathbb{P}_1^\alpha(x, y)\partial_y Z_V^{(2)}(v, y; \tau_1, \tau_2, \epsilon) \\ & + \partial_y \mathbb{P}_1^\alpha(x, y)\, \mathrm{wt}[v] Z_V^{(2)}(v, y; \tau_1, \tau_2, \epsilon), \end{aligned}$$

for a specific $\mathbb{P}_1^\alpha(x, y)$ which depends on $\alpha = 2\,\mathrm{wt}[v] - 2$.