Quantum state tomography of a qubit

We choose the following observables, i.e. self-adjoint operators representing measurable physical quantities,

$$\frac{I}{\sqrt{2}}, \frac{X}{\sqrt{2}}, \frac{Y}{\sqrt{2}}, \frac{Z}{\sqrt{2}}.$$

These operators form an orthonormal set, or a basis, with respect to the Hilbert-Schmidt inner product defined as

$$(A, B) = \mathrm{tr}\,(A^{\dagger} B).$$

Since the Pauli operators are traceless and satisfy

$$\begin{array}{lll} XY = iZ & YZ = iX & ZX = iY \\ YX = -iZ & ZY = -iX & XZ = -iY \end{array}$$

we can explicitly verify that the operators above form an orthonormal set

$$\begin{aligned} \left(X/\sqrt{2}, Y/\sqrt{2}\right) &= \frac{1}{2}\mathrm{tr}\,(X^\dagger Y) = \frac{1}{2}\mathrm{tr}\,(XY) = \frac{i}{2}\mathrm{tr}\,Z = 0, \qquad \left(Y/\sqrt{2}, X/\sqrt{2}\right) = 0, \\ \left(Y/\sqrt{2}, Z/\sqrt{2}\right) &= \frac{i}{2}\mathrm{tr}\,X = 0, \qquad \left(Z/\sqrt{2}, Y/\sqrt{2}\right) = 0, \\ \left(Z/\sqrt{2}, X/\sqrt{2}\right) &= \frac{i}{2}\mathrm{tr}\,Y = 0, \qquad \left(X/\sqrt{2}, Z/\sqrt{2}\right) = 0. \end{aligned}$$

and also

$$\left(X/\sqrt{2}, X/\sqrt{2}\right) = \frac{1}{2}\mathrm{tr}\,(X^2) = \frac{1}{2}\mathrm{tr}\,(I) = 1, \qquad \left(Y/\sqrt{2}, Y/\sqrt{2}\right) = 1, \qquad \left(Z/\sqrt{2}, Z/\sqrt{2}\right) = 1.$$

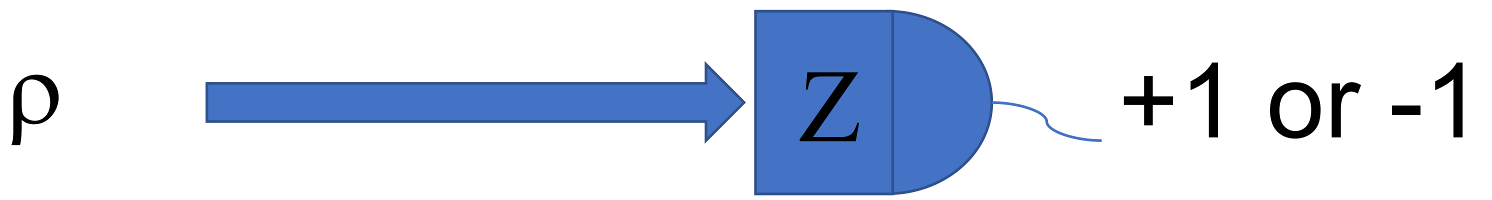
Since the observables form a complete orthonormal set of operators, or a basis, they can be used to expand the density matrix as follows

$$\begin{aligned}\rho &= \left[\mathrm{tr}\left(\frac{I}{\sqrt{2}}\rho\right)\frac{I}{\sqrt{2}} + \mathrm{tr}\left(\frac{X}{\sqrt{2}}\rho\right)\frac{X}{\sqrt{2}} + \mathrm{tr}\left(\frac{Y}{\sqrt{2}}\rho\right)\frac{Y}{\sqrt{2}} + \mathrm{tr}\left(\frac{Z}{\sqrt{2}}\rho\right)\frac{Z}{\sqrt{2}}\right] \\ &= \frac{1}{2}\left[\mathrm{tr}\,(\rho)I + \mathrm{tr}\,(X\rho)X + \mathrm{tr}\,(Y\rho)Y + \mathrm{tr}\,(Z\rho)Z\right]\end{aligned}$$

where $\mathrm{tr}\,(\rho) = 1$ and the quantities $\mathrm{tr}\,(X\rho)$, $\mathrm{tr}\,(Y\rho)$, and $\mathrm{tr}\,(Z\rho)$ have the interpretation of the expectation values, or average value of the observables X, Y and Z for the system in the state ρ, respectively.

To get estimates of these quantities, the measurements of X, Y and Z need to be performed repeatedly on a large number m of equally prepared states ρ. The uncertainty of the result is decreasing as $1/\sqrt{m}$ via the central limit theorem.

The density matrix can be reconstructed from the measurement results.



Repeat m times with the same state ρ and calculate the average

$$\mathrm{Tr}(Z\rho) = \frac{(+1) + (-1) + (-1) + \ldots (-1) + (+1)}{m}$$

Example:

Consider the following results of the repeated measurements of the following physical quantities, that is, the expectation values

$$\mathrm{tr}\,(X\rho) = 0, \qquad \mathrm{tr}\,(Y\rho) = \frac{1}{2}, \qquad \mathrm{tr}\,(Z\rho) = 0$$

The density matrix is then given as

$$\rho = \frac{1}{2}\left(I + \frac{1}{2}\,Y\right) = \begin{pmatrix} \frac{1}{2} & -\frac{i}{4} \\ \frac{i}{4} & \frac{1}{2} \end{pmatrix}.$$

Note that the corresponding Bloch representation has the Bloch vector $(0, 1/2, 0)$.

Bit-flip error in three-qubit error correcting code

We consider a symmetric binary error channel with the probability p of a bit-flip error.

If we encode the initial pure state of one logical qubit into three physical qubits of a bit-flip error-correcting code, we get

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle \quad \rightarrow \quad c_0|000\rangle + c_1|111\rangle.$$

The initial density matrix is then

$$\rho = \left|c_0\right|^2 |000\rangle\langle 000| + c_0 c_1^*|000\rangle\langle 111| + c_0^* c_1|111\rangle\langle 000| + |c_1|^2 |111\rangle\langle 111|.$$

The quantum operation associated with all possible bit-flip errors is then given as

$$\begin{aligned}\mathcal{E}(\rho) &= (1-p)^3\,\rho \\ &+ p(1-p)^2\,X_1\rho X_1 + p(1-p)^2\,X_2\rho X_2 + p(1-p)^2\,X_3\rho X_3 \\ &+ p^2(1-p)\,X_1X_2\rho X_1X_2 + p^2(1-p)\,X_2X_3\rho X_2X_3 + p^2(1-p)\,X_1X_3\rho X_1X_3 \\ &+ p^3\,X_1X_2X_3\rho X_1X_2X_3\end{aligned}$$

where for example $X_1X_2 = X_1 \otimes X_2 \otimes I_3$ with X_1 acting on the first physical qubit, X_2 acting on the second and the identity operator on the third qubit. Also $X = X^\dagger$.

The probabilities for the individual types of errors are then given as follows

no error	$(1-p)^3$
one correctable error on any physical qubit	$3p\,(1-p)^2$
two correlated errors on any two qubits	$3p^2\,(1-p)$
three correlated errors	p^3

The three-qubit bit-flip code can correct only single qubit bit flip errors:

The logical qubit is encoded in a two-dimensional subspace of the three-qubit Hilbert space which has the dimension $2^3 = 8$ and thus has 4 orthogonal two-dimensional subspaces.

Flipping one of the physical qubits causes the qubit state to rotates to another orthogonal two-dimensional subspace. The corrupted state can still be distinguished and transformed back to the original subspace.

	100	011	
000	010	101	111
	001	110	

Considering only the single qubit errors, we can write the relevant quantum operation as

$$
\begin{aligned}
\mathcal{E}(\rho) &= (1-p)^3\,\rho + p(1-p)^2\,X_1\rho X_1 + p(1-p)^2\,X_2\rho X_2 + p(1-p)^2\,X_3\rho X_3 \ldots \\
&= (1-p)^3\left(\left|c_0\right|^2|000\rangle\langle 000| + c_0c_1^*|000\rangle\langle 111| + c_0^*c_1|111\rangle\langle 000| + |c_1|^2\,|111\rangle\langle 111|\right) \\
&+ p(1-p)^2\left(\left|c_0\right|^2|100\rangle\langle 100| + c_0c_1^*|100\rangle\langle 011| + c_0^*c_1|011\rangle\langle 100| + |c_1|^2\,|011\rangle\langle 011|\right) \\
&+ p(1-p)^2\left(\left|c_0\right|^2|010\rangle\langle 010| + c_0c_1^*|010\rangle\langle 101| + c_0^*c_1|101\rangle\langle 010| + |c_1|^2\,|101\rangle\langle 101|\right) \\
&+ p(1-p)^2\left(\left|c_0\right|^2|001\rangle\langle 001| + c_0c_1^*|001\rangle\langle 110| + c_0^*c_1|110\rangle\langle 001| + |c_1|^2\,|110\rangle\langle 110|\right)\ldots
\end{aligned}
$$

To perform the error syndrome measurement, we define four operators, that project the state onto one of the four orthogonal two-dimensional subspaces

$$
\begin{aligned}
P_0 &= |000\rangle\langle 000| + |111\rangle\langle 111| \\
P_1 &= |100\rangle\langle 100| + |011\rangle\langle 011| \\
P_2 &= |010\rangle\langle 010| + |101\rangle\langle 101| \\
P_3 &= |001\rangle\langle 001| + |110\rangle\langle 110|.
\end{aligned}
$$

We can get one of the four measurement outcomes which correspond to the following probabilities and states after the measurement

$$p_0 = (1-p)^3 \quad \rho_0 = P_0\mathcal{E}(\rho)P_0^\dagger/\mathrm{tr}\left(P_0\mathcal{E}(\rho)P_0^\dagger\right) = P_0\mathcal{E}(\rho)P_0/\mathrm{tr}\left(P_0\mathcal{E}(\rho)P_0\right)$$
$$= \left|c_0\right|^2|000\rangle\langle 000| + c_0c_1^*|000\rangle\langle 111| + c_0^*c_1|111\rangle\langle 000| + |c_1|^2|111\rangle\langle 111|$$

$$p(1-p)^2 \quad \rho_1 = P_1\mathcal{E}(\rho)P_1/\mathrm{tr}\left(P_1\mathcal{E}(\rho)P_1\right)$$
$$= \left|c_0\right|^2|100\rangle\langle 100| + c_0c_1^*|100\rangle\langle 011| + c_0^*c_1|011\rangle\langle 100| + |c_1|^2|011\rangle\langle 011|$$

$$p(1-p)^2 \quad \rho_2 = P_2\mathcal{E}(\rho)P_2/\mathrm{tr}\left(P_2\mathcal{E}(\rho)P_2\right)$$
$$= \left|c_0\right|^2|010\rangle\langle 010| + c_0c_1^*|010\rangle\langle 101| + c_0^*c_1|101\rangle\langle 010| + |c_1|^2|101\rangle\langle 101|$$

$$p(1-p)^2 \quad \rho_3 = P_3\mathcal{E}(\rho)P_3/\mathrm{tr}\left(P_3\mathcal{E}(\rho)P_3\right)$$
$$= \left|c_0\right|^2|001\rangle\langle 001| + c_0c_1^*|001\rangle\langle 110| + c_0^*c_1|110\rangle\langle 001| + |c_1|^2|110\rangle\langle 110|$$

Note that the effect of the measurement is that it produced one of the four pure states corresponding to the individual terms of the convex combination of pure states $\mathcal{E}(\rho)$ obtained after the error process took place.

We can now **recover** the original state ρ as follows

$$\begin{aligned}
\rho &= \rho_0 = \left|c_0\right|^2 |000\rangle\langle 000| + c_0 c_1^* |000\rangle\langle 111| + c_0^* c_1 |111\rangle\langle 000| + |c_1|^2 |111\rangle\langle 111| \\
\rho &= X_1 \rho_1 X_1 = \rho_0 \\
\rho &= X_2 \rho_1 X_2 = \rho_0 \\
\rho &= X_3 \rho_1 X_3 = \rho_0.
\end{aligned}$$