Condensed Matter Theory (MP473) Assignment 2

Please hand in your solutions no later than Tuesday, February 25. If you have questions about this assignment, please ask your lecturer,

Joost Slingerland, (`joost-at-thphys-dot-nuim-dot-ie`), Office 1.7D, Mathematical Physics

Ex. 2.1: Creation and annihilation operators

We look at bosonic creation and annihilation operators a_i and fermionic creation and annihilation operators c_i

a. State the canonical commutation and anticommutation relations for the a_i and $a_i^\dagger$ and for the c_i and $c_i^\dagger$.

b. We define $\hat{n}_i = a_i^\dagger a_i$ for bosons and $\hat{n}_i = c_i^\dagger c_i$ for fermions.
Show that $[\hat{n}_i, a_j^\dagger] = \delta_{ij} a_i^\dagger$ for bosons and that $[\hat{n}_i, c_j^\dagger] = \delta_{ij} c_i^\dagger$ for fermions (notice that this is a commutator, not an anticommutator!).
Use only the canonical commutation and anticommutation relations.

We now consider the N-particle states $a_{i_1}^\dagger a_{i_2}^\dagger \ldots a_{i_N}^\dagger |\, 0 \,\rangle$ for bosons and $c_{i_1}^\dagger c_{i_2}^\dagger \ldots c_{i_N}^\dagger |\, 0 \,\rangle$ for fermions (here $|\, 0 \,\rangle$ denotes the vacuum state). Note that some of the indices i_p can be the same for bosons (why not fer fermions?)

c. Show that $\hat{n}_j a_{i_1}^\dagger a_{i_2}^\dagger \ldots a_{i_N}^\dagger |\, 0 \,\rangle = \left(\sum_p \delta_{j,i_p}\right) a_{i_1}^\dagger a_{i_2}^\dagger \ldots a_{i_N}^\dagger |\, 0 \,\rangle$.
Argue similarly that $\hat{n}_j c_{i_1}^\dagger c_{i_2}^\dagger \ldots c_{i_N}^\dagger |\, 0 \,\rangle = \left(\sum_p \delta_{j,i_p}\right) c_{i_1}^\dagger c_{i_2}^\dagger \ldots c_{i_N}^\dagger |\, 0 \,\rangle$.
Finally argue that in fact $\hat{n}_j a_{i_1}^\dagger a_{i_2}^\dagger \ldots a_{i_N}^\dagger |\, 0 \,\rangle = n_j a_{i_1}^\dagger a_{i_2}^\dagger \ldots a_{i_N}^\dagger |\, 0 \,\rangle$.
Where the unhatted n_j is the occupation number of the single particle state ψ_j.
Similarly for fermions, $\hat{n}_j c_{i_1}^\dagger c_{i_2}^\dagger \ldots c_{i_N}^\dagger |\, 0 \,\rangle = n_j c_{i_1}^\dagger c_{i_2}^\dagger \ldots c_{i_N}^\dagger |\, 0 \,\rangle$.
where now $n_j \in \{0, 1\}$

d. Show that the states $c_{i_1}^\dagger c_{i_2}^\dagger \ldots c_{i_N}^\dagger |\, 0 \,\rangle$ for fermions form an orthonormal basis for the Hilbert space. More specifically, show that these states are normalized and that two such states are orthogonal unless they have the same number of particles N and the same indices $i_1, \ldots i_N$ for the occupied single particle states. Again, you can calculate the inner products using only the canonical anticommutation rules.

e. Show that the states for bosons are also orthogonal. They are not necessarily normalized however. Calculate the norms of the bosonic states.
Hint: start by showing that $[a_i, (a_i^\dagger)^n] = n (a_i^\dagger)^{n-1}$

Ex. 2.2: Potentials in terms of creation and annihilation operators

We consider a system of fermions with single particle wave functions ψ_k and corresponding creation and annihilation operators $c_k^\dagger$, c_k. In this system the particles interact through a two particle potential $V(\vec{x}_1, \vec{x}_2)$.

a. The states $c_k^\dagger c_{k'}^\dagger |\,0\,\rangle$, with $k < k'$, form a basis for the space of two-particle states. Find the action of the operator $c_m^\dagger c_{m'}^\dagger c_l\, c_{l'}$ on this basis. More precisely, show that

$$c_m^\dagger c_{m'}^\dagger c_l\, c_{l'} c_k^\dagger c_{k'}^\dagger |\,0\,\rangle = (\delta_{k,l'}\delta_{k',l} - \delta_{k,l}\delta_{k',l'})\, c_m^\dagger c_{m'}^\dagger |\,0\,\rangle$$

b. Show that

$$\hat{V} c_k^\dagger c_{k'}^\dagger |\,0\,\rangle = \frac{1}{2} \sum_{l,l',m,m'} V_{m',m,l,l'} c_m^\dagger c_{m'}^\dagger c_l\, c_{l'} c_k^\dagger c_{k'}^\dagger |\,0\,\rangle$$

where

$$V_{m',m,l,l'} = \int d\vec{x}_1\, d\vec{x}_2 \bar{\psi}_{m'}(\vec{x}_1)\bar{\psi}_m(\vec{x}_2) V(\vec{x}_1, \vec{x}_2)\psi_l(\vec{x}_1)\psi_{l'} m(\vec{x}_2)$$

c. Now let the ψ_k be momentum eigenfunctions. To keep things simple, we consider the case, where the particles live in one dimension, on a line segment of length L, with periodic boundary conditions, so $\psi_k(x + L) = \psi_k(x)$. In this case we can write $\psi_k(x) = \frac{1}{\sqrt{L}} e^{ikx}$, with $k = \frac{2\pi n}{L}$ for some integer n. We now assume that $V(x_1, x_2)$ depends only on the difference $x_1 - x_2$ (in fact it will usually depend only on the absolute value of this difference). More concretely, we write $V(x_1, x_2) = \mathcal{V}(x_1 - x_2)$. We require that $\mathcal{V}(x + L) = \mathcal{V}(x)$. Show that in this case,

$$V_{m',m,l,l'} = \delta_{m+m',l+l'} \frac{1}{L} \int_0^L \mathcal{V}(x) e^{i(m-l')x}\, dx$$

d. For extra points (and kudos!) repeat parts **a.** and **b.** of this exercise for states with more than 2 particles. In other words, show that we actually have

$$\hat{V} = \frac{1}{2} \sum_{l,l',m,m'} V_{m',m,l,l'} c_m^\dagger c_{m'}^\dagger c_l\, c_{l'}$$

on the entire Fock space.