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MATHEMATICAL PHYSICS

SEMESTER 2

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Condensed Matter Theory

Interactions, Magnetism and Superconductivity

MP473

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Time allowed: $1\frac{1}{2}$ hours

Answer ALL questions

1. N non-relativistic non-interacting spinless bosons of mass m are confined to a square pipe with walls at $x = 0$ and $x = L$ and at $y = 0$ and $y = L$. They also experience a harmonic oscillator potential $V(z) = \frac{1}{2}m\omega^2 z^2$ in the z-direction.

   (a) Show that the density of states for this system is $g(\epsilon) \approx \frac{2\pi^2 mL^2}{3h^3\omega}\epsilon$.
   You may assume that $\epsilon \gg \hbar\omega$ and $\epsilon \gg \frac{h^2}{2mL^2}$ **[15 marks]**
   $g(E)$ is defined so that the number of states with energies between E and $E + dE$ is equal to $g(E)dE$. Here, we have $E_{\vec{n}} = \frac{h^2(n_x^2+n_y^2)}{2mL^2} + \hbar\omega(n_z + \frac{1}{2})$, where n_x, n_y and n_z are positive integers. Let $f(E)$ be the number of states with energy less than or equal to E. $f(E)$ is equal to the number of points with positive integer coordinates (n_x, n_y, n_z) which lie inside a right circular cone. The base of this cone is a disk of radius $\sqrt{2mE}L/h$ in the (n_x, n_y)-plane and the apex lies on the n_z axis at $n_z = E/(\hbar\omega) - \frac{1}{2}$. Since there is one point with integer coordinates per unit area in $\vec{n}$-space, $f(E)$ is (approximately) just the volume of the intersection of this cone with the positive quadrant, so one quarter of the volume of the cone. This gives $f(E) = \frac{1}{12}\left(\frac{E}{\hbar\omega} - \frac{1}{2}\right)\pi(2mEL^2/h^2) \approx \frac{\pi^2 mL^2}{3h^3\omega}E^2$. (We are making a small relative error by replacing $E/\hbar\omega - \frac{1}{2}$ by $E/\hbar\omega$ as $E \gg \hbar\omega$.) By definition of g, we have $f(E) = \int_{\epsilon_0}^{E} g(E')dE'$ where ϵ_0 is the single particle ground state energy. Hence $g(E) = \frac{df}{dE}$ and indeed $g(\epsilon) \approx \frac{2\pi^2 mL^2}{3h^3\omega}\epsilon$ as required. Of course we are making an error due to the fact that individual points in $\vec{n}$-space may fall just inside or outside the cone, but this is small (in relative terms) for large enough energies compared to $\hbar\omega$ and $h^2/2mL^2$.

   (b) Argue that this system exhibits a Bose condensation transition and find the critical temperature T_C. **[15 marks]**
   You may use that $\int_0^\infty \frac{x}{e^x-1}\,dx = \frac{\pi^2}{6}$.
   We treat the system in the grand canonical formalism, so we can in principle exchange particles with a particle bath, but the chemical potential μ of this bath is set to make sure that the average total number of particles in the system satisfies the formula

$$N = \langle N \rangle = \sum_s n_{BE}(s) = \sum_s \frac{1}{e^{\beta(\epsilon(s)-\mu)} - 1}.$$

   Here the sum is over all single particle states s of the system. Note that we must have $\mu \leq \epsilon_0 = \frac{1}{2}\hbar\omega$, since otherwise there would be states with $\epsilon < \mu$ and for these states the average number of particles n_{BE} would be negative. In the limit $N \to \infty$, if no single state contains a finite fraction of the number of particles, we can replace the sum by an integral over energies using the density of states and we get

$$N = \langle N \rangle = \int_{\epsilon_0}^{\infty} g(\epsilon)n_{BE}(\epsilon, \mu)\,d\epsilon = D\int_{\epsilon_0}^{\infty} \frac{\epsilon}{e^{\beta(\epsilon-\mu)} - 1}\,d\epsilon,$$

where $D = \frac{2\pi^2 m L^2}{3h^3\omega}$. If Bose condensation occurs, there will be a finite fraction of the number of particles in the single particle ground state and this integral formula breaks down, that is, the equation does not allow a solution for μ with $\mu < \epsilon_0$. The temperature T_C at which the transition occurs will be the temperature for which the equation is satisfied with $\mu = \epsilon_0$ (below this temperature a condensate forms). Setting $\mu = \epsilon_0$ and $T = T_C$, we obtain

$$\begin{aligned} N &\approx D\int_{\epsilon_0}^{\infty} \frac{\epsilon}{e^{(\epsilon-\epsilon_0)/(kT_C)}-1}\,d\epsilon = D(kT_C)^2\int_0^{\infty}\frac{x+\frac{\epsilon_0}{kT_C}}{e^x-1}\,dx \\ &\approx D(kT_C)^2\int_0^{\infty}\frac{x}{e^x-1}\,dx = \frac{\pi^2}{6}Dk^2T_C^2 = \frac{\pi^4 m L^2 k^2}{9h^3\omega}T_C^2 \end{aligned}$$

The second equality uses substitution of $x = (\epsilon-\epsilon_0)/(kT_C)$. Going to the second line can be done by using $kT_c \gg \epsilon_0$ or in a number of other ways. For example changing ϵ to $\epsilon - \epsilon_0$ in $g(\epsilon)$ clearly improves $g(\epsilon)$ for small ϵ and makes little difference for $\epsilon \gg \epsilon_0$. We now find $T_C = \frac{3h}{\pi^2 k}\sqrt{\frac{h\omega}{m}}\sqrt{N/L^2}$.

(c) Show that, for temperatures $T < T_C$, the energy $E(T)$ of the system satisfies $E(T) = \frac{N_0}{N}E_0 + C(N-N_0)T$.
Here, E_0 is the energy at $T=0$, N_0 is the number of condensed particles and C is a constant, independent of T, L, m and ω. **[15 marks]**

The energy of the system equals the energy of the N_0 condensed particles, which is $N_0\epsilon_0 = \frac{N_0}{N}N\epsilon_0 = \frac{N_0}{N}E_0$, plus the energy of particles in excited states. The number of particles in states at higher energies, $N - N_0$, is given by a very similar integral to that performed in part **(b)**,

$$N - N_0 = D\int_{\epsilon_0}^{\infty}\frac{\epsilon}{e^{(\epsilon-\epsilon_0)/(kT)}-1}\,d\epsilon \approx D(kT)^2\int_0^{\infty}\frac{x}{e^x-1}\,dx = \frac{\pi^2}{6}D(kT)^2.$$

Here we set $\mu = \epsilon_0$ which is justified since for $T < T_C$, the occupation of the ground state is of order N and we must have $\frac{1}{e^{\beta(\epsilon_0-\mu)}-1} \sim N$ so $e^{\beta(\epsilon_0-\mu)} - 1 \sim \frac{1}{N}$ and $\epsilon_0 - \mu \sim \frac{1}{\beta N}$) which is small at large N. We see that $N - N_0 = N(\frac{T}{T_C})^2$ and hence $N_0 = N(1-(\frac{T}{T_C})^2)$. For the energy in the excited states we write similarly that

$$E_{exc} = D\int_{\epsilon_0}^{\infty}\frac{\epsilon^2}{e^{(\epsilon-\epsilon_0)/(kT)}-1}\,d\epsilon \approx D(kT)^3\int_0^{\infty}\frac{x^2}{e^x-1}\,dx = 2\zeta(3)D(kT)^3.$$

Here $2\zeta(3)$ is the value of the integral, but the only important thing here is to realize that this is a constant. We now see that $E_{exc} = \frac{12\zeta(3)}{\pi^2}(N-N_0)kT$ and we have the desired result, with $C = \frac{12\zeta(3)}{\pi^2}$

2. A large object moves through a fluid at a nonrelativistic velocity $\vec{v}$. The motion of the object excites a quantum excitation of the fluid, with energy E_{exc} and momentum $\vec{p}_{exc}$. Total energy and momentum are conserved in this process. The final velocity of the object is $\vec{v'}$ and you may assume that $|\vec{v}-\vec{v'}| \ll |\vec{v}|$.

   (a) Show that we must have $|\vec{v}| \geq \frac{E_{exc}}{|\vec{p}_{exc}|}$ **[10 marks]**

   Conservation of energy and momentum give $m\vec{v} = m\vec{v'} + \vec{p}_{exc}$ (or $\vec{p}_{exc} = m(\vec{v}-\vec{v'})$) and $\frac{1}{2}m|\vec{v}|^2 = \frac{1}{2}m|\vec{v'}|^2 + E_{exc}$. We then have that

$$\begin{aligned} E_{exc} &= \frac{1}{2}m|\vec{v'}|^2 - \frac{1}{2}m|\vec{v}|^2 = \frac{1}{2}m\left(|\vec{v} + (\vec{v'}-\vec{v})|^2 - |\vec{v}|^2\right) \\ &= \frac{1}{2}m\left(|\vec{v}|^2 + 2\vec{v'}\cdot(\vec{v'}-\vec{v}) + O(|(\vec{v}-\vec{v'})|^2) - |\vec{v}|^2\right) \\ &\approx \frac{1}{2}m\left(2\vec{v'}\cdot(\vec{v'}-\vec{v})\right) \approx -\frac{1}{2}m\left(2\vec{v}\cdot\frac{\vec{p}_{exc}}{m}\right) = -\vec{v}\cdot\vec{p}_{exc} \leq |\vec{v}||\vec{p}_{exc}| \end{aligned}$$

   and hence, indeed $|\vec{v}| \geq \frac{E_{exc}}{|\vec{p}_{exc}|}$.

   (b) Suppose the fluid's dispersion relation satisfies $E_{exc}/|\vec{p}_{exc}| \geq v_c$ for some speed v_c. What conclusions can we draw about the fluid? Particularly about friction between this fluid and macroscopic objects? **[10 marks]**

   The fluid is a superfluid with critical velocity v_c, that is, it will flow past macroscopic objects without any friction at all as long as the relative velocity of fluid and object is less than v_c. The mechanicsm for friction is transfer of energy between the object and fluid and this has to take place through creation of excitations in the fluid. Since this is not possible at speeds below v_c, according to part **(a)**, we find that there is no friction at those speeds. At higher speeds there will be friction.

3. A system of fermions hopping on a one-dimensional lattice is described by the following Hamiltonian

$$H = -t \sum_l \left(c_l c_{l+1}^\dagger - c_l^\dagger c_{l+1} \right) - s \sum_l \left(c_l c_{l+1} - c_l^\dagger c_{l+1}^\dagger \right),$$

Here, the $c_l^\dagger$ and c_l are fermionic creation and annihilation operators at site l. The constants s and t are energies and the sum ranges over all $l \in \mathbb{Z}$.

(a) We can define spin operators in terms of the fermionic creation operators as follows,

$$\sigma_l^z = 2c_l^\dagger c_l - 1 \qquad \sigma_l^x = \left(\prod_{j<l} \sigma_j^z \right) (c_l + c_l^\dagger) \qquad \sigma_l^y = i\sigma_l^z \sigma_l^x$$

Check that the spin operators at different sites commute. Also show that at any fixed site l we have $(\sigma_l^x)^2 = (\sigma_l^y)^2 = (\sigma_l^z)^2 = 1$ as well as the equation $\sigma_l^x \sigma_l^y = i\sigma_l^z$ and its cyclic permutations. **[15 marks]**

For different sites l and m, we see that $[\sigma_l^z, \sigma_m^z] = 0$, since the operators c_l and $c_l^\dagger$ both anticommute with c_m and $c_m^\dagger$. This means that exchanging $c_l^\dagger c_l$ with $c_m^\dagger c_m$ yields an overall plus sign (4 minus signs from the individual exchanges of the creation/annihilation operators). We then see that $[\sigma_l^z, \sigma_m^x] = 0$ if $[\sigma_l^z, c_m + c_m^\dagger] = 0$, since we already know that σ_l^z commutes with σ_j^z for all $j < m$. But $[\sigma_l^z, c_m + c_m^\dagger] = 2[c_l^\dagger c_l, c_m] + 2[c_l^\dagger c_l, c_m^\dagger] = 0$, since $c_m^\dagger$ and c_m both anticommute with both $c_l^\dagger$ and c_l. We then see that $[\sigma_l^z, \sigma_m^y] = [\sigma_l^x, \sigma_m^y] = [\sigma_l^y, \sigma_m^y] = 0$ since $\sigma_m^y = i\sigma_m^z \sigma_m^x$ and we already checked that the σ^x and σ^z operators at different sites commute. For the relations at a particular site l, we first note that $\sigma_l^z = 2n_l - 1 = (-1)^{n_l+1}$ where $n_l \in \{0, 1\}$ is the occupation number at site l. This immediately gives us $(\sigma_l^z)^2 = 1$. We then have $(\sigma_l^x)^2 = (c_l^\dagger + c_l)^2 = \frac{1}{2}\{c_l, c_l\} + \frac{1}{2}\{c_l^\dagger, c_l^\dagger\} + \{c_l, c_l^\dagger\} = 0 + 0 + 1 = 1$. The operator $c_l^\dagger + c_l$ in σ_l^x changes n_l to $n_l \pm 1$ so it changes the action of σ_l^z by a minus sign and we find $\sigma_l^x \sigma_l^z = -\sigma_l^z \sigma_l^x$. This gives $(\sigma_l^y)^2 = -\sigma_l^x \sigma_l^z \sigma_l^x \sigma_l^z = (\sigma_l^x)^2 (\sigma_l^z)^2 = 1$. Finally, we *defined* $\sigma_l^y = i\sigma_l^z \sigma_l^x$ and we can now check that

$$i\sigma_l^x \sigma_l^y = -\sigma_l^x \sigma_l^z \sigma_l^x = \sigma_l^z \quad \text{and} \quad i\sigma_l^y \sigma_l^z = -\sigma_l^z \sigma_l^x \sigma_l^z = \sigma_l^x$$

(b) We now set $s = t$. Show that $H = -t \sum_l \sigma_l^x \sigma_{l+1}^x$ **[10 marks]**

We calculate

$$\begin{aligned}
\sigma_l^x \sigma_{l+1}^x &= (c_l + c_l^\dagger)\sigma_l^z (c_{l+1} + c_{l+1}^\dagger) = (c_l + c_l^\dagger)(2c_l^\dagger c_l - 1)(c_{l+1} + c_{l+1}^\dagger) \\
&= (2c_l c_l^\dagger c_l - c_l - c_l^\dagger)(c_{l+1} + c_{l+1}^\dagger) = (c_l - c_l^\dagger)(c_{l+1} + c_{l+1}^\dagger) \\
&= c_l c_{l+1}^\dagger - c_l^\dagger c_{l+1} + c_l c_{l+1} - c_l^\dagger c_{l+1}^\dagger
\end{aligned}$$

and the result follows directly on comparison with the formula for H given initially (setting $s = t$ in that formula).

(c) Describe the ground state or ground states of the system with $s = t$ and $t > 0$ in the spin language. What is the expectation value of the number of fermions occupying site l in the ground state(s)? **[10 marks]**

The eigenstates of H can be chosen to be simultaneous eigenstates of all σ_l^x. We can denote these eigenstates $|\rightarrow\rangle_l$ and $|\leftarrow\rangle_l$, with eigenvalues ± 1. The system has ferromagnetic interactions. The terms $-t\sigma_l^x \sigma_{l+1}^x$ yield energy contributions $+t$ if the arrows at sites l and $l+1$ point in opposite directions and $-t$ if they point in the same direction. There are two ground states, with all spins pointing in the same direction, either all to the left or all to the right, i.e. $\prod_l |\leftarrow\rangle_l$ and $\prod_l |\rightarrow\rangle_l$. Looking at the fermions, we note that $\langle n_l \rangle = \langle \frac{1}{2}(\sigma_l^z + 1) \rangle$. This equals $\frac{1}{2}$ since the expectation value $\langle \sigma^z \rangle = 0$ in the eigenstates of σ^x. Thus we see that in the ground state at $t = s$, we have a 50% chance of finding a fermion at any site.