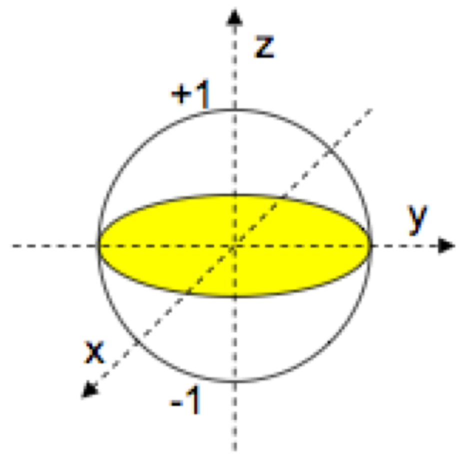
OPEN QUANTUM SYSTEMS
QUANTUM OPERATIONS

Examples of quantum noise and operations

We are now about to consider sending one qubit in a general quantum state characterized by a density matrix ρ in the Bloch representation through various noisy quantum channels characterized by operation elements $\{E_k\}$. These will introduce with some probability certain elementary errors onto the quantum state of the qubit, mapping the state ρ into a new state given by $\mathcal{E}(\rho)$.

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = (1/2) \begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{pmatrix}$$



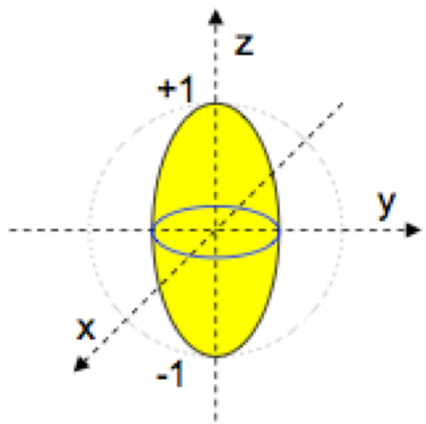
Phase flip error channel

For the initial state is $\rho = \frac{1}{2}(I + \vec{r}.\vec{\sigma})$ and the operation elements $E_0 = \sqrt{1-p}\, I$, $E_1 = \sqrt{p}\, Z$ we get

$$\begin{aligned}
\mathcal{E}(\rho) &= E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger = (1-p)\,\rho + p\, Z\rho Z \\
&= \frac{1}{2}[(1-p)\,(I + \vec{r}.\vec{\sigma}) + p\, Z(I + \vec{r}.\vec{\sigma})Z] \\
&= \frac{1}{2}[(1-p)I + (1-p)(r_x X + r_y Y + r_z Z) + pI + p(r_x ZXZ + r_y ZYZ + r_z ZZZ)] \\
&= \frac{1}{2}[I + (1-p)(r_x X + r_y Y + r_z Z) + p(-r_x X - r_y Y + r_z Z)] \\
&= \frac{1}{2}[I + (1-2p) r_x X + (1-2p) r_y Y + r_z Z]
\end{aligned}$$

Phase flips contracts the state on the Bloch sphere in the x-y direction:

- relative phase between qubit basis states is being lost;
- coherences, off-diagonal elements of ρ, decay;
- populations, diagonal elements of ρ, remain unchanged.



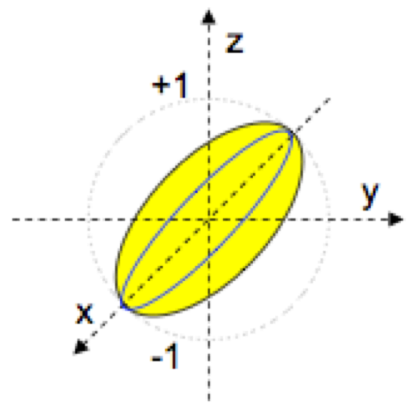
Bit flip error channel

For the initial state is $\rho = \frac{1}{2}(I + \vec{r}.\vec{\sigma})$ and the operation elements $E_0 = \sqrt{1-p}\, I$, $E_1 = \sqrt{p}\, X$ we get

$$\begin{aligned}
\mathcal{E}(\rho) &= E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger = (1-p)\,\rho + p\, X\rho X \\
&= \frac{1}{2}[(1-p)\,(I + \vec{r}.\vec{\sigma}) + p\, X(I + \vec{r}.\vec{\sigma})X] \\
&= \frac{1}{2}[(1-p)I + (1-p)(r_x X + r_y Y + r_z Z) + pI + p(r_x XXX + r_y XYX + r_z XZX)] \\
&= \frac{1}{2}[I + (1-p)(r_x X + r_y Y + r_z Z) + p(r_x X - r_y Y - r_z Z)] \\
&= \frac{1}{2}[I + r_x X + (1-2p) r_y Y + (1-2p) r_z Z]
\end{aligned}$$

Phase flips contracts the state on the Bloch sphere in the y-z direction:

- populations flip randomly under this process while reducing the difference between both populations;
- imaginary part of coherences is being lost.



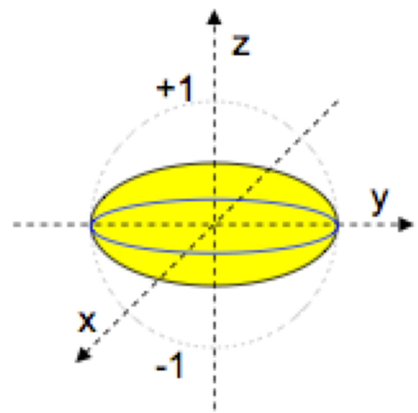
Bit-phase flip error channel

For the initial state is $\rho = \frac{1}{2}(I + \vec{r}.\vec{\sigma})$ and the operation elements $E_0 = \sqrt{1-p}\,I$, $E_1 = \sqrt{p}\,Y$ we get

$$\begin{aligned}
\mathcal{E}(\rho) &= E_0\rho E_0^\dagger + E_1\rho E_1^\dagger = (1-p)\,\rho + p\,Y\rho Y \\
&= \frac{1}{2}[(1-p)\,(I + \vec{r}.\vec{\sigma}) + p\,Y(I + \vec{r}.\vec{\sigma})Y] \\
&= \frac{1}{2}[(1-p)I + (1-p)(r_xX + r_yY + r_zZ) + pI + p(r_xYXY + r_yYYY + r_zYZY)] \\
&= \frac{1}{2}[I + (1-p)(r_xX + r_yY + r_zZ) + p(-r_xX + r_yY - r_zZ)] \\
&= \frac{1}{2}[I + (1-2p)r_xX + r_yY + (1-2p)r_zZ]
\end{aligned}$$

Phase flips contracts the state on the Bloch sphere in the y-z direction:

- both populations and phase flip randomly under this process,
  the difference between both populations is reduced;
- real part of coherences is being lost.



Depolarizing channel

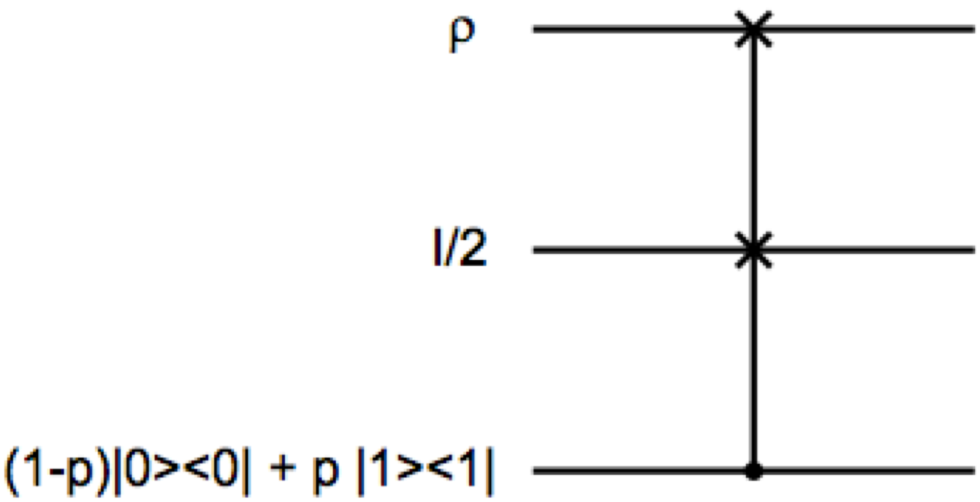
The qubit is with the probability p is depolarized, that is, replaced by the completely mixed state $I/2$:

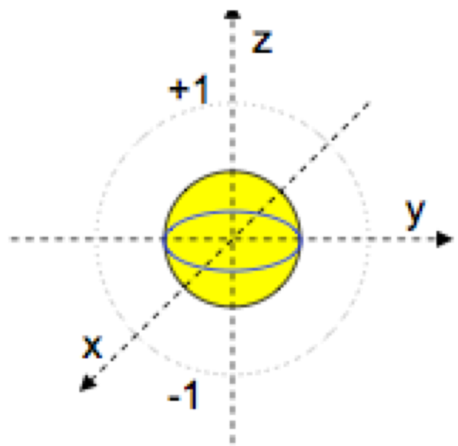
$$\rho \;\rightarrow\; \mathcal{E}(\rho) = (1-p)\,\rho + p\,\frac{I}{2}$$

Note that $\frac{I}{2} = \frac{1}{4}(\rho + X\rho X + Y\rho Y + Z\rho Z)$, then

$$\mathcal{E}(\rho) \;=\; \left(1 - \frac{3p}{4}\right)\rho + \frac{p}{4}(X\rho X + Y\rho Y + Z\rho Z) = (1-q)\rho + \frac{q}{3}(X\rho X + Y\rho Y + Z\rho Z)$$

where $q = 3p/4$.





Amplitude damping

Amplitude damping describes energy dissipation, that is, a loss of energy from the system, like due to spontaneous emission:

$$\rho \quad \rightarrow \quad \mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$$

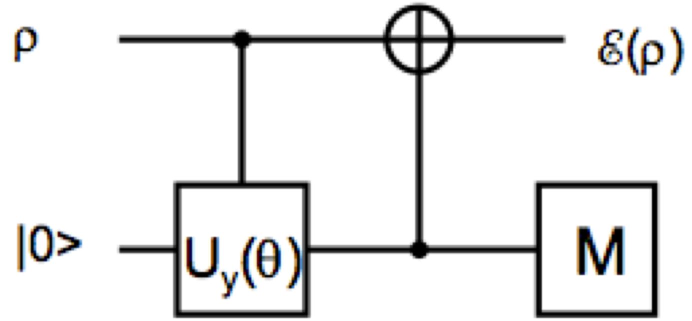
The operation elements are defined as

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \qquad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

where $\gamma = \sin^2\theta$ can be thought as probability of jumping from the state $|1\rangle$ to $|0\rangle$, for example a probability of emitting a photon.

The effect of amplitude damping on the Bloch sphere:

$$(r_x, r_y, r_z) \quad \rightarrow \quad \left(r_x\sqrt{1-\gamma}, r_y\sqrt{1-\gamma}, \gamma + r_z(1-\gamma)\right)$$



Phase damping

This describes loss of quantum information without a loss of energy.

$$\rho \quad \rightarrow \quad \mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$$

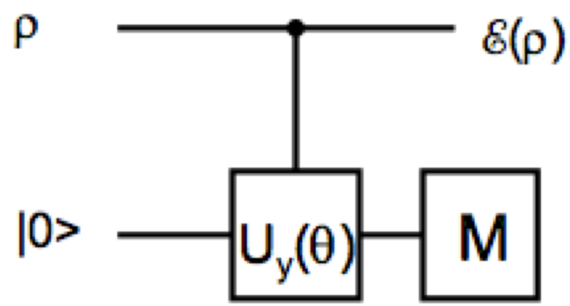
The operation elements are defined as

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix} \qquad E_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}$$

where $\lambda = 1 - \cos^2(\chi \Delta t/2)$ can be thought as probability of a jump of the relative phase, like a probability of elastic photon scattering.

The effect of phase damping on the Bloch sphere:

$$(r_x, r_y, r_z) \quad \rightarrow \quad \left(r_x \sqrt{1-\lambda}, r_y \sqrt{1-\lambda}, r_z\right)$$



Main approaches to dynamics of open quantum systems

1) Master equation approaches

Liouville-von Neumann equation for density matrix dynamics with suitably chosen dissipative superoperator, for example:

- Lindblad superoperator,
- Redfield equations, etc.

2) Stochastic wavefunction dynamics

These techniques are based on the idea that a suitable stochastic process generates quantum trajectories in the Hilbert space. It often requires stochastic integration of the Schrödinger equation using for example:

- Stratonovich approach,
- Ito approach

An average over stochastic quantum trajectories reproduces the dynamics as given by the master equation approaches.

3) Hamiltonian approaches

These techniques are based on full system-environment dynamics with complete or effective description of the environment. They include for example:

- full Hamiltonian dynamics for systems with small environments,
- short-time surrogate Hamiltonian dynamics based on an effective environment model.

Liouville - von Neumann equation

Liouville - von Neumann equation is a generalized Schrödinger equation, a master equation, that describes generally non-unitary dynamics of open quantum systems

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{i}{\hbar}[H, \rho] + \mathcal{L}(\rho)$$

where $\mathcal{L}(\rho)$ is a dissipative superoperator, H is the Hamiltonian of the system, and ρ is the density matrix characterizing the state of the system.

Lindblad dissipative superoperator

Generally $\mathcal{L}(\rho)$ can be given in the form

$$\mathcal{L}(\rho) = \sum_k \lambda_k \left[2L_k \rho L_k^\dagger - \left\{L_k^\dagger L_k, \rho\right\}\right]$$

where $\{x, y\} = xy + yx$ is an anticommutator, and the Lindblad operators L_k are suitably chosen generators of the dissipative dynamics, which represent system-environment interaction, and λ is a given rate constant.

This dissipative superoperator was derived by Lindblad within the axiomatic framework for quantum operations, with the central role being played by the complete positivity axiom.

Example: Pure dephasing of a harmonic oscilator

The Hamiltonian of a quantum harmonic oscillator $H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) = \hbar\omega\left(n + \frac{1}{2}\right)$ satisfies the eigenvalue equation $H|n\rangle = E_n|n\rangle$ where $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ where n is some nonnegative integer.

Pure dephasing may come from fluctuations of the Hamiltonian for example due to inelastic collisions with stray particles and fields. We therefore choose the Hamiltonian as the Lindblad generator of pure dephasing $L = H$. Since the Hamiltonian is self-adjoint, the Lindblad dissipative superoperator simplifies as follows

$$\mathcal{L}(\rho) = \frac{\lambda}{\hbar^2}\left[2H\rho H^\dagger - \left\{H^\dagger H, \rho\right\}\right] = -\frac{\lambda}{\hbar^2}[H, [H, \rho]]$$

The quadratic character of the resulting generator characterizes the Gaussian type of dissipative process.

The master equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{i}{\hbar}[H,\rho] - \frac{\lambda}{\hbar^2}[H,[H,\rho]]$$

has, for the time-independent Hamiltonian, the solution

$$\rho(t) = e^{-\left(\frac{i}{\hbar}[H,\,.] + \frac{\lambda}{\hbar^2}\,[H,\,[H,\,.]]\right)\,t}\rho(0)$$

In the energy representation, the elements of the density matrix are explicitely

$$\rho_{mn}(t) = e^{-i\,\omega_{mn}\,t\,-\,\lambda\,\omega_{mn}^2\,t}\rho_{mn}(0)$$

where $\omega_{mn} = \omega(m-n)$.

$$\rho_{mn}(t) = e^{-\,i\,\omega_{mn}\,t\,-\,\lambda\,\omega_{mn}^2\,t}\,\rho_{mn}(0)$$

We observe that

• the diagonal elements of ρ, the populations, are constant with time,

• the coherences oscillate with time due to coherent part of the dynamics with the rate proportional to $(m-n)$,

• moduli of the coherences decay with time due to dephasing with the rate proportional to $(m-n)^2$.