Collapse of a self-similar cylindrical scalar field with non-minimally coupling

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Gravitational Collapse in GR

- Gravitational collapse is one of the most interesting aspects of gravitational physics, leading to many exotic phenonmenon such as neutron stars, supernovae and black holes
- Some theoretical models even have 'naked singularities'
- Penrose's 'Cosmic Censorship Hypothesis' claims that naked singularities shouldn't really exist in our universe
- Einstein equation :

$$G_{ab} = 8\pi T_{ab}$$

The Model

- Self-Similarity of the first kind: $\mathcal{L}_\xi g_{ab} = 2g_{ab}$
- The line element for cylindrical symmetry with self-similarity in double null coords with $\eta = v/u$

$$ds^2 = -|u|^{-1}e^{2\gamma(\eta)+2\phi(\eta)}dudv + |u|e^{2\phi(\eta)}S^2(\eta)d\theta + |u|e^{-2\phi(\eta)}dz^2. \qquad (1)$$

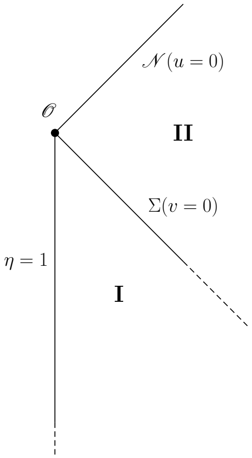
- The energy-momentum tensor for self-interacting scalar field:

$$T_{ab} = \nabla_a\psi\nabla_b\psi - \frac{1}{2}g_{ab}\nabla^c\psi\nabla_c\psi - g_{ab}V(\psi) \qquad (2)$$

$$\psi = F(\eta) + \frac{k}{2}\ln|u| \qquad V(\psi) = V_0 e^{-\frac{2}{k}\psi}$$

- Different values of V_0 and k represent different matter models
- Initial data given along a regular axis

Spacetime Diagram



Initial Value problem

- New variables

$$\tau = -\log\eta, \quad R = e^{\tau/2} S/S_0, \quad I = 2F/k + \tau/2, \quad \lambda = \frac{k^2}{2} - 1$$

$$\ddot{R} = R\left(\frac{1}{4} - V_0 e^{\lambda I}\right), \tag{3a}$$

$$2\gamma + 2\phi - \frac{k^2}{2} I = \frac{1}{2}\tau, \tag{3b}$$

$$R\ddot{I} + \dot{R}\dot{I} = R\left(\frac{1}{4} - \frac{2}{k^2} V_0 e^{\lambda I}\right), \tag{3c}$$

$$2R\left(\dot{\phi} - \frac{1}{4}\right) + \dot{R} = \dot{R}(0), \tag{3d}$$

$$R(0) = 0, \quad \dot{R}(0) = 1, \quad I(0) = I_0, \quad \dot{I}(0) = 0. \tag{3e}$$

Initial value problem

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$$2R\left(\dot{\phi} - \frac{1}{4}\right) + \dot{R} = 1,$$

$$R(0) = 0, \quad \dot{R}(0) = 1, \quad I(0) = I_0, \quad \dot{I}(0) = 0.$$

Existence & Uniqueness of Solutions

System as a set of first order integral equations:

$$\vec{x} = \vec{x}_0 + \int_0^{\tau} f(\tau, \tau', \vec{x}) d\tau' \tag{5}$$

Proposition 1

For τ_* sufficiently small, the mapping T defined above is a contraction mapping from χ to χ and thus has a unique fixed point.

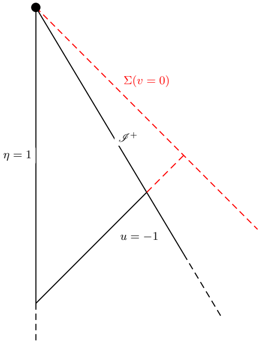
Theorem 1

Let $k, V_0 \in \mathbb{R}$. For each $l_0, \phi_0 \in \mathbb{R}$ there exists a unique solution of the equations (3a)-(3d) on an interval $[0, \tau_*]$ corresponding to a spacetime with line element (1) and energy-momentum tensor (2).

Evolution of solutions to the past null cone of origin Σ

Proposition 2

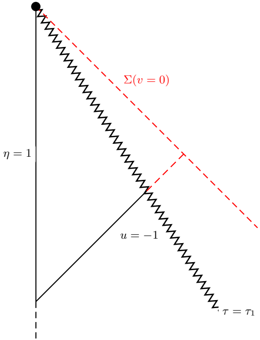
For $V_0 < 0$, $\lambda < 0$, the surface corresponding to $\tau = \tau_1$ in lemma 1 represents future null infinity and the Ricci scalar decays to zero there.



Evolution of solutions to the past null cone of origin Σ

Proposition 3

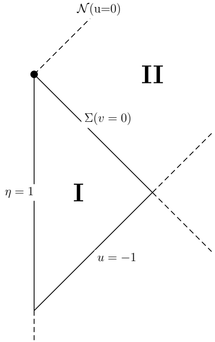
For $V_0 > 0, \lambda < 0, u_2(0) < k^2/8$ *and for* $V_0 > 0, \lambda > 0$*, there exists* $\tau_1 < \infty$ *such that* $\lim_{\tau \to \tau_1} u_1 = -\infty$ *(*$R \to 0$*) and the Ricci scalar blows up there.*



Evolution of solutions to the past null cone of origin Σ

Proposition 4

For $V_0 > 0, \lambda < 0, u_2(0) > k^2/8$ and for $V_0 > 0, \lambda < 0$, Σ is reached in finite affine time along outgoing radial null geodesic and the radial function S, the scalar field ψ and Ricci scalar are non-zero and finite there.



Field equations in region II

- Yet another change of variables

$$t = \ln(-\eta), \quad \mu_1 = \frac{R'(t)}{R} + \frac{1}{2}, \quad \mu_2 = |V_0| e^{\lambda I}, \quad \mu_3 = I'(t) + \frac{1}{2}$$

Field equations in region II

$$\mu_1'(t) = \mu_1 - \epsilon\mu_2 - \mu_1^2 \tag{6a}$$

$$\mu_2'(t) = |\lambda|\mu_2 \left(\frac{1}{2} - \mu_3\right) \tag{6b}$$

$$\mu_3'(t) = \frac{\mu_1}{2} + \frac{\mu_3}{2} - \epsilon\frac{2\mu_2}{k^2} - \mu_1\mu_3 \tag{6c}$$

$$\lim_{t\to-\infty} e^{-|\lambda|t/2}(\mu_1, \mu_2, \mu_3) = C\left(\frac{4}{2+k^2}, 1, \frac{16}{k^4(2+k^2)}\right), \quad C > 0 \tag{6d}$$

Three possible outcomes

Case I

$\lim_{t\to\infty} \vec{\mu} = (1, 0, 1)$

- Corresponds to naked singularity

Case II

$\lim_{t\to\infty} \vec{\mu} = \left(1/2 - \sqrt{|\lambda|}/2, k^2/8, 1/2\right)$

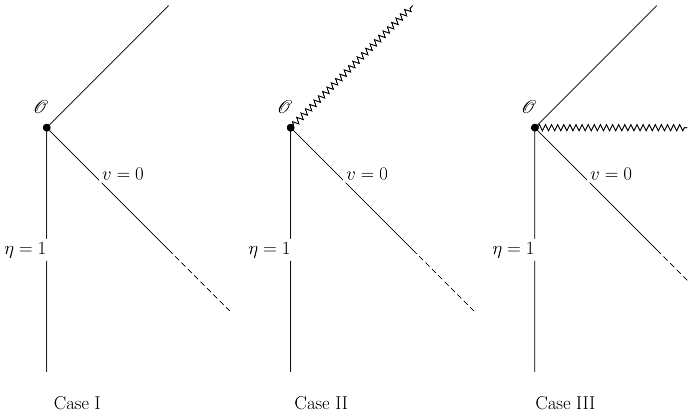
- Corresponds to censored singularity

Case III

$|\vec{\mu}| \to \infty$ in finite time

- Corresponds to censored singularity

The three pictures



Analytic solutions

$$\mu_1 = \sum_{n=1}^{\infty} a_n e^{n|\lambda|\bar{t}/2}, \qquad \mu_2 = \sum_{n=1}^{\infty} b_n e^{n|\lambda|\bar{t}/2}, \qquad \mu_3 = \sum_{n=1}^{\infty} c_n e^{n|\lambda|\bar{t}/2} \tag{7}$$

$$a_1 = \frac{4}{2+k^2}, \qquad b_1 = 1, \qquad c_1 = \frac{16}{k^4(2+k^2)}. \tag{8}$$

Recurrence relations for coefficients

$$a_n = \left(\frac{n|\lambda|}{2} - 1\right)^{-1} \left(-b_n - \sum_{j=1}^{n-1} a_j a_{n-j}\right), \tag{9a}$$

$$b_n = -\frac{2}{n-1} \sum_{j=1}^{n-1} b_j c_{n-j}, \tag{9b}$$

$$c_n = \left(\frac{n|\lambda|}{2} - \frac{1}{2}\right)^{-1} \left(\frac{a_n}{2} - \frac{2b_n}{k^2} - \sum_{j=1}^{n-1} a_j c_{n-j}\right). \tag{9c}$$

Theorem 6

Suppose that $|a_j|, |b_j|, |c_j| \leq \kappa\delta^{j-1}/j^2$ *for all* $j < n_*$, *for some* n_*, κ, δ. *Suppose further that*

$$n_* > \frac{2}{L}\left(1 + \frac{2}{k^2} + \frac{\kappa}{\delta}s_{n_*}\right), \tag{10}$$

where

$$s_{n_*} = 2H_{2,n_*-1} + \frac{4}{n_*}H_{1,n_*-1}. \tag{11}$$

Then $|a_{n_*}|, |b_{n_*}|, |c_{n_*}| \leq \kappa\delta^{n_*-1}/n_*^2$.

Corollary 7

If theorem 8 holds for some κ, δ, n_*, *then it holds for* κ, δ *and all* $n > n_*$.

Uniform convergence & truncation error

Theorem 8

Let $\hat{\eta} = e^{|\lambda|t/2}$*. Suppose that there exist* κ, δ *such that* $|a_n|, |b_n|, |c_n| \leq \kappa\delta^{n-1}/n^2$ *for all* $n \geq 1$*.*
Then the series (4) converge uniformly on $\hat{\eta} \in [0, 1/\delta)$*.*

Lemma 9

Suppose that there exist κ, δ such that $|a_n|, |b_n|, |c_n| \leq \kappa\delta^{n-1}/n^2$ for all $n \geq 1$, and that the series (10) are approximated by the first N terms. Then the truncation errors e_i for these approximations satisfy the bound

$$e_i \leq \frac{\kappa}{N^2\delta}\left(\frac{(\delta\tau)^{N+1}}{1-\delta\tau}\right). \tag{12}$$

Some results for $V_0 > 0$

Lemma 10

Suppose there exists $t_1 > -\infty$ *such that* $\mu_1(t_1) = 0$*. Then there exists* $t_2 > t_1$ *such that* $\lim_{t \to t_2} \mu_1 = -\infty$*.*

Lemma 11

Suppose there exists t_1 *such that* $\mu_1(t_1), \mu_3(t_1) < 1/2$ *and* $\mu'(t_1), \mu_3'(t_1) < 0$*. Then there exists* $t_2 > t_1$ *such that* $\mu_1(t_2) = 0$*.*

...

Lemma 16

Suppose that for $1 < k^2 < 4/3$ *there exists some* t_1 *such that* $\mu_3'(t_1), \mu_3''(t_1), \mu_3'''(t_1), \mu_1'''(t_1) < A\mu_2'''(t_1)$*, and* $|\lambda|\mu_3(t_1)/2 < \mu_1(t_1) < 2\mu_2(t_1)$*,* $\mu_1'(t_1) < \mu_2'(t_1)$*,* $\mu_1''(t_1) < \mu_2''(t_1)$*.*
Then there exists some time $t_2 > t_1$ *such that* $\mu_1(t_2) = 0$*.*

Numerical data

Let

$$m = \max_{j \in [1,N]} \{ j^2 (|a_j|, b_j|, |c_j|) / \kappa \delta^{j-1} \} \tag{13}$$

k^2	δ	κ	N	m	$e\hat{t}a_1$	$\bar{\mu}_1(e\hat{t}a_1)$	$e(e\hat{t}a_1) <$
0.01	10^6	10^5	412	0.796	9×10^{-7}	-2.4×10^{-5}	10^{-24}
0.02	10^5	$25,000$	224	0.792	9×10^{-6}	-3.04×10^{-4}	10^{-14}
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
0.32	90	80	80	0.846	0.008	-0.0527	10^{-13}
0.33	80	80	80	0.976	0.01	-0.0832	10^{-9}
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.

Summary

- For $k^2 > 2$, all solutions terminate in region I
- For $V_0 < 0$, $0 < k^2 < 1.19$ we have censored singularity in region II
- For $V_0 > 0$, $0 < k^2 < 1.43$ we have censored singularity in region II
- Unlikely that these methods will succeed for all k^2
- Question of cosmic censorship violation thus far goes unanswered

Thanks for listening.
Any questions?