OLLSCOIL NA hÉIREANN MÁ NUAD

THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

MATHEMATICAL PHYSICS

SEMESTER 2

2016-2017

Condensed Matter Theory

Interactions, Magnetism and Superconductivity

MP473

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Time allowed: $1\frac{1}{2}$ hours

Answer ALL questions

1. N non-relativistic non-interacting spinless bosons of mass m are confined to a square pipe with walls at $x = 0$ and $x = L$ and at $y = 0$ and $y = L$. They also experience a harmonic oscillator potential $V(z) = \frac{1}{2}m\omega^2 z^2$ in the z-direction.

   (a) Show that the density of states for this system is $g(\epsilon) \approx \frac{2\pi^2 m L^2}{3h^3\omega}\epsilon$.
   You may assume that $\epsilon \gg \hbar\omega$ and $\epsilon \gg \frac{h^2}{2mL^2}$ **[15 marks]**

   (b) Argue that this system exhibits a Bose condensation transition and find the critical temperature T_C. **[15 marks]**
   You may use that $\int_0^\infty \frac{x}{e^x - 1}\,dx = \frac{\pi^2}{6}$.

   (c) Show that, for temperatures $T < T_C$, the energy $E(T)$ of the system satisfies $E(T) = \frac{N_0}{N}E_0 + C(N - N_0)T$.
   Here, E_0 is the energy at $T = 0$, N_0 is the number of condensed particles and C is a constant, independent of T, L, m and ω. **[15 marks]**

2. A large object moves through a fluid at a nonrelativistic velocity $\vec{v}$. The motion of the object excites a quantum excitation of the fluid, with energy E_{exc} and momentum $\vec{p}_{exc}$. Total energy and momentum are conserved in this process. The final velocity of the object is $\vec{v'}$ and you may assume that $|\vec{v} - \vec{v'}| \ll |\vec{v}|$.

   (a) Show that we must have $|\vec{v}| \geq \frac{E_{exc}}{|\vec{p}_{exc}|}$ **[10 marks]**

   (b) Suppose the fluid's dispersion relation satisfies $E_{exc}/|\vec{p}_{exc}| \geq v_c$ for some speed v_c. What conclusions can we draw about the fluid? Particularly about friction between this fluid and macroscopic objects? **[10 marks]**

Question 3 is on the next page

3. A system of fermions hopping on a one-dimensional lattice is described by the following Hamiltonian

$$H = -t \sum_l \left(c_l c_{l+1}^\dagger - c_l^\dagger c_{l+1} \right) - s \sum_l \left(c_l c_{l+1} - c_l^\dagger c_{l+1}^\dagger \right),$$

Here, the $c_l^\dagger$ and c_l are fermionic creation and annihilation operators at site l. The constants s and t are energies and the sum ranges over all $l \in \mathbb{Z}$.

(a) We can define spin operators in terms of the fermionic creation operators as follows,

$$\sigma_l^z = 2c_l^\dagger c_l - 1 \qquad \sigma_l^x = \left(\prod_{j<l} \sigma_j^z \right) (c_l + c_l^\dagger) \qquad \sigma_l^y = i \sigma_l^z \sigma_l^x$$

Check that the spin operators at different sites commute. Also show that at any fixed site l we have $(\sigma_l^x)^2 = (\sigma_l^y)^2 = (\sigma_l^z)^2 = 1$ as well as the equation $\sigma_l^x \sigma_l^y = i\sigma_l^z$ and its cyclic permutations. **[15 marks]**

(b) We now set $s = t$. Show that $H = -t \sum_l \sigma_l^x \sigma_{l+1}^x$ **[10 marks]**

(c) Describe the ground state or ground states of the system with $s = t$ and $t > 0$ in the spin language. What is the expectation value of the number of fermions occupying site l in the ground state(s)? **[10 marks]**