Quantum Quenches in 1D systems

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Pisa University

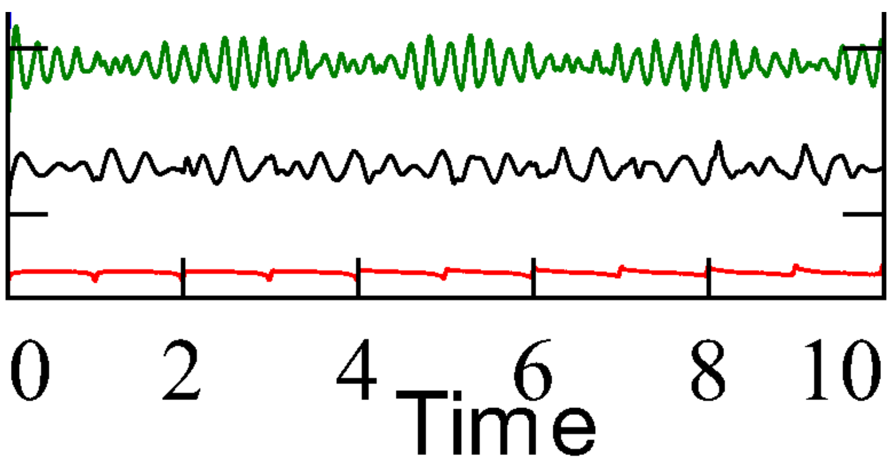
Irish Quantum Foundations, May 2013

Mainly joint works with J. Cardy, F. Essler, M. Fagotti

P. Calabrese and J. Cardy, Phys. Rev. Lett. 96, 136801 (2006); J. Stat. Mech. P06008 (2007).
P. Calabrese, F. H. L. Essler, M. Fagotti, Phys. Rev. Lett. 106, 227203 (2011); J. Stat. Mech. P07016 (2012); P07022 (2012).

Non equilibrium quantum dynamics

- A many-body quantum system prepared in a state $|\psi_0\rangle$ that is not an eigenstate of the Hamiltonian
- From *t=0* it evolves unitarily: $|\psi(t)\rangle = e^{-iHt}|\psi_0\rangle$
- No contact with "external" world
- How can we describe the dynamics?
- What about a "stationary state"?



von Neumann in 1929 posed the question [1003.2133]

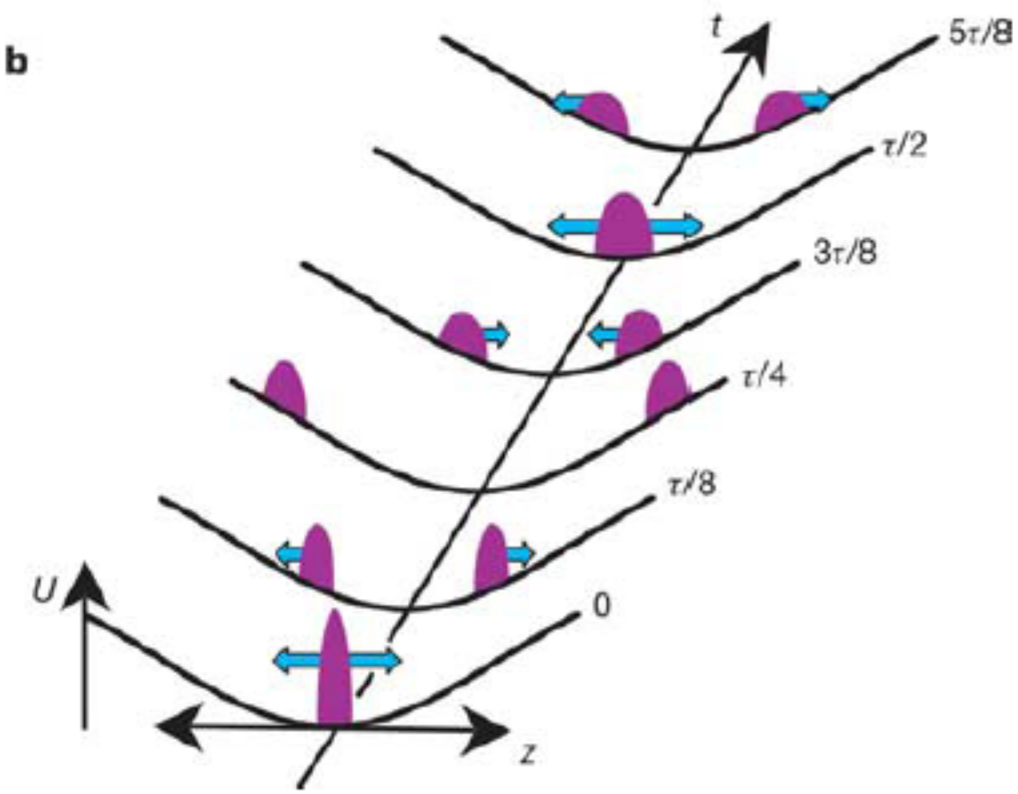
It stayed a purely academic question: for condensed matter systems the coupling to the environment is unavoidable

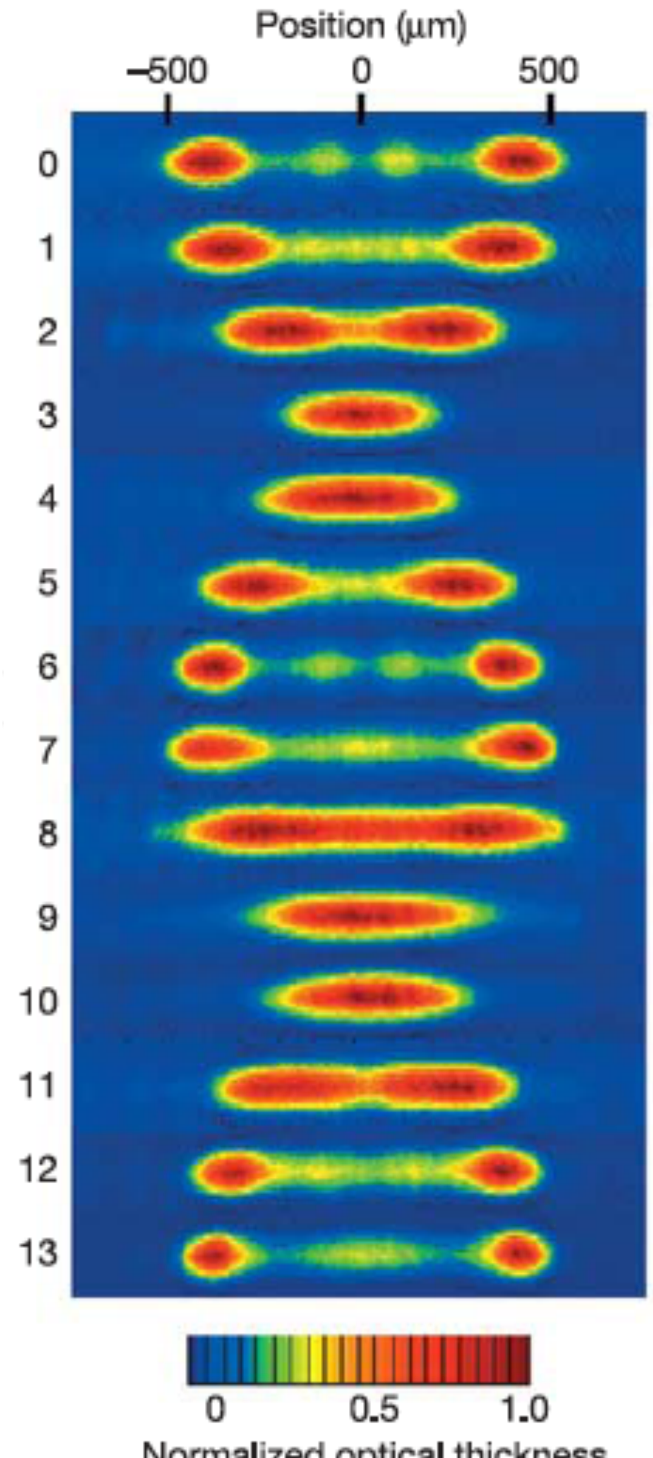
Quantum Quench: $|\psi_0\rangle$ is the Ground state of $H_0 \neq H$

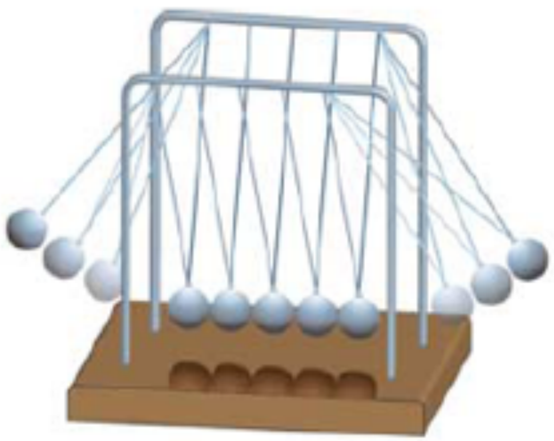
Quantum Newton Cradle

T. Kinoshita, T. Wenger and D.S. Weiss, Nature 440, 900 (2006)

40-250 ^{87}Rb atoms in a 1D optical trap



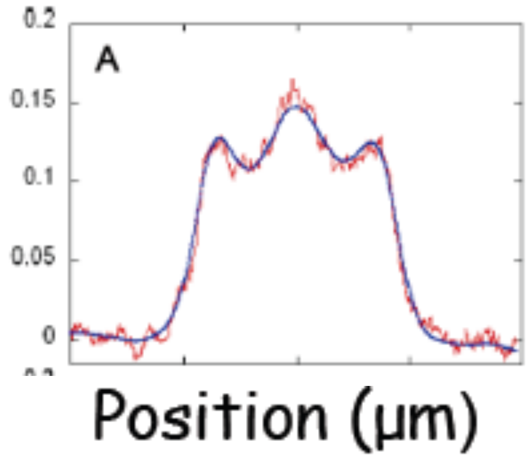




Essentially unitary time evolution

Can a steady state be attained? Surprisingly, YES

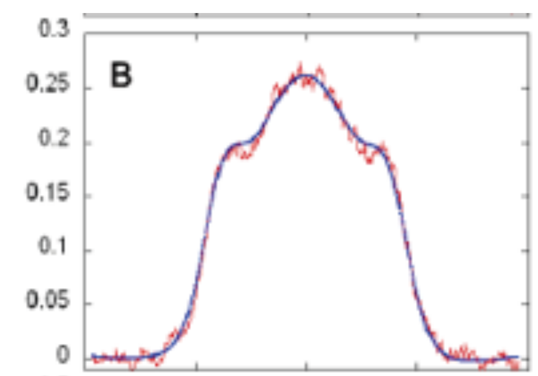
- 1D system “relaxes” very slowly in time, to a strange distribution.



$\gamma=18$

$\underline{\tau_{th}}$

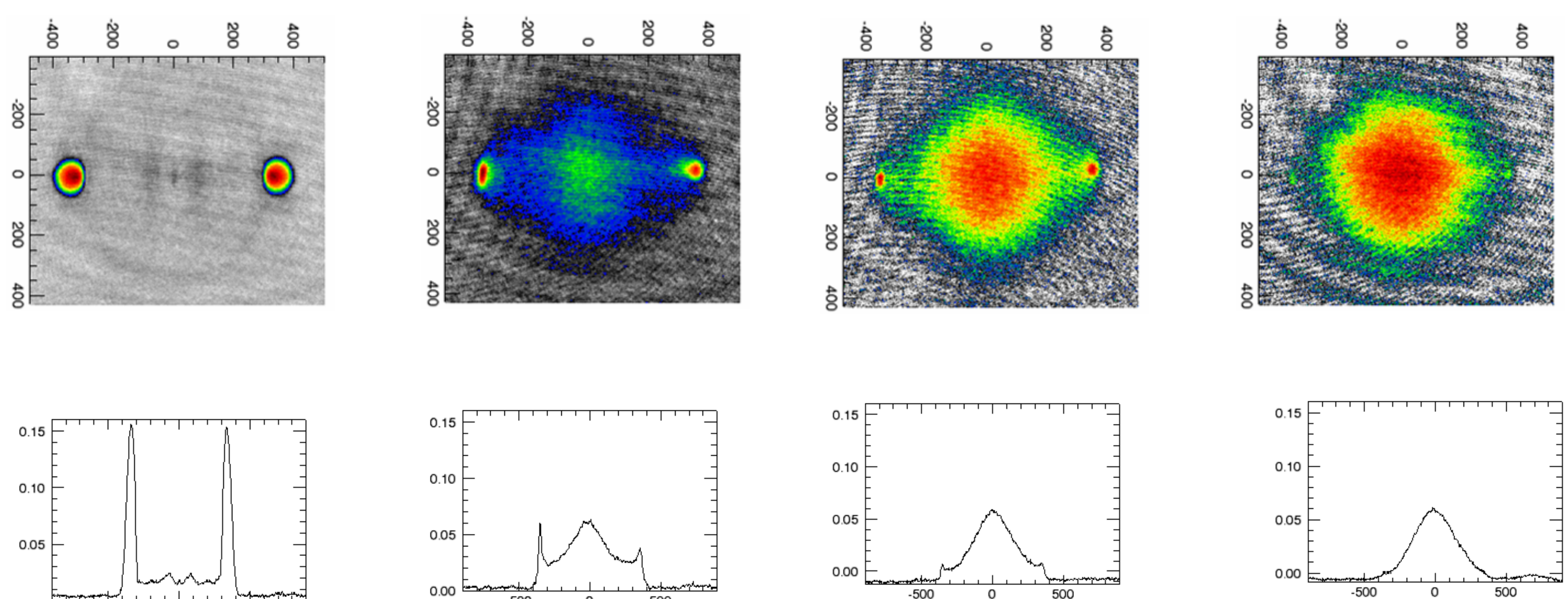
$>390\tau$



$\gamma=3.2$

$>1910\tau$

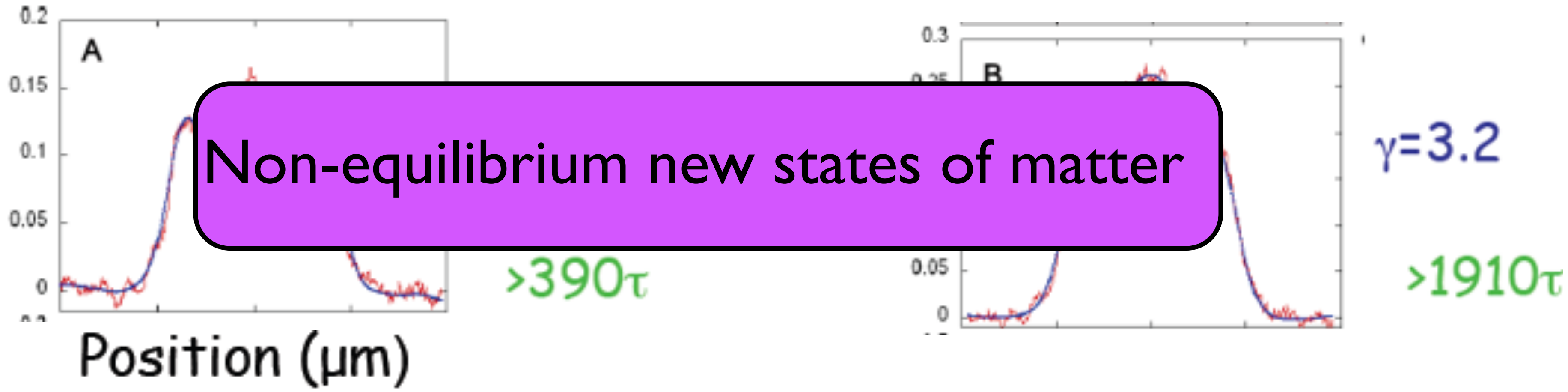
- 2D and 3D systems relax quickly:



$0\ \tau$ $2\ \tau$ $4\ \tau$ $9\ \tau$

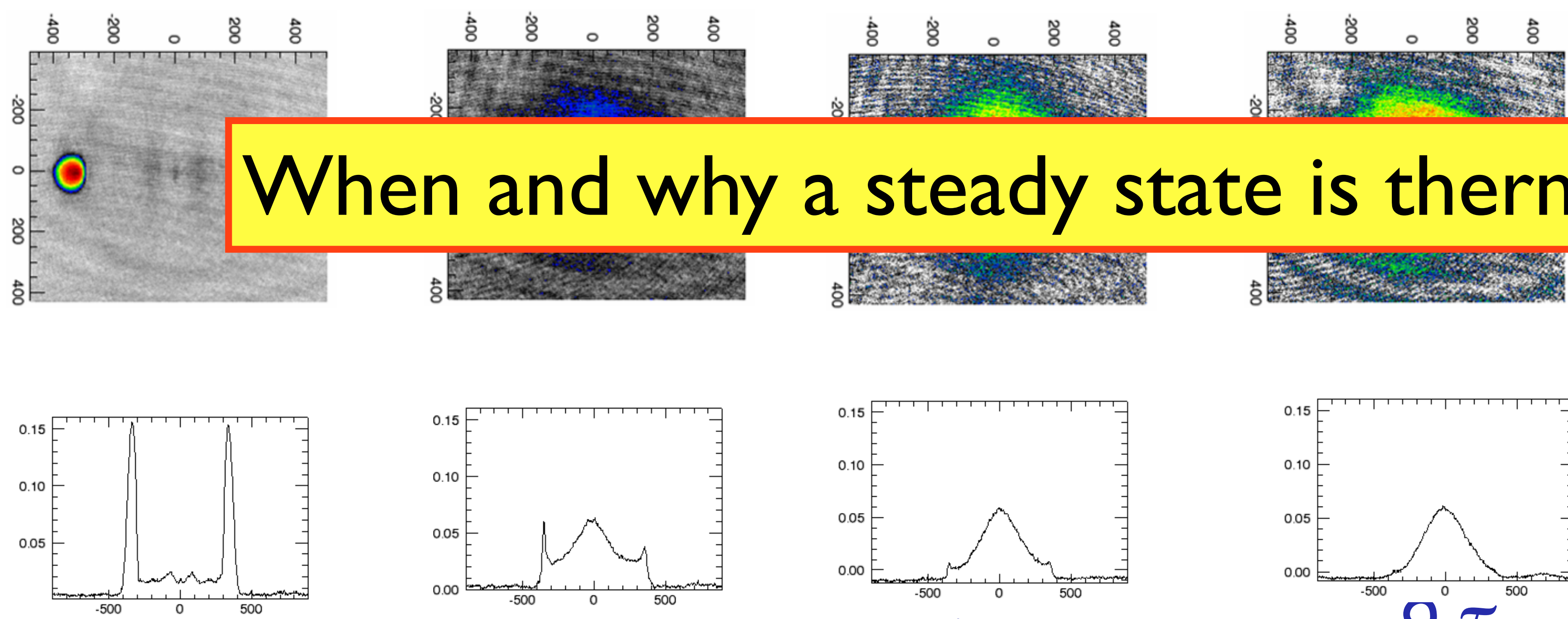
Can a steady state be attained? Surprisingly, YES

- 1D system "relaxes" very slowly in time, to a strange distribution.



Non-equilibrium new states of matter

- 2D and 3D systems relax quickly:

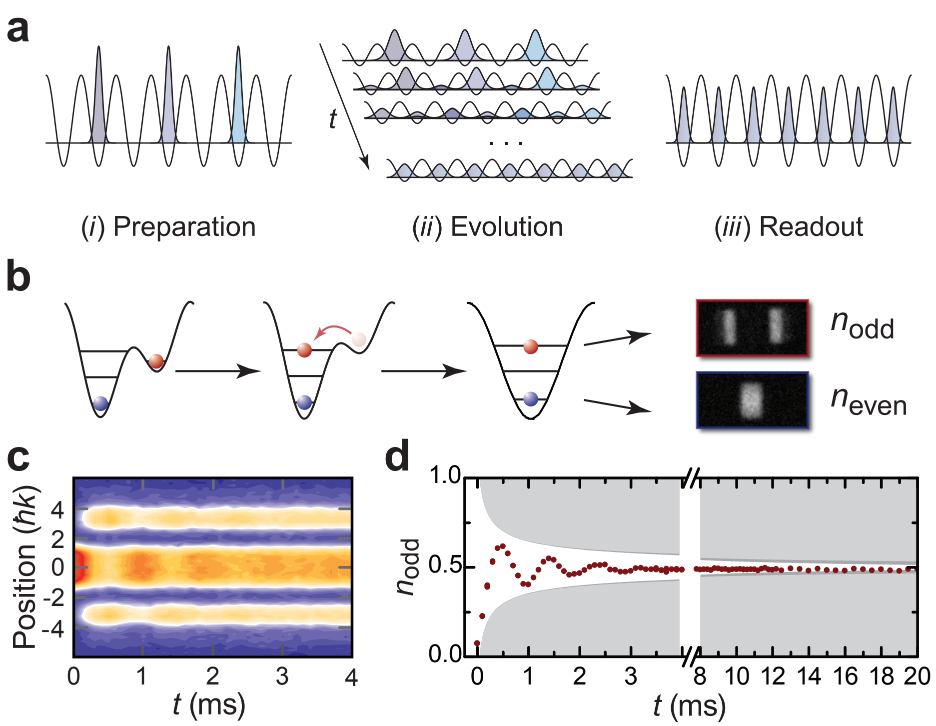


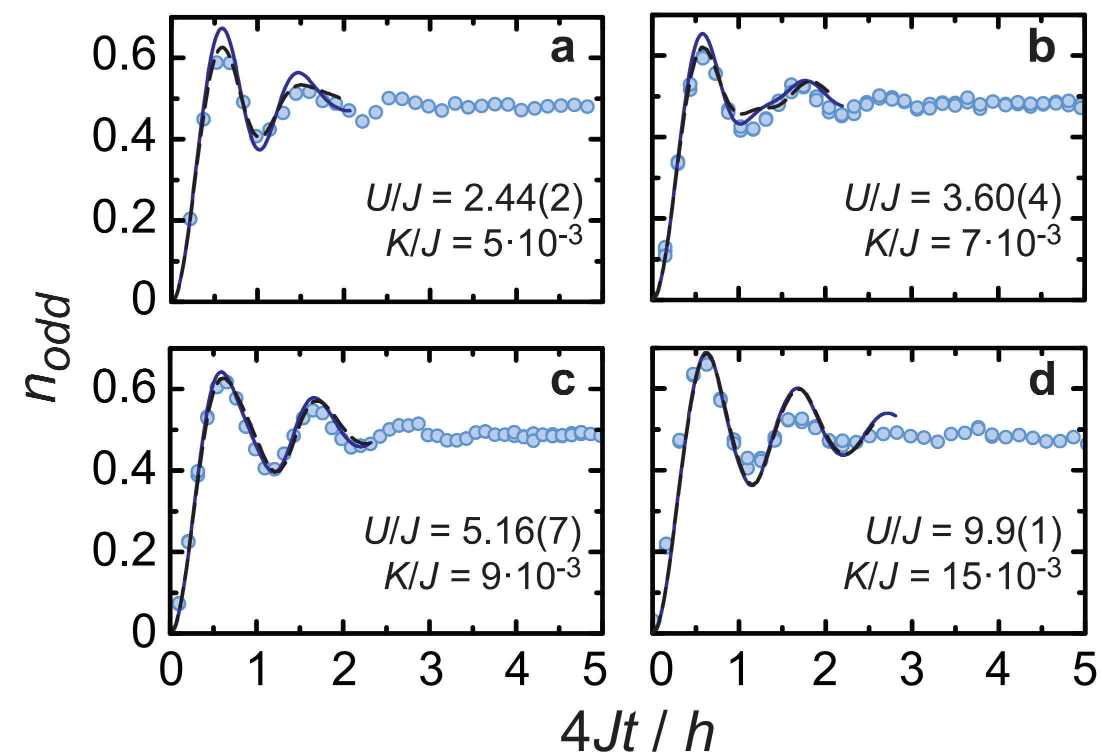
When and why a steady state is thermal??

The 1D case is special because the system is almost integrable

Probing Relaxation

S Trotzky et al, Nature Phys. 8, 325 (2012)





- Numerical DMRG and experiment agree perfectly
- The stationary state looks thermal

Probing Relaxation

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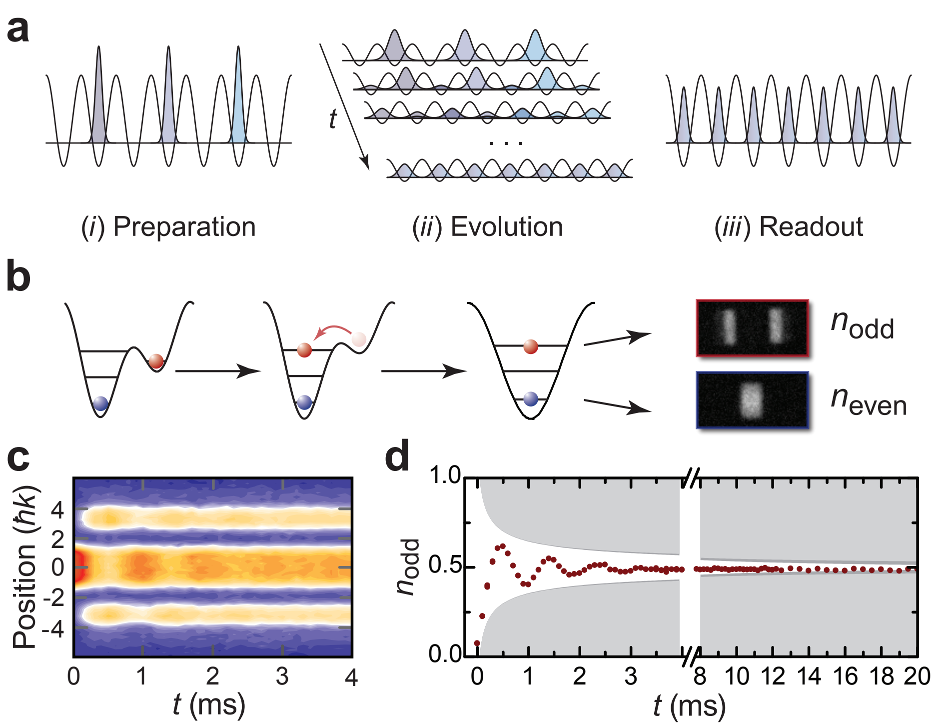
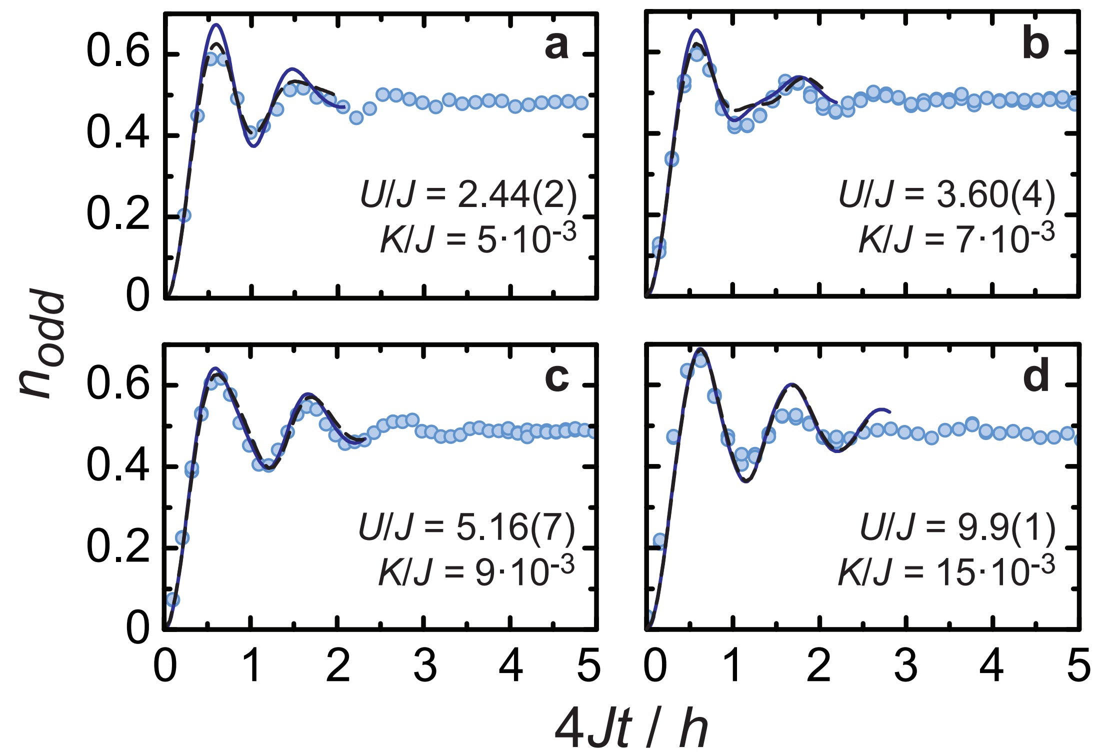




- Numerical DMRG and experiment agree perfectly
- The stationary state looks thermal

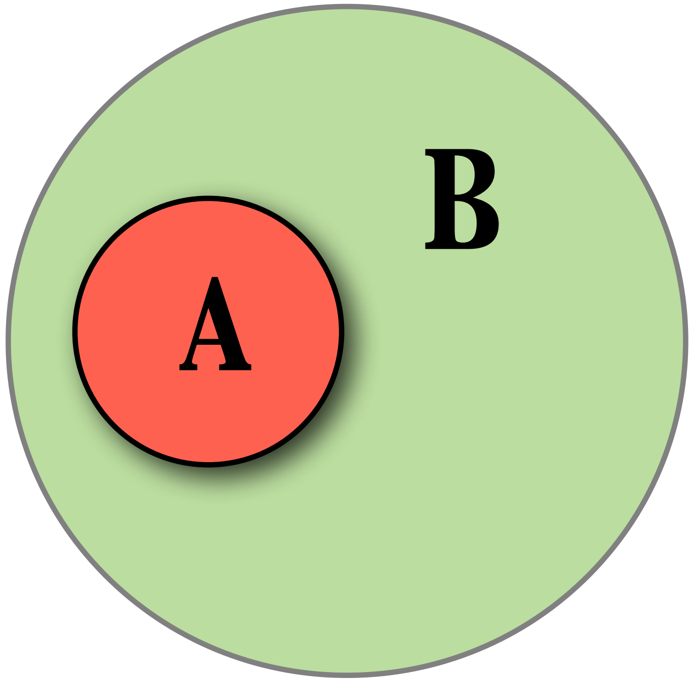
COMMON BELIEF: - Generic systems "thermalizes" — Deutsch '91, Srednicki '95

- Integrable systems are different — Rigol et al '07

But the system is always in a pure state!

Reduced density matrix

$|\psi(t)\rangle$ time dependent pure state



$\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ density matrix of $A \cup B$ (Infinite)

Reduced density matrix: $\rho_A(t) = \mathrm{Tr}_B\, \rho(t)$

The expectation values of all local observables in A are

$$\langle\psi(t)|O_A(x)\,|\psi(t)\rangle = \mathrm{Tr}[\rho_A(t)\, O_A(x)]$$

Stationary state: If for any finite subsystem A it exists the limit

$$\lim_{t\to\infty} \rho_A(t) = \rho_A(\infty)$$

Thermalization

Consider the Gibbs ensemble for the whole system A∪B

$\rho_T = e^{-H/T_{eff}} / Z$ with $\boxed{\langle\psi_0| H |\psi_0\rangle = \mathrm{Tr}[\rho_T H]}$

T_{eff} "is" the energy in the initial state: no free parameter!!

Reduced density matrix for subsystem A: $\boxed{\rho_{A,T} = \mathrm{Tr}_B\, \rho_T}$

The system thermalizes if for any finite subsystem A

$$\rho_{A,T} = \rho_A(\infty)$$

[Landau-Lifshitz vol 5]
[Barthel-Schollwock '08]
[Cramer, Eisert, et al '08] + ...

The infinite part B of the system "acts as an heat bath for A"

Generalized Gibbs Ensemble

[Rigol et al 2007]

What about integrable systems?

I_m is a complete set of local (in space) integrals of motion

$[I_m, I_n]=0 \quad [I_m, H]=0 \qquad I_m=\sum_x O_m(x)$

The GGE density matrix is

$\boxed{\rho_{GGE}= e^{-\sum \lambda_m I_m} /Z}$ with λ_m fixed by $\langle\psi_0| I_m |\psi_0\rangle = \mathrm{Tr}[\rho_{GGE}\, I_m]$

Again no free parameter!!

Reduced density matrix for subsystem A: $\rho_{A,GGE}=\mathrm{Tr}_B\, \rho_{GGE}$

The system is described by GGE if for any finite subsystem A

$$\rho_{A,GGE} = \rho_A(\infty)$$

[Barthel-Schollwock '08]
[Cramer, Eisert, et al '08] + ...

B is not a standard heat bath for A:
infinite information on the initial state is retained!

A toy theory: CFT in 1D

PC, J Cardy 2006/07

Let us make a long story short (v=1):

1. One-point function of a primary operator with $\langle\psi_0|O(x)|\psi_0\rangle \neq 0$:

$$\langle O(t,x)\rangle \propto e^{-\pi x_0 t/2\tau_0}$$

Exponential relaxation!

τ_0 related to the initial correlation

2. Two-point function of a primary operator with $\langle\psi_0|O(x)|\psi_0\rangle \neq 0$:

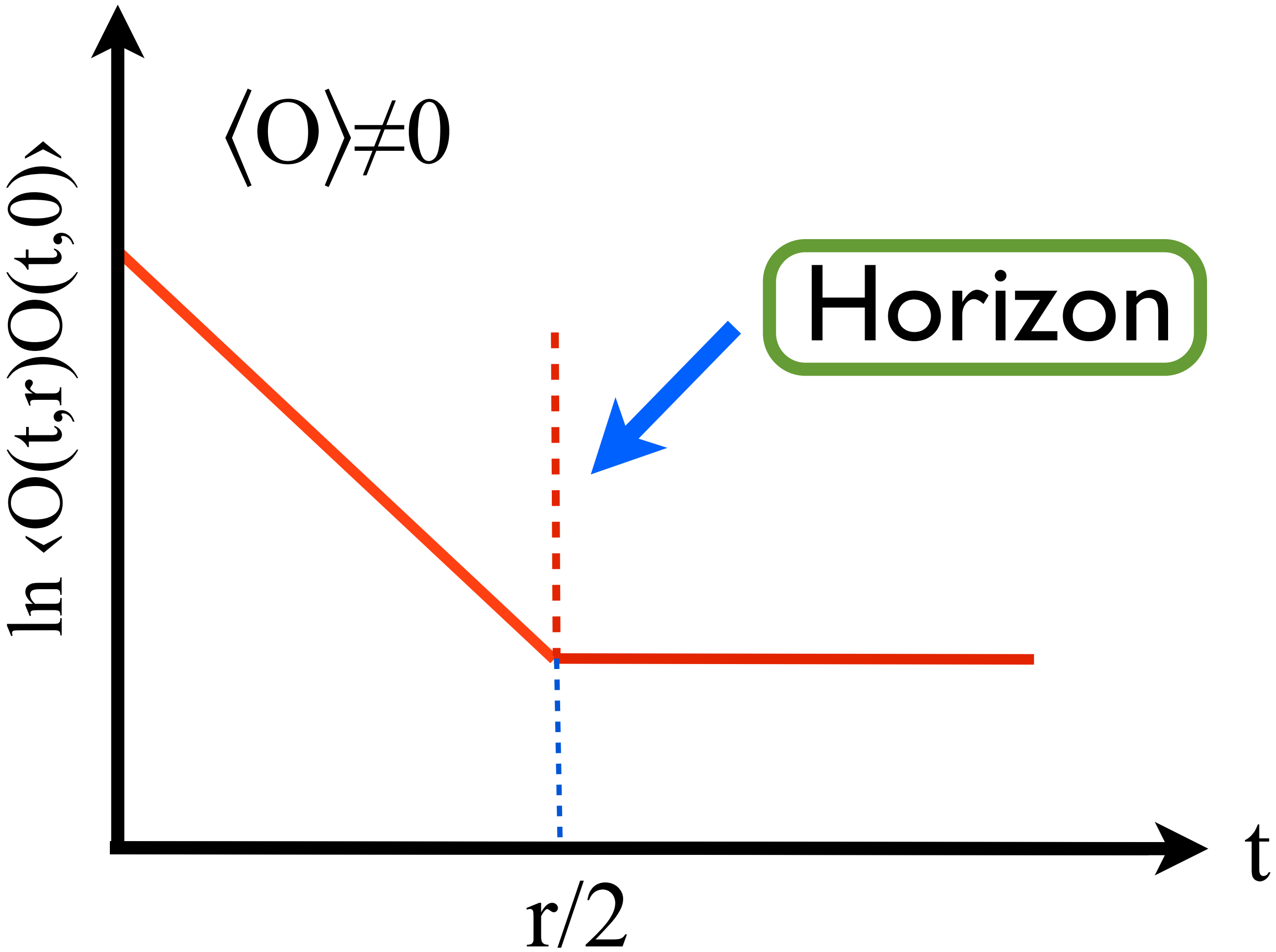
$$\Rightarrow \quad \langle O(t,r)O(t,0)\rangle \propto \begin{cases} e^{-\pi x_0 r/2\tau_0} & \text{for} \quad t>r/2 \\ e^{-\pi x_0 t/\tau_0} & \text{for} \quad t<r/2 \end{cases}$$

If $\langle\psi_0|O(x)|\psi_0\rangle \neq 0$, for $t<r/2 \implies \langle O(t,r)O(t,0)\rangle = \langle O(t,0)\rangle^2$

$\Rightarrow$ Connected correlations vanish for $t<r/2$

Focus: Two-point functions

PC, J Cardy 2006/07



Correlations saturate to
t-independent values for $t > r/2$

Asymptotic r-decay is exponential

Finite T

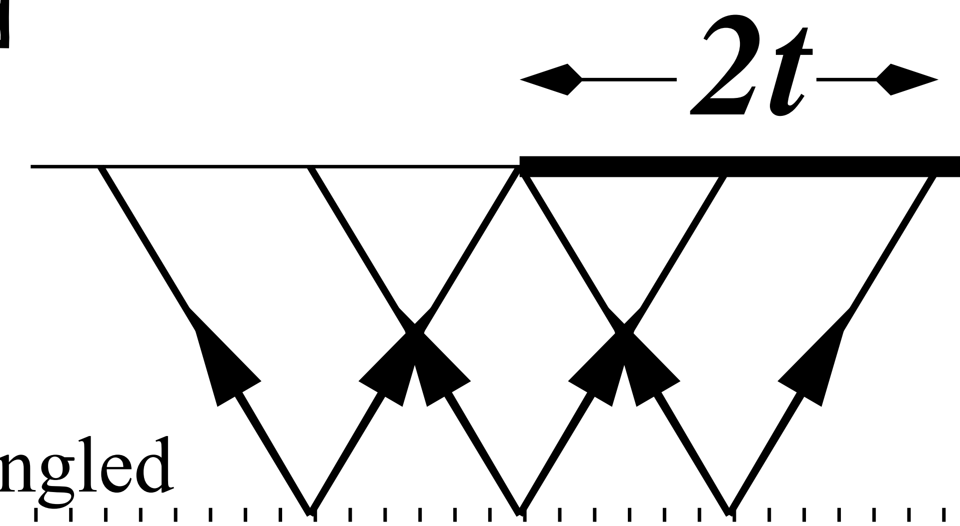
Thermalization of CFT

Sharp horizon and thermalization are consequences
of perfectly linear dispersion relation and specific initial state
Not true in general

CFT calculations have been generalized to different situations such as two-time correlations, systems with boundaries, different initial states, etc....

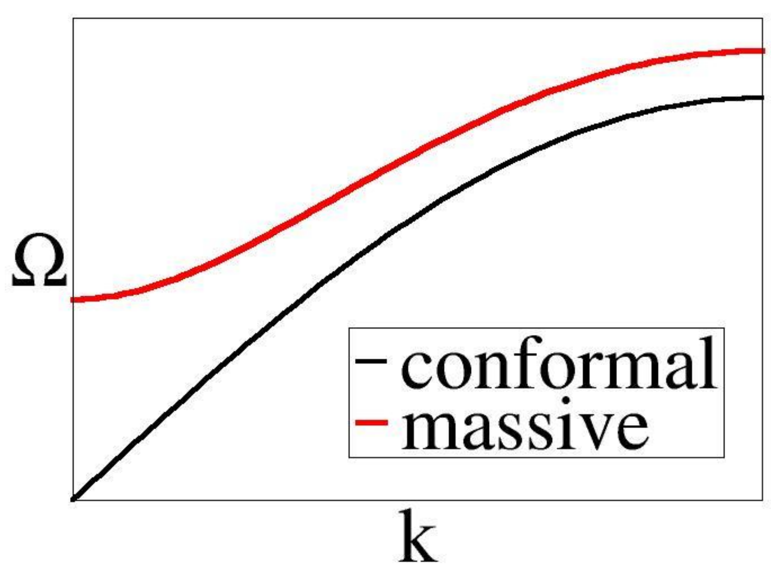
Physical Interpretation

PC, J Cardy 2006/07



- $|\psi_0\rangle$ has an extensive excess of energy
- $|\psi_0\rangle$ acts as a source of quasi-particle at t=0
- particles emitted from regions of size $\sim\tau_0$ are entangled
- For t>0 quasi-particles move at fixed velocity ±v (linear dispersion)
- Horizon: points at separation r become correlated when left- and right-moving particles originating from the same spatial region $\sim\tau_0$ first reach them
- If all particles move at speed v, correlations are then frozen for t > r/2v

General lattice dispersion:



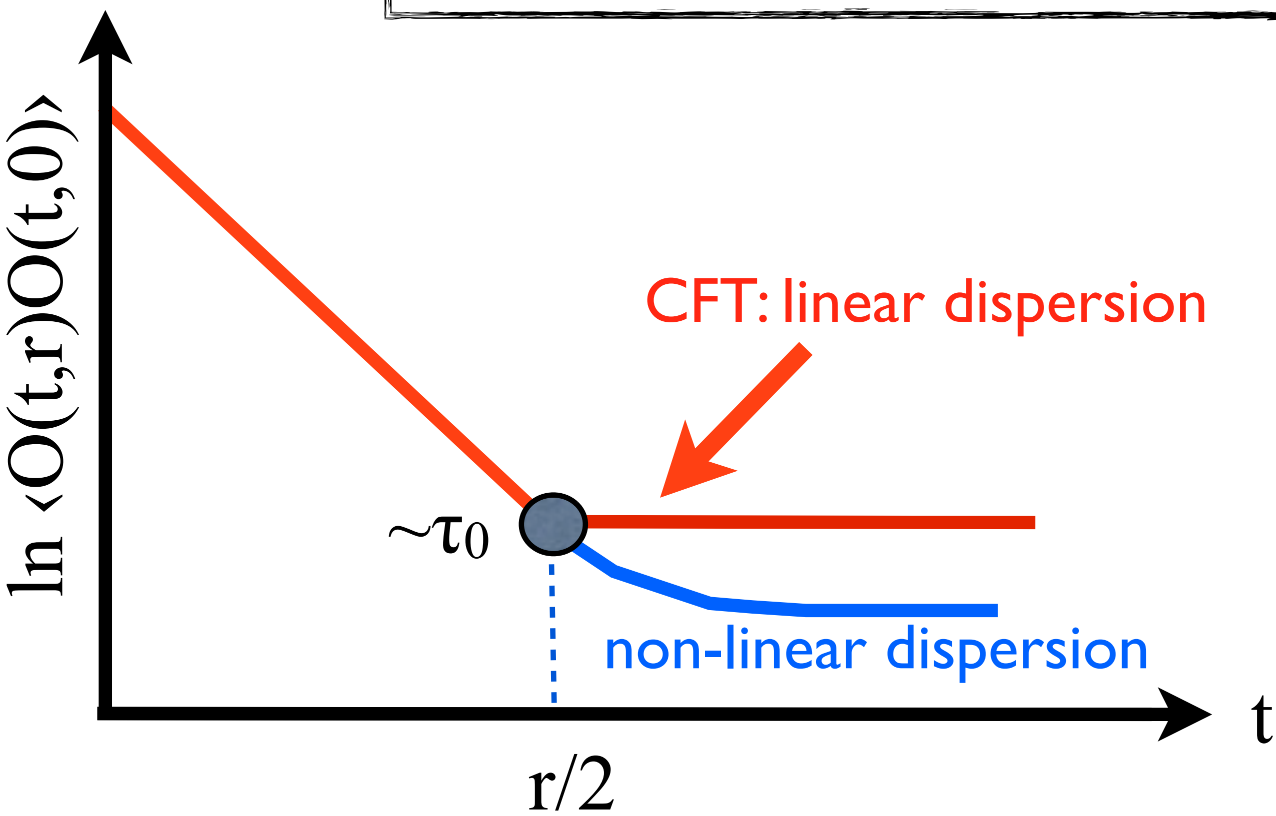
$$v_k = \frac{d\Omega_k}{dk}$$

v_{max} exists

correlations form at $t = r/2v_{max}$

Slower particles change correlations after $t = r/2v_{max}$
large t is driven by slowest particles

Horizon in lattice models

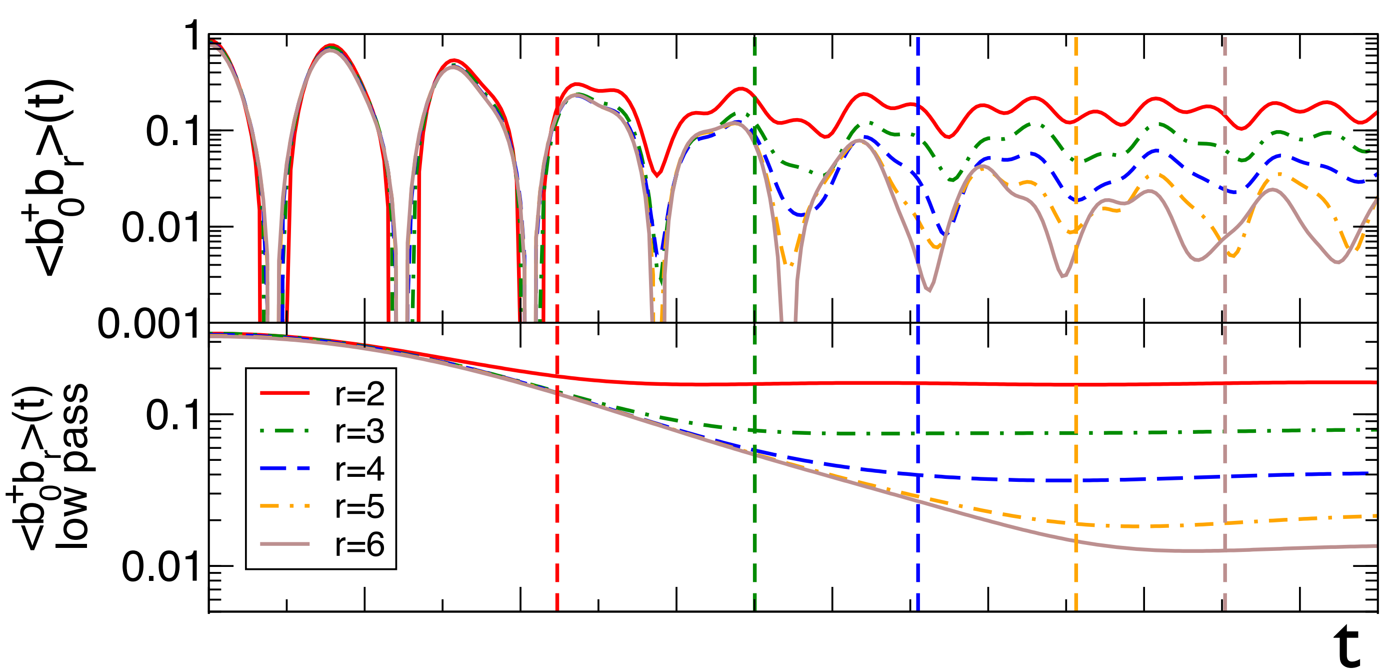


+ a bunch of slowly decaying oscillations
  - Lattice
  - non-zero momentum v_{max}

Numerical checks:

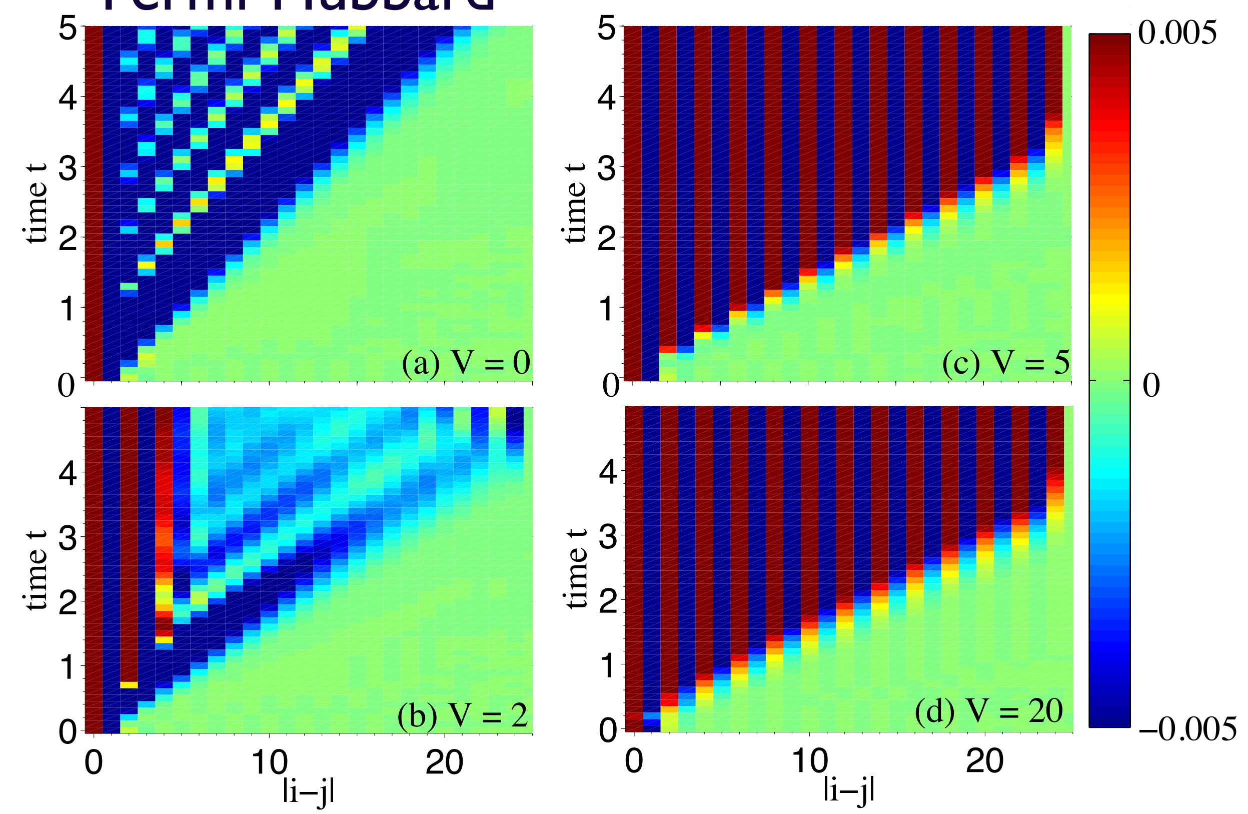
Kollath-Lauechli '08

Bose-Hubbard



Manmana et al '08

Fermi-Hubbard



Light cone in experiment

M. Cheneau et al, Nature 481, 484 (2012)

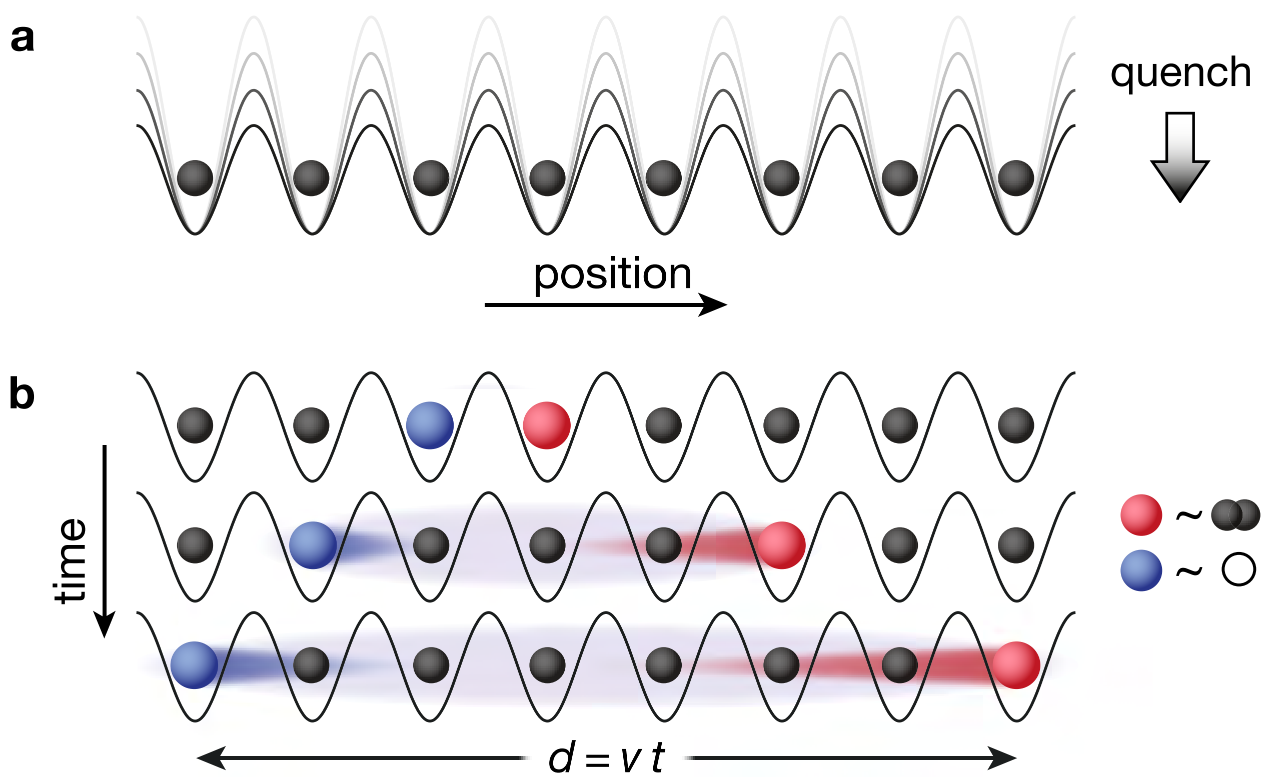
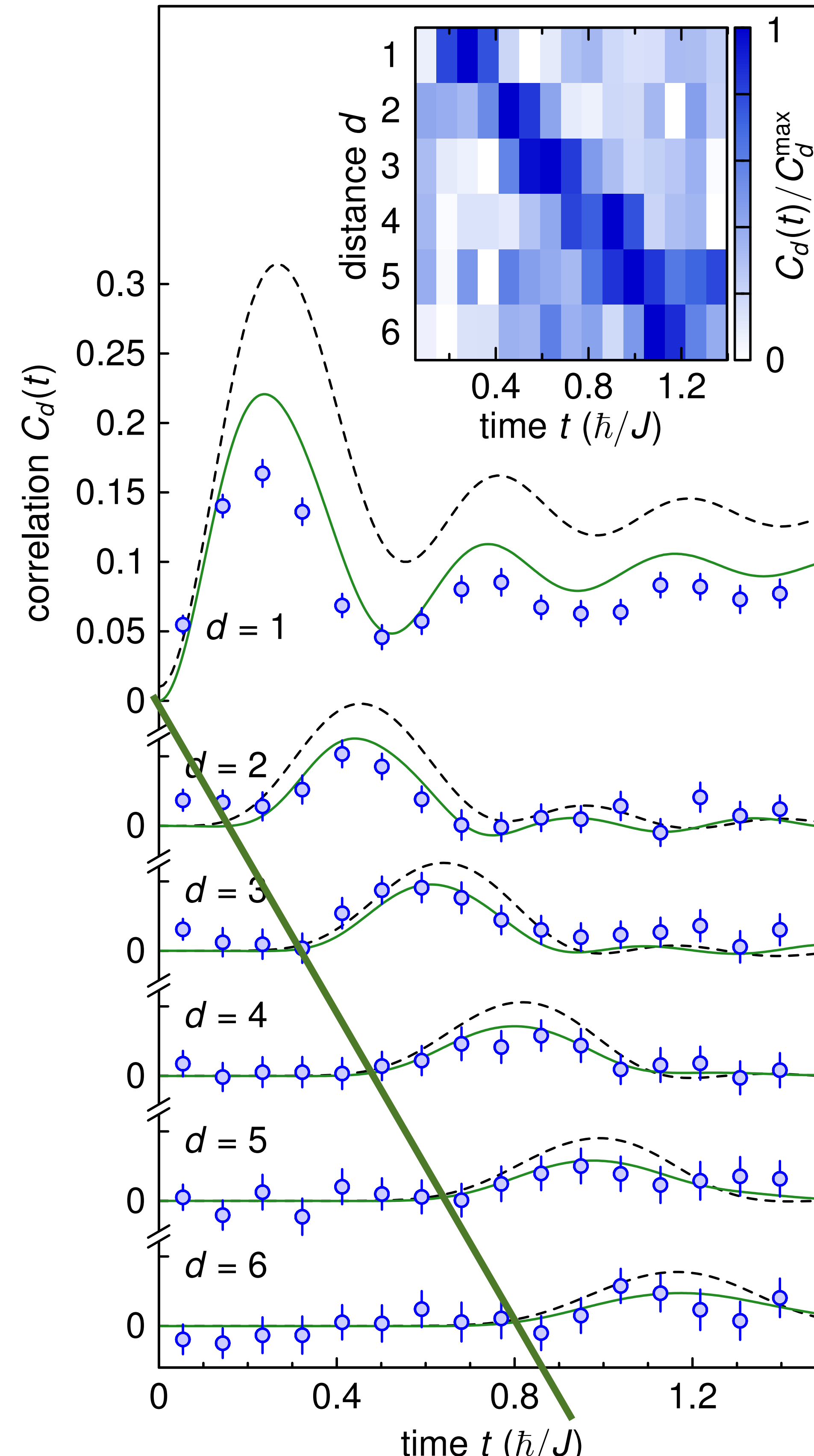


FIG. 1. **Spreading of correlations in a quenched atomic Mott insulator.** **a**, A 1D ultracold gas of bosonic atoms (black balls) in an optical lattice is initially prepared deep in the Mott-insulating phase with unity filling. The lattice depth is then abruptly lowered, bringing the system out of equilibrium. **b**, Following the quench, entangled quasiparticle pairs emerge at all sites. Each of these pairs consists of a doublon (red ball) and a holon (blue ball) on top of the unity-filling background, which propagate ballistically in opposite directions. It follows that a correlation in the parity of the site occupancy builds up at time t between any pair of sites separated by a distance $d = vt$, where v is the relative velocity of the doublons and holons.



Transverse field Ising chain

$$H(h) = -\frac{1}{2}\sum_{l=-\infty}^{\infty}\left[\sigma_l^x\sigma_{l+1}^x + h\sigma_l^z\right]$$

$\langle\sigma^x\rangle \neq 0$ | $\langle\sigma^x\rangle = 0$ → h

$h_c = 1$

Jordan-Wigner	Fourier	Bogoliubov
$\begin{cases}\sigma_j^z = 2c_j^\dagger c_j - 1 \\ \sigma_j^x = e^{i\pi\sum_{l<j} n_l} c_j + h.c.\end{cases}$	$c(k) = \frac{1}{\sqrt{L}}\sum_j c_j\, e^{-ikj}.$	$\begin{pmatrix} c(k) \\ c^\dagger(-k)\end{pmatrix} = R_h(k)\begin{pmatrix}\alpha_k \\ \alpha^\dagger_{-k}\end{pmatrix}$

$$H = \sum_k \epsilon_h(k)\left[\alpha_k^\dagger\alpha_k - \frac{1}{2}\right]$$

$$\epsilon_h(k) = 2J\sqrt{1+h^2-2h\cos(k)}.$$

free fermions!

Ground State: $\alpha_k|0\rangle = 0.$

Non-equilibrium dynamics not so easy analytically

Barouch-McCoy '70, Igloi-Rieger '00, Sengupta et al 04....... many more

Quantum Quench h→h'

New Hamiltonian: $H(h') = \sum_k \epsilon_{h'}(k) \left[\beta_k^\dagger \beta_k - \frac{1}{2}\right]$ $\qquad \beta_k(t) = e^{-i\epsilon_{h'}(k)t}\beta_k$

New vs old Bogoliubov fermions: $\begin{pmatrix} \beta_k \\ \beta^\dagger_{-k} \end{pmatrix} = U(k) \begin{pmatrix} \alpha_k \\ \alpha^\dagger_{-k} \end{pmatrix}$ $\qquad U(k) = R^\dagger_{h'}(k) R_h(k)$

★ An old calculation [Barouch, McCoy & Dresden '70] shows that σ^z does not thermalize [but a posteriori GGE works]

★ σ^z is quite **special** (non generic): it is **local** w.r.t. to the fermion excitations and couples only to 2-particle states.

★ σ^x is **non-local** (couples to states with arbitrary number of fermions)

$$\langle \sigma^x_j(t) \sigma^x_{j+n}(t) \rangle \xrightarrow{\text{Wick's thm}} \mathrm{Pf(T)} \qquad T_{ln} = \begin{pmatrix} f_{l-n} & -g_{n-l} \\ g_{l-n} & -f_{l-n} \end{pmatrix}$$

Block-Toeplitz matrix

$$f_l = i \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ikl} \sin(\Delta_k) \ \sin(2\epsilon'_h(k)t)$$

$$g_l = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ik(l-1)} \left[\cos(\Delta_k) + i \sin(\Delta_k) \cos(2\epsilon'_h(k)t)\right]$$

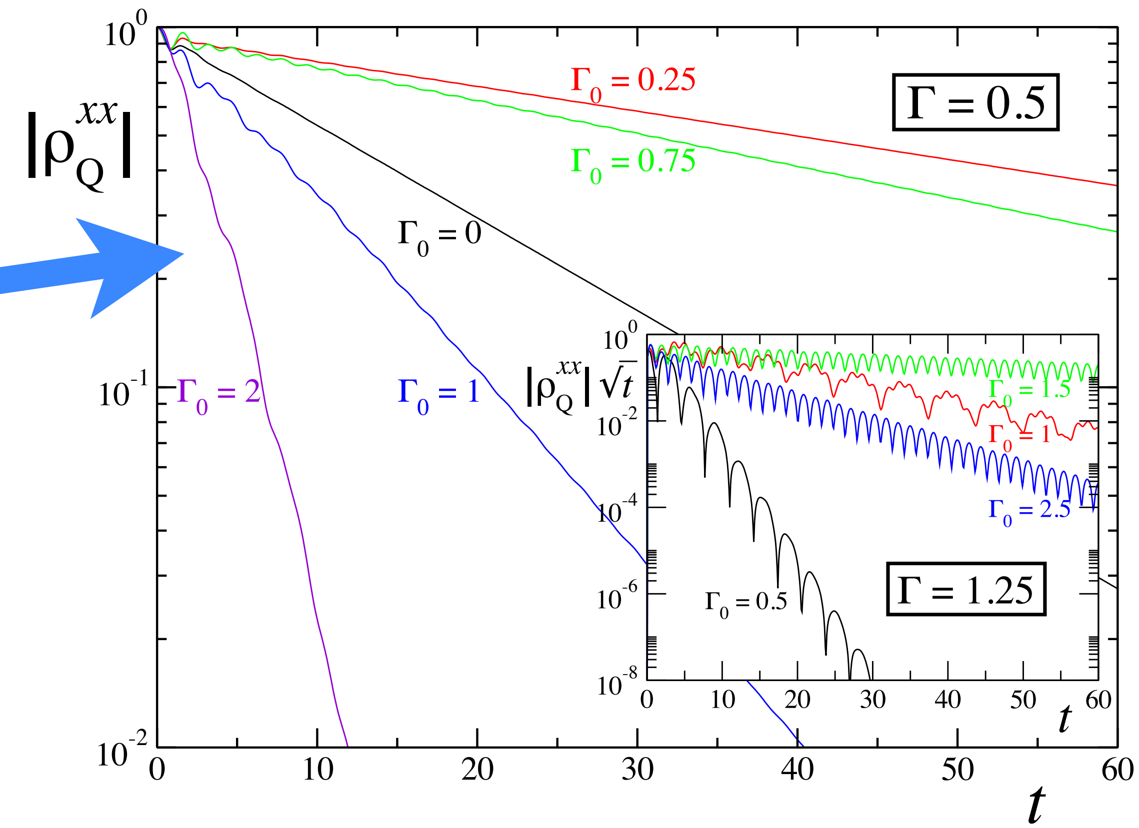
Questions

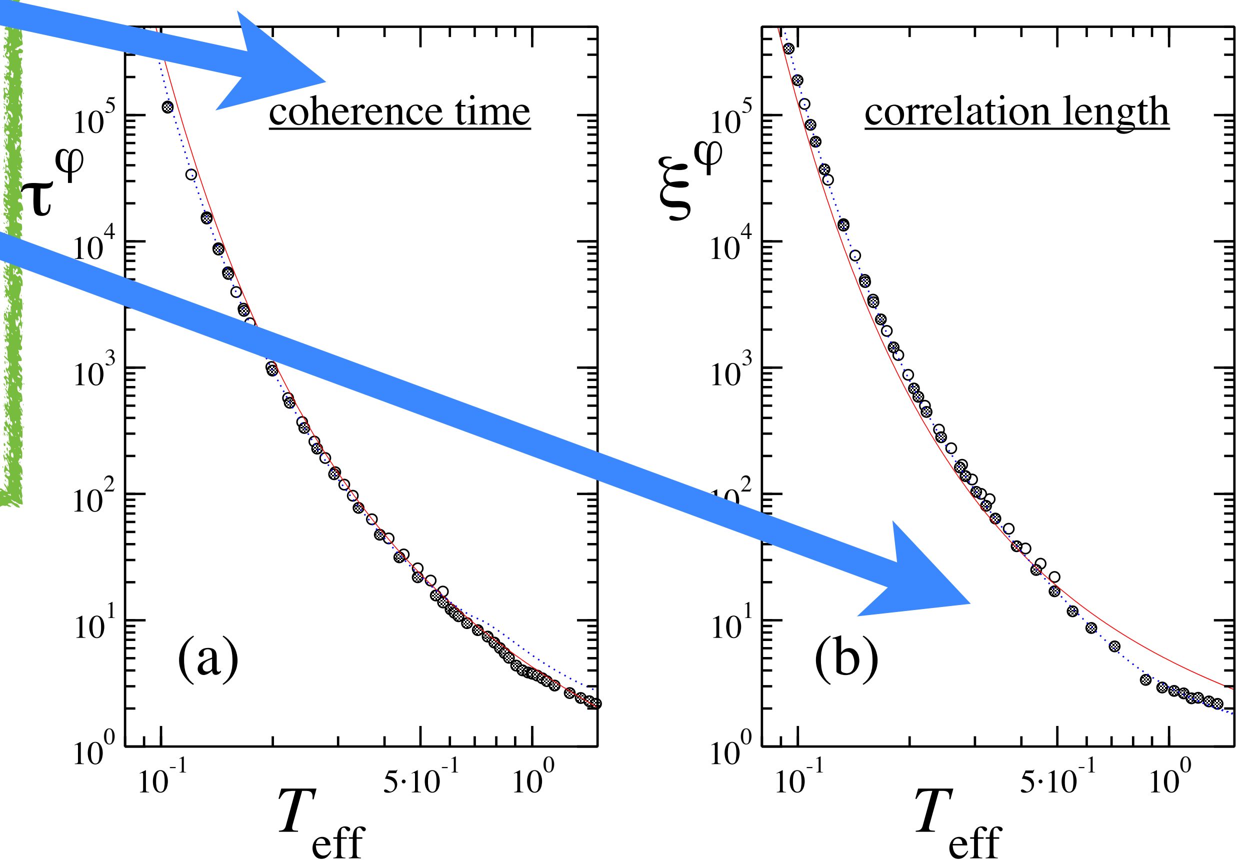
1. What happens for $t \to \infty$?
   a. Is the stationary state described by GGE?
   b. Is it possible that certain operators thermalize and others don't?
   c. Can we calculate expectation values of local observables?
2. What is the behavior of local observables for finite t?
3. How fast the $t \to \infty$ limit is approached?

Numerical results

Rossini, Silva, Mussardo, Santoro '09-'10

- $\langle \sigma_l^x(t)\sigma_{l+\ell}^x(t)\rangle$ decays exponentially in time
- For large time, it decays exponentially with separation
- Allow to define coherence time and correlation length
- Numerically they are the same as in finite T





Can we say smth analytically?

Exact solution

PC, Essler, Fagotti 11/12

Calculations are **difficult**. Developed two analytic methods based on (a) determinants (b) form factors.

Result 1: For t=∞ GGE holds

We showed that: $$\boxed{\rho_{A,\mathrm{GGE}} = \rho_A(\infty)}$$

Result 2: t=∞ behavior for arbitrary h,h'

$$\lim_{t\to\infty} \langle \sigma_j^x(t)\sigma_{j+\ell}^x(t)\rangle \sim \exp(-\ell/\xi) \quad \text{for } \ell \gg 1$$

$$\xi^{-1} = \int_0^\pi \frac{dk}{\pi}\xi^{-1}(k) = -\int_0^\pi \frac{dk}{\pi}\epsilon_{h'}(k)\tanh\frac{\epsilon_{h'}(k)}{2T_{\rm eff}(k)}\,.$$

(h,h'<1)

thermal correlation length: $\xi_T^{-1} = -\int_{-\pi}^{\pi}\frac{dk}{2\pi}\ln\left|\tanh\frac{\epsilon_h(k)}{2T}\right|.$

Compatible with **GGE**, but **not with thermalization!**

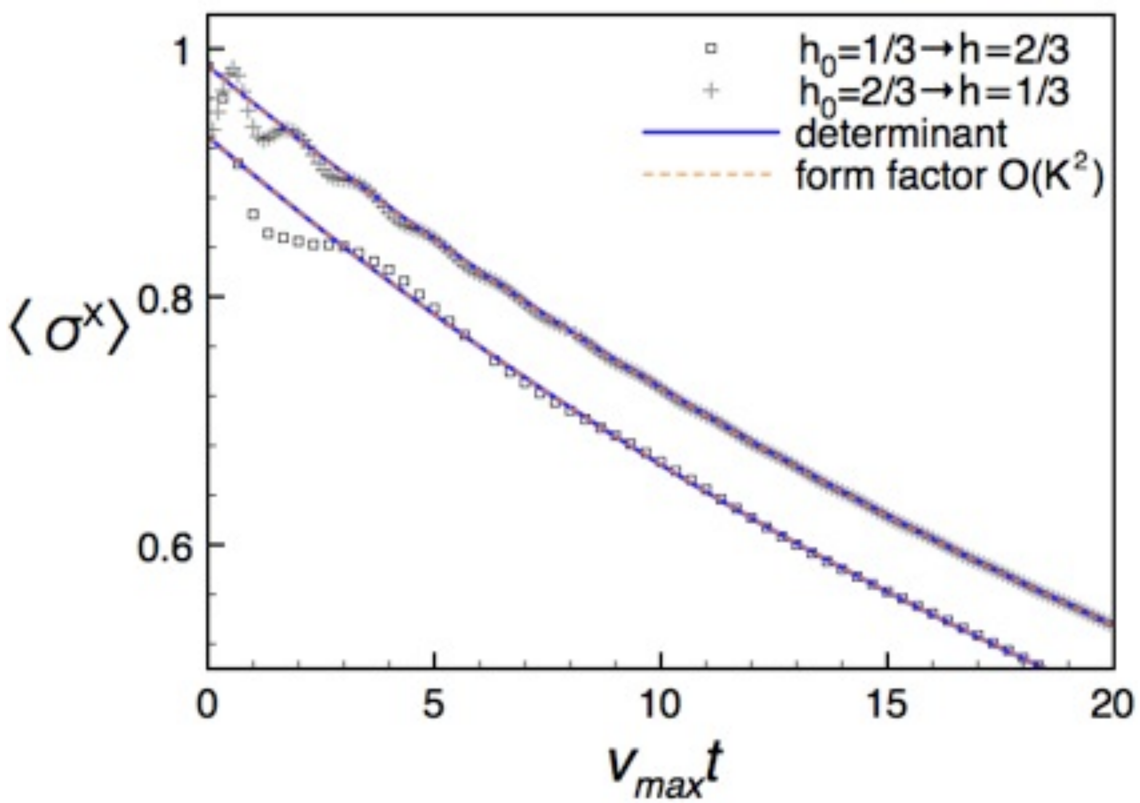
Interpretation in terms mode-dependent temperature

Result 3: Time dependence of 1-pt function

For quenches within the ordered phase (h<1 to h′<1):

$$\langle \sigma_l^x(t) \rangle \propto \exp\left[t \int_0^\pi \frac{\mathrm{d}k}{\pi} \epsilon'_h(k) \ln|\cos\Delta_k| \right] \qquad \cos\Delta_k = \frac{h'h - (h'+h)\cos k + 1}{\epsilon_h(k)\epsilon_{h'}(k)}.$$

(approaches **zero** although we remain in the ordered phase).



Mode-dependent decay rate:

$$\tau^{-1} = \int_0^\pi \frac{dk}{\pi} \tau^{-1}(k) = \int_0^\pi \frac{dk}{\pi} \epsilon_{h'}(k) \ln\left[\cos\Delta_k\right].$$

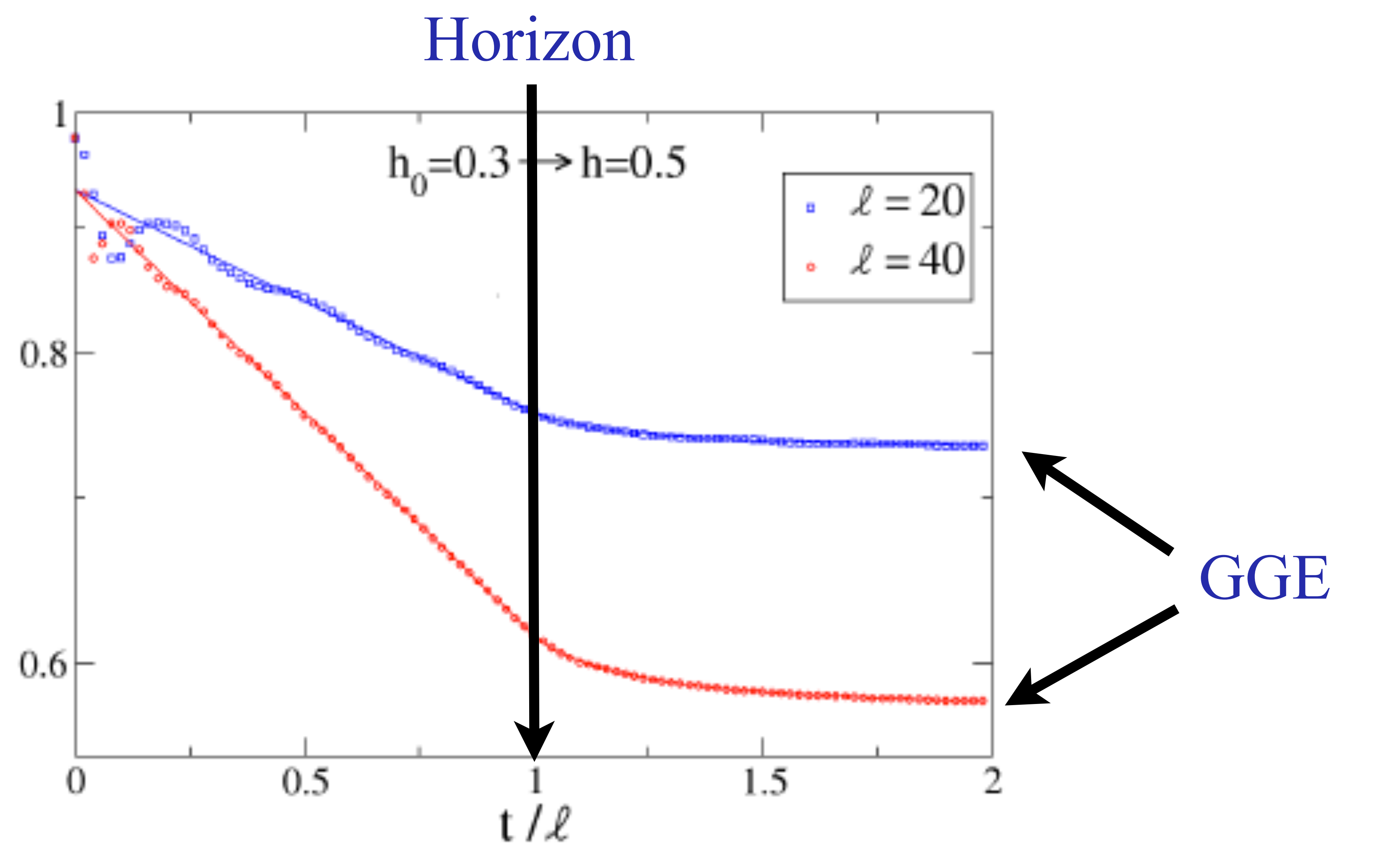
$$\xi(k) = \epsilon_{h'}(k)\, \tau(k).$$

Result 4: FULL time-dependence for quenches within the ordered phase (h<1 to h'<1):

Multi-dimensional stationary phase approach

$$\langle \sigma_j^x(t)\sigma_{j+\ell}^x(t)\rangle \sim \exp\left(t \int_{2t\epsilon'_{h'}(k)<\ell} \frac{dk}{\pi} 2\epsilon'_{h'}(k) \ln\left[\cos(\Delta_k)\right] + \ell \int_{2t\epsilon'_{h'}(k)>\ell} \frac{dk}{\pi} \ln\left[\cos(\Delta_k)\right] \right)$$

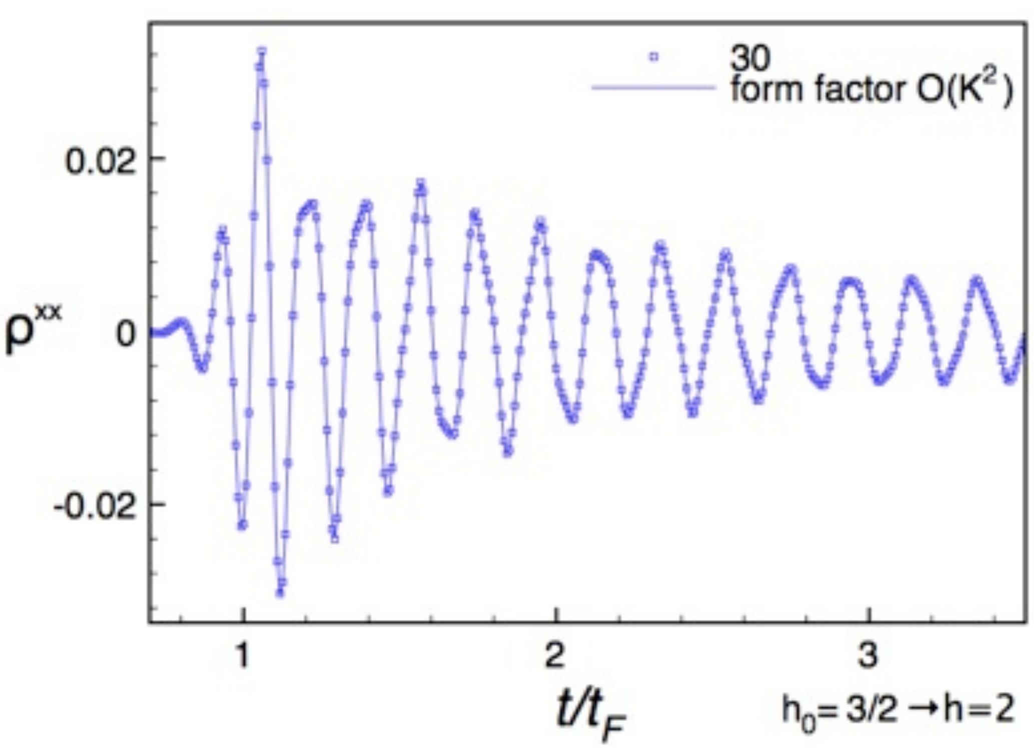
Asymptotics
vs
Numerics:

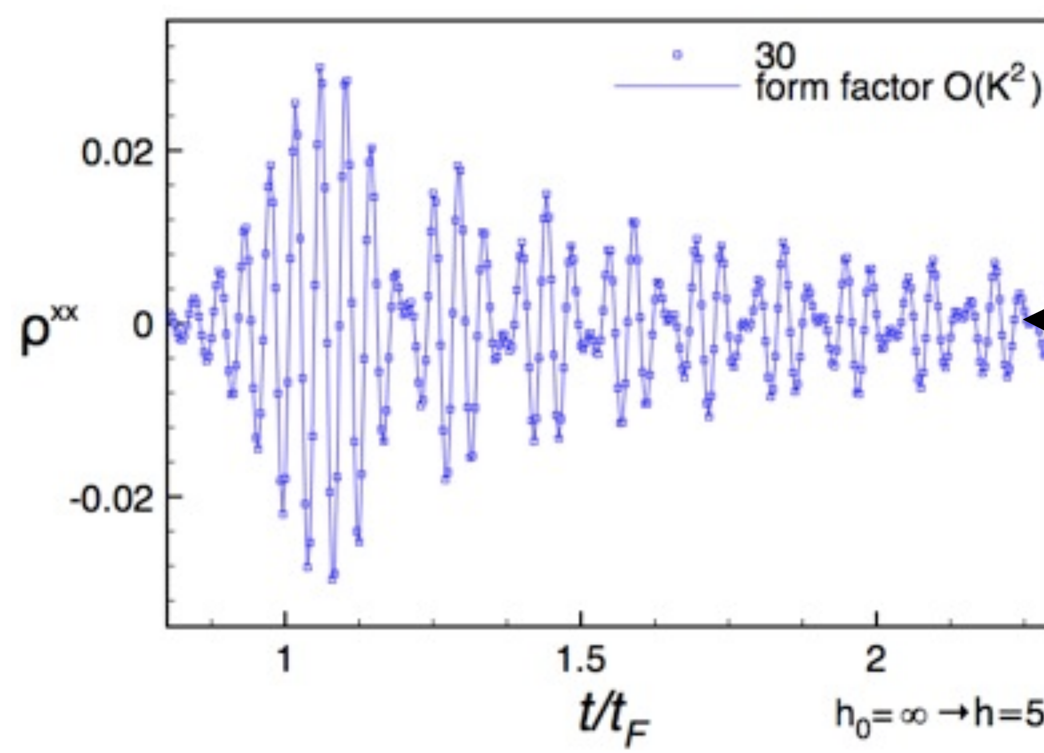


Result 5: FULL time-dependence for quenches within the disordered phase (h>1 to h'>1):

$$\rho^{xx}(\ell,t) \simeq \left[\mathcal{C}^x_{\rm PP}(\ell) + (h^2-1)^{\frac14}\sqrt{4J^2h}\int_{-\pi}^{\pi}\frac{dk}{\pi}\frac{K(k)}{\varepsilon_k}\sin(2t\varepsilon_k - k\ell) + \dots\right]$$
$$\times\exp\left[-\int_0^{\pi}\frac{dp}{\pi}\ln\left[\frac{1+K^2(p)}{1-K^2(p)}\right]\left(\ell + \theta_H(\ell - 2t\varepsilon'_p)[2t\varepsilon'_p - \ell]\right)\right] + \dots\ .$$

Asymptotics vs Numerics:





GGE

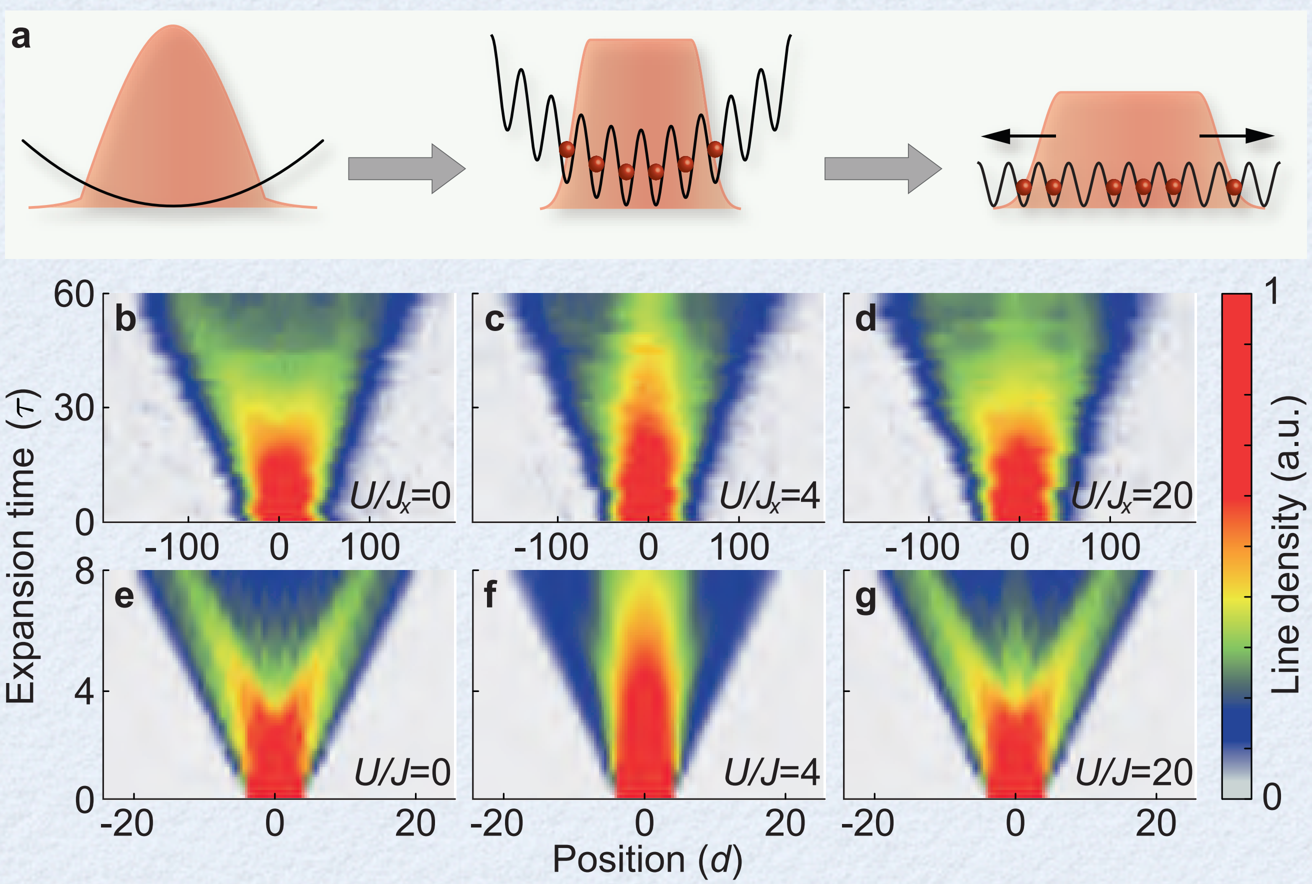
More complicated relaxational mechanism

A NON HOMOGENOUS INITIAL STATE

Expansion of an interacting gas

Expansion of initially localized ultracold bosons in 1D and 2D optical lattices.

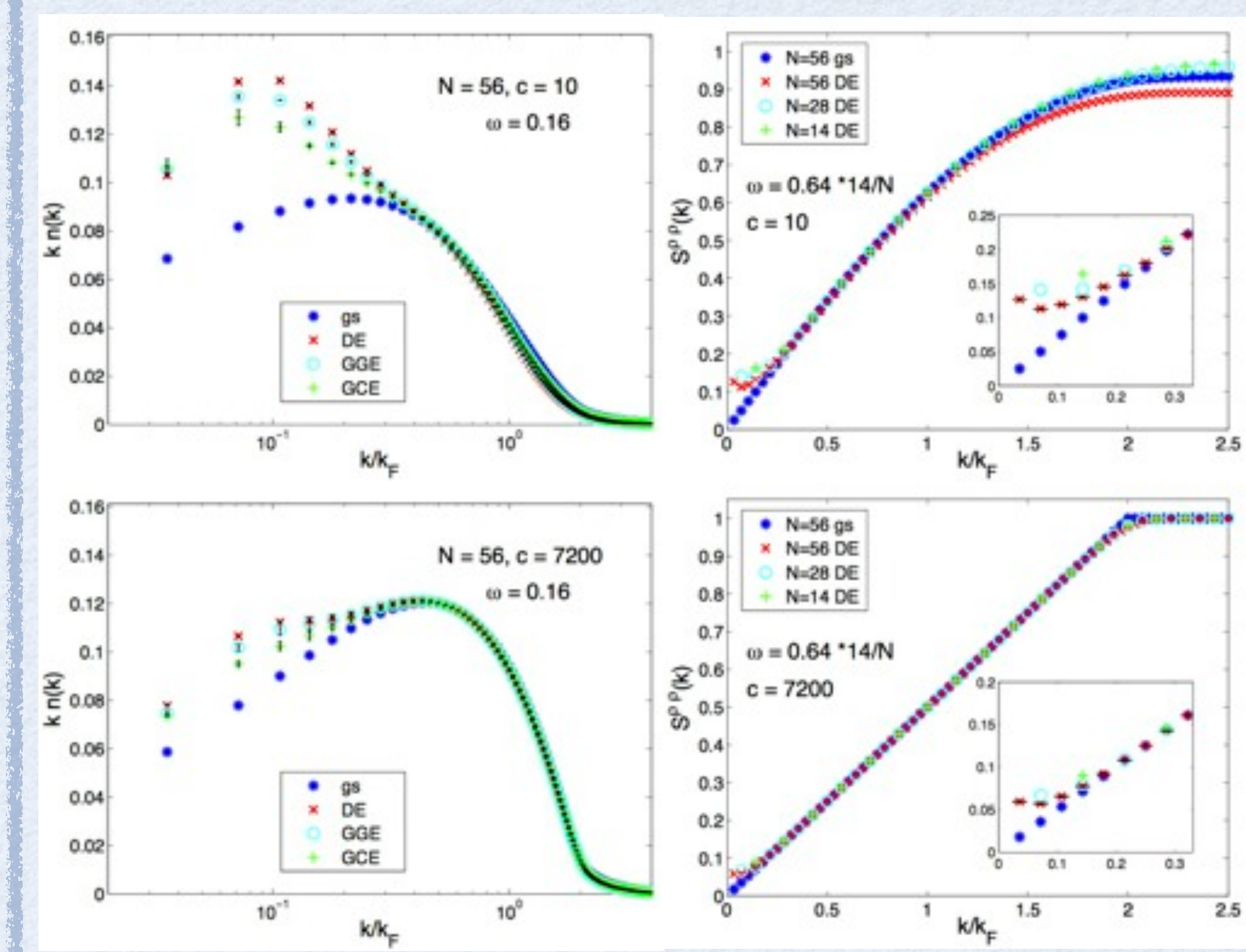
J.P.Ronzheimer ***et al*****,** arXiv:1301.5329



1) Integrable system: Ballistic Expansion

2) Not-integrable: Diffusive Expansion

Caux and **Konik** developed a new numerical method for the non-equilibrium dynamics of the integrable 1D Bose gas (**Lieb-Liniger)** after the release of a parabolic trap [**PRL 109, 175301 (2012)**].



The Strongly Interacting Regime

Model & Quench

Collura, Sotiradis, PC 13

1D Bos gas with delta pairwise interaction and in a harmonic external potential (Lieb-Liniger):

$$H = -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \frac{1}{2}\sum_{j=1}^{N}\omega^2 x_j^2 + c\sum_{i\neq j}\delta(x_i - x_j)$$

Tonks-Girardeau limit ($c\rightarrow\infty$) in 2° quantization:

$$H = \int dx\, \hat{\Psi}^\dagger(x)\left[-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\omega^2 x^2\right]\hat{\Psi}(x)$$

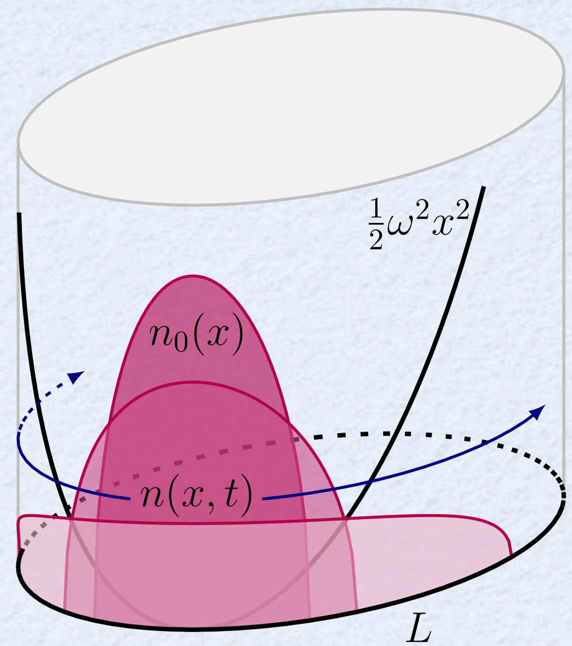
via the **Jordan-Wigner** transformation

$$\hat{\Psi}(x) = \exp\left\{i\pi\int_0^x dz\, \hat{\Psi}^\dagger(z)\hat{\Psi}(z)\right\}\hat{\Phi}(x)$$

QUENCH PROTOCOL

At time t=0 we release the harmonic trap. The evolution in governed by the free-particle Hamiltonian with PBC:

$$H_0 = \int dx\, \hat{\Psi}^\dagger(x)\left[-\frac{1}{2}\frac{\partial^2}{\partial x^2}\right]\hat{\Psi}(x)$$



The 1 particle solution with Periodic BC:

$$\phi_j(x,t) = \sum_{p=-\infty}^{\infty}\phi_j^\infty(x+pL,t)$$

is written in terms of the one in infinite space:

Minguzzi and Gangardt, Phys. Rev. Lett. **94**, 240404 (2005)

$$\phi_j^\infty(x,t) = \frac{1}{\sqrt{1+i\omega t}}\left(\frac{1-i\omega t}{1+i\omega t}\right)^{j/2} \mathrm{e}^{-i\frac{t\omega^2x^2}{2(1+\omega^2t^2)}}\chi_j\left(\frac{x}{\sqrt{1+\omega^2t^2}}\right)$$

TWO-POINT FERMIONIC CORRELATORS

$$C_F(x,y;t) = \langle\hat{\Psi}^\dagger(x)\hat{\Psi}(y)\rangle_t = \sum_{j=0}^{N-1}\phi_j^*(x,t)\phi_j(y,t)$$

Density profile & the TD limit

In terms of the TD limit of the particle density at initial time

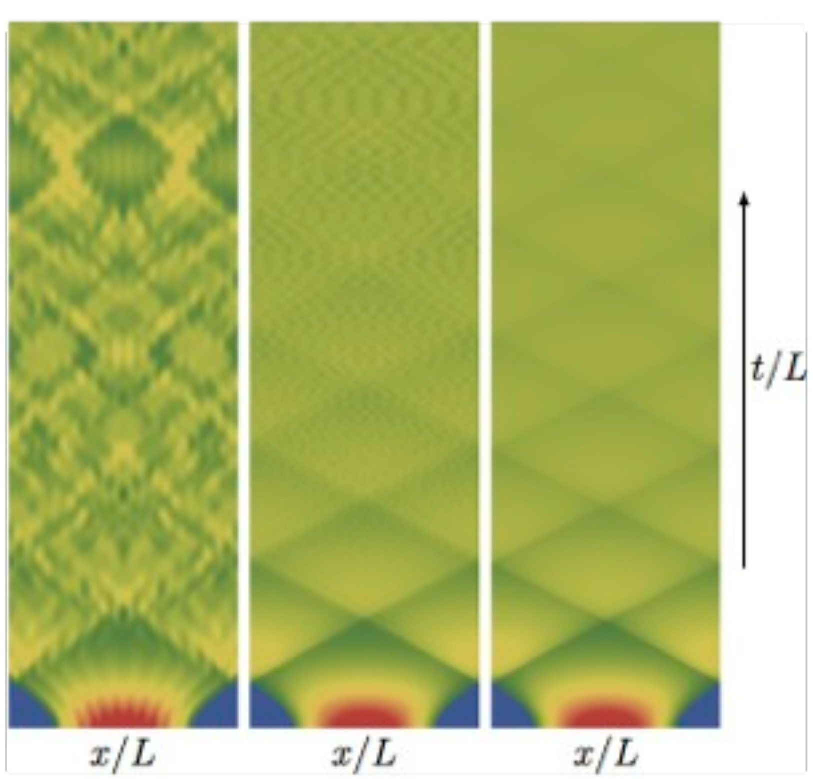
$$n_0(x) = \sqrt{2N\omega - \omega^2 x^2}/\pi$$

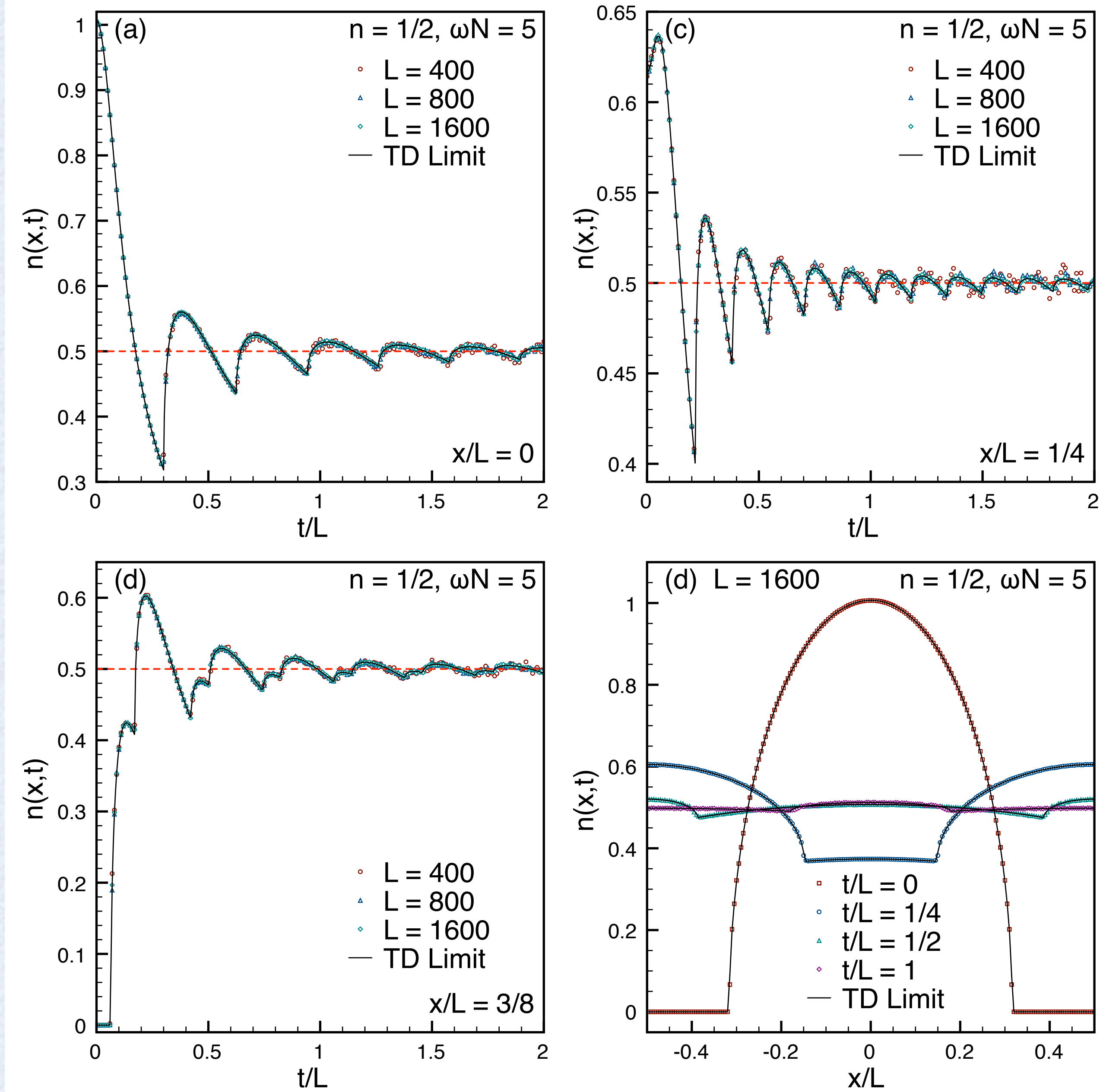
one has the **TD behavior:**

$$n(x,t) = \frac{1}{\sqrt{1+\omega^2 t^2}} \sum_{p=-\infty}^{\infty} n_0\left(\frac{x+qL}{\sqrt{1+\omega^2 t^2}}\right)$$

n = N/L = 1/2, ωN = 5

N = 10 N = 100 N = ∞





Numerical evidence it approaches to the TD Limit as N and L increase

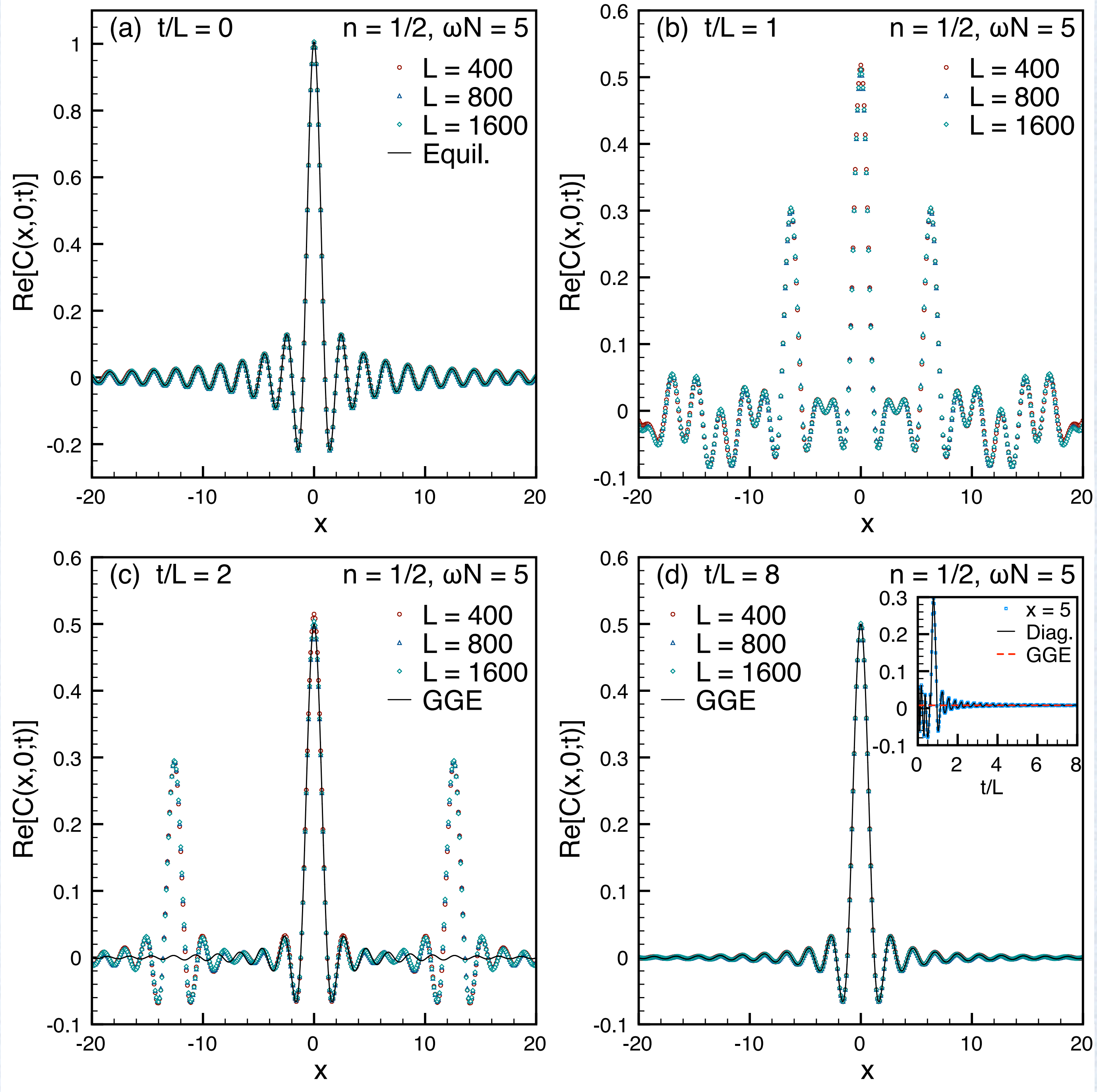
AGAIN THE GGE

In the **TD** and **large-time** limits the **traslational invariance is recovered** and the fermionic correlation function is:

$$C_F(x,y;t\to\infty) = 2n\frac{J_1\left[\sqrt{2\omega N}(x-y)\right]}{\sqrt{2\omega N}(x-y)}$$

Again this can be understood in terms of a GGE

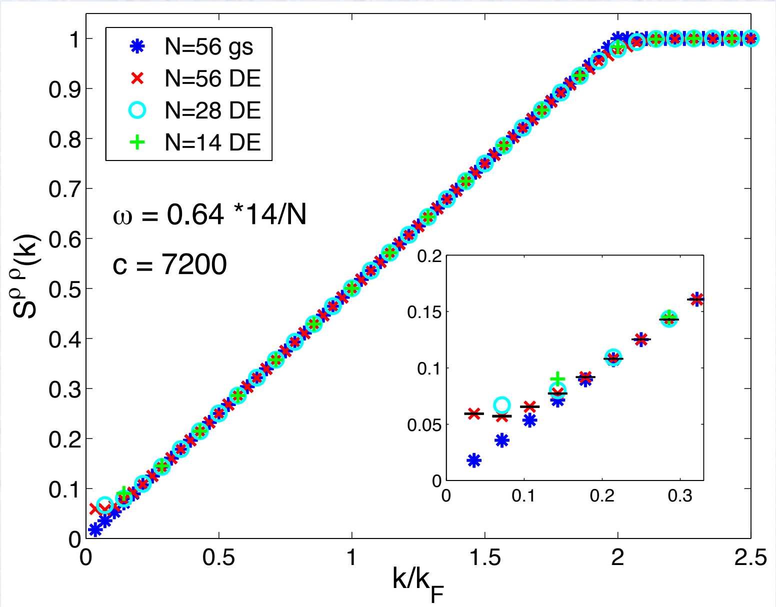
Via Wick theorem all local observables are GGE!



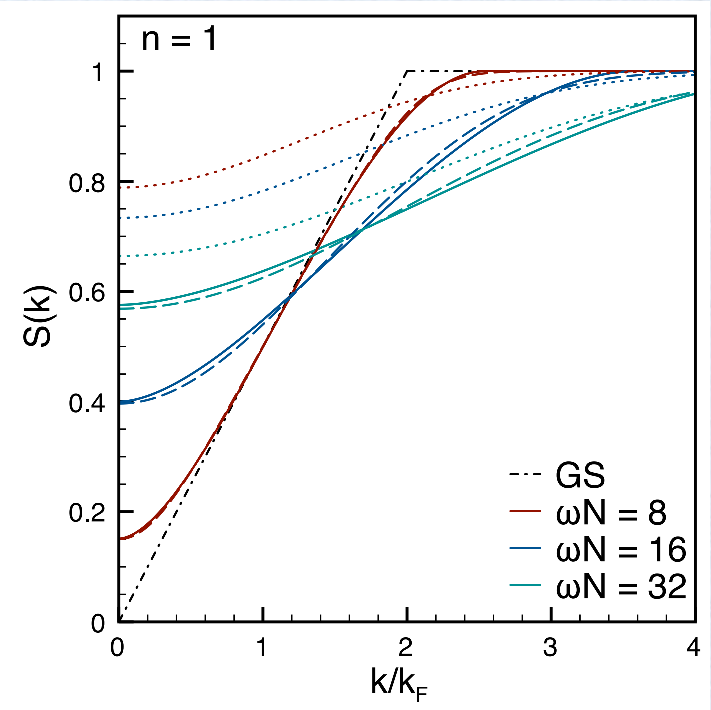
OBSERVABLE WITHIN THE GGE

DENSITY-DENSITY CORRELATION

(In Fourier space)



from J.S. Caux and R.M. Konik, Phys. Rev. Lett. **109**,



BOSONIC CORRELATORS

$$C_B(x,y;t\to\infty) = C_F(x,y;t\to\infty)\mathrm{e}^{-2n|x-y|} = 2n\frac{J_1\left[\sqrt{2\omega N}(x-y)\right]}{\sqrt{2\omega N}(x-y)}\mathrm{e}^{-2n|x-y|}$$

RÈNYI ENTANGLEMENT ENTROPIES

$$S_\alpha(\ell;t\to\infty) = \frac{\ln \mathrm{Tr}\rho^\alpha_{[\ell;t\to\infty]}}{1-\alpha} = \frac{N}{1-\alpha}\ln\left[\left(\frac{\ell}{L}\right)^\alpha + \left(1-\frac{\ell}{L}\right)^\alpha\right]$$

Conclusions

- Quantum quenches display a rich phenomenology
- Here, only a small portion of it!
- Ideal candidates for novel phases of matter
- Many open problems

Thank you for your attention