Assignment 7: selected solutions

Problem 3:

Show that the amplitude damping defined by the quantum operation $\rho \to E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$ with the operation elements

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \qquad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

transforms the Bloch vector of the state as follows:

$$(r_x, r_y, r_z) \quad \to \quad \left(r_x \sqrt{1-\gamma}, r_y \sqrt{1-\gamma}, \gamma + r_z(1-\gamma)\right).$$

Solution:

$$\begin{aligned}
\mathcal{E}(\rho) &= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}\begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}\right. \\
&\quad \left. + \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix}\right] \\
&= \frac{1}{2}\left[\begin{pmatrix} 1+r_z & \sqrt{1-\gamma}(r_x - ir_y) \\ \sqrt{1-\gamma}(r_x + ir_y) & (1-\gamma)(1-r_z) \end{pmatrix} + \begin{pmatrix} \gamma(1-r_z) & 0 \\ 0 & 0 \end{pmatrix}\right] \\
&= \frac{1}{2}\begin{pmatrix} 1+r_z+\gamma(1-r_z) & \sqrt{1-\gamma}(r_x - ir_y) \\ \sqrt{1-\gamma}(r_x + ir_y) & (1-\gamma)(1-r_z) \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1+[\gamma+(1-\gamma)r_z] & \sqrt{1-\gamma}(r_x - ir_y) \\ \sqrt{1-\gamma}(r_x + ir_y) & 1-[\gamma+(1-\gamma)r_z] \end{pmatrix}
\end{aligned}$$

The Bloch vector of the state $\mathcal{E}(\rho)$ is then $\left(\sqrt{1-\gamma}r_x,\ \sqrt{1-\gamma}r_y, \gamma + (1-\gamma)r_z\right)$.

Problem 5:

The Liouville-von Neumann equation with the Lindblad dissipative superoperator for pure dephasing of a harmonic oscillator is given as

$$d\rho/dt = -(i/\hbar)[\hat{H}, \rho] - \lambda[\hat{H}, [\hat{H}, \rho]]$$

where $\hat{H}$ is a time-independent Hamiltonian. Calculate the evolution operator in the representation given by the energy eigenstates $|n\rangle$ for the components of ρ.

Solution:

The **Schrödinger equation** for a density operator:

We start with the Schrödinger equation for a ket $|\psi\rangle$ and its adjoint $\langle\psi|$

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle = -\frac{i}{\hbar}\, H\,|\psi\rangle \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}t}\langle\psi| = \frac{i}{\hbar}\,\langle\psi|\,H$$

and combine these as follows

$$\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle\langle\psi| = \left(\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle\right)\langle\psi| + |\psi\rangle\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi|\right) \\
&= -\frac{i}{\hbar}\left(H|\psi\rangle\langle\psi| - |\psi\rangle\langle\psi|H\right) = -\frac{i}{\hbar}\left(H\rho - \rho H\right) \\
&= -\frac{i}{\hbar}\,[H,\rho]
\end{aligned}$$

This holds also when the state is mixed, i.e. $\rho = \sum_i p_i\, \rho_i$, as $\frac{\mathrm{d}\rho}{\mathrm{d}t} = \sum_i p_i\, \frac{\mathrm{d}\rho_i}{\mathrm{d}t}$.

Liouville - von Neumann equation

Liouville - von Neumann equation is a generalized Schrödinger equation, a master equation, that describes generally non-unitary dynamics of open quantum systems

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{i}{\hbar}[H, \rho] + \mathcal{L}(\rho)$$

where ρ is the density matrix characterizing the state of the system, H is the Hamiltonian of the system, and $\mathcal{L}(\rho)$ is a dissipative superoperator.

The first term on *r.h.s.* corresponds to a unitary evolution. The corresponding evolution superoperator has in the absence of the dissipative term the following form:

$$\rho(t) = e^{-\frac{i}{\hbar}[H,\ .\ ]t}\,\rho(0).$$

Lindblad dissipative superoperator

Generally $\mathcal{L}(\rho)$ can be given in the form

$$\mathcal{L}(\rho) = \sum_k \lambda_k \left[2L_k \rho L_k^\dagger - \left\{L_k^\dagger L_k, \rho\right\}\right]$$

where $\{x, y\} = xy+yx$ is an anticommutator, and the Lindblad operators L_k are suitably chosen generators of the dissipative dynamics, which represent system-environment interaction, and λ is a given rate constant.

This dissipative superoperator was derived by Göran Lindblad within the axiomatic framework for quantum operations, with the central role being played by the complete positivity axiom.

Harmonic oscillator

The Hamiltonian of a quantum harmonic oscillator

$$H = \hbar\omega\left(a^{\dagger}a + \frac{1}{2}\right) = \hbar\omega\left(n + \frac{1}{2}\right)$$

satisfies the eigenvalue equation

$$H\left|n\right\rangle = E_n\left|n\right\rangle$$

where

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

where $n \geq 0$ is some nonnegative integer.

The density matrix in the representation given by the energy eigenstates is explicitly

$$\rho = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \rho_{mn} \, |m\rangle\langle n|$$

and, since this representation is given by the energy eigenstates, the Hamiltonian acts as

$$\begin{aligned} H\rho &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \rho_{mn} \, H \, |m\rangle\langle n| = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \hbar\omega \left(m + \frac{1}{2} \right) \rho_{mn} \, |m\rangle\langle n| \\ \rho H &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \rho_{mn} \, |m\rangle\langle n| \, H = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \hbar\omega \left(n + \frac{1}{2} \right) \rho_{mn} \, |m\rangle\langle n| \end{aligned}$$

and for the commutator

$$[H, \rho] = H\rho - \rho H = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \hbar\omega \, (m - n) \; \rho_{mn} \, |m\rangle\langle n|.$$

Unitary evolution of the density operator is then given as

$$\rho(t) = e^{-\frac{i}{\hbar}[H,\,.\,]t}\rho(0) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e^{-i\omega(m-n)t}\,\rho_{mn}(0)\,|m\rangle\langle n| = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e^{-i\omega_{mn}t}\,\rho_{mn}(0)\,|m\rangle\langle n|$$

where $\omega_{mn} = \omega(m-n)$.

This is completely equivalent to the expression

$$\begin{aligned}\rho(t) &= U\rho(0)U^{\dagger} = e^{-iHt/\hbar}\rho(0)e^{iHt/\hbar} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\rho_{mn}(0)\,e^{-i\omega\left(m+\frac{1}{2}\right)t}\,|m\rangle\langle n|\,e^{i\omega\left(n+\frac{1}{2}\right)t} \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e^{-i\omega(m-n)t}\rho_{mn}(0)\,|m\rangle\langle n| = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e^{-i\omega_{mn}t}\,\rho_{mn}(0)\,|m\rangle\langle n|.\end{aligned}$$

Pure dephasing may come from fluctuations of the Hamiltonian for example due to inelastic collisions with stray particles and fields.

We therefore choose the Hamiltonian as the Lindblad generator of pure dephasing $L = H$. Since the Hamiltonian is self-adjoint, the Lindblad dissipative superoperator simplifies as follows

$$\mathcal{L}(\rho) = \frac{\lambda}{\hbar^2}\left[2H\rho H^\dagger - \left\{H^\dagger H, \rho\right\}\right] = -\frac{\lambda}{\hbar^2}\left[H, \left[H, \rho\right]\right]$$

The quadratic character of the resulting superoperator characterizes the Gaussian type of dissipative process.

The master equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{i}{\hbar}[H,\rho] - \frac{\lambda}{\hbar^2}[H,[H,\rho]]$$

has, for the time-independent Hamiltonian, the solution

$$\rho(t) = e^{-\left(\frac{i}{\hbar}[H,\,.\,] - \frac{\lambda}{\hbar^2}\,[H,\,[H,\,.\,]]\right)\,t}\rho(0)$$

which can be written as

$$\rho(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e^{-i\,\omega_{mn}\,t\,-\,\lambda\omega_{mn}^2\,t}\,\rho_{mn}(0)\,|m\rangle\langle n|.$$

or for each matrix element separately as

$$\rho_{mn}(t) = e^{-i\,\omega_{mn}\,t\,-\,\lambda\omega_{mn}^2\,t}\,\rho_{mn}(0)$$

where $\omega_{mn} = \omega(m-n)$.

$$\rho_{mn}(t) = e^{-\, i\, \omega_{mn}\, t\, -\, \lambda\, \omega_{mn}^2\, t}\, \rho_{mn}(0)$$

We observe that

- the diagonal elements of ρ, the populations, are constant with time, as $\omega_{nn} = \omega(n-n) = 0$.

- the off-diagonal elements, the coherences, oscillate with time due to coherent part of the dynamics with the rate proportional to $(m-n)$,

- moduli of the coherences decay with time due to dephasing with the rate proportional to $(m-n)^2$.