Condensed Matter Theory (MP473) — Assignment 4

Please hand in your solutions no later than Tuesday, March 24, 11:05 pm. If you have questions about this assignment, please ask your lecturer,

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Ex. 4.1 Bose condensates in harmonic traps (old exam question)

N non-relativistic non-interacting spinless bosons of mass m are subject to an anisotropic harmonic oscillator type potential,

$$V(x,y,z) = \frac{1}{2}m\omega(x^2+y^2) + \frac{1}{2}m\omega_z z^2 - \frac{1}{2}\hbar(\omega_z + 2\omega).$$

(The constant last term sets the oscillator's ground state energy to zero.)

(a) Show that, when $\epsilon \gg \hbar\omega$ and $\epsilon \gg \hbar\omega_z$, the density of states for this system is given to good approximation by $g(\epsilon) \approx \frac{\epsilon^2}{2\hbar^3\omega_z\omega^2}$.

(b) In this system, a Bose condensate will form below a certain critical temperature T_0. Find T_0 in terms of N, ω and ω_z.

You may use that, for $\alpha > 1$, $\int_0^\infty \frac{x^{\alpha-1}}{e^x-1}\,dx = \Gamma(\alpha)\zeta(\alpha)$, where Γ and ζ are the Euler Gamma and Riemann zeta functions, see also part (d).

(c) It is given that the momentum space groundstate of a one-dimensional harmonic oscillator with angular frequency ω is $\psi_0(p) = (\pi m\omega\hbar)^{-1/4}\, e^{\frac{-p^2}{2\hbar m\omega}}$.
Calculate the probability to observe a particle in our three-dimensional Bose condensate with momentum $\vec{p} = (p_x, p_y, p_z)$. Compare this probability to the probability of observing a classical nonrelativistic particle with this momentum. Indicate two important observable differences in the behavior of condensed vs. classical particles.

(d) (extra part) The Euler Γ and Riemann ζ functions are defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\Gamma(z) = \int_0^\infty x^{(z-1)} e^{-x}\,dx$$

Prove the integral formula that was given in part **(b)**