OPEN QUANTUM SYSTEMS

Open quantum systems

No physical system is *closed* or *isolated*. It interacts with its environment formed by other particles and physical fields.

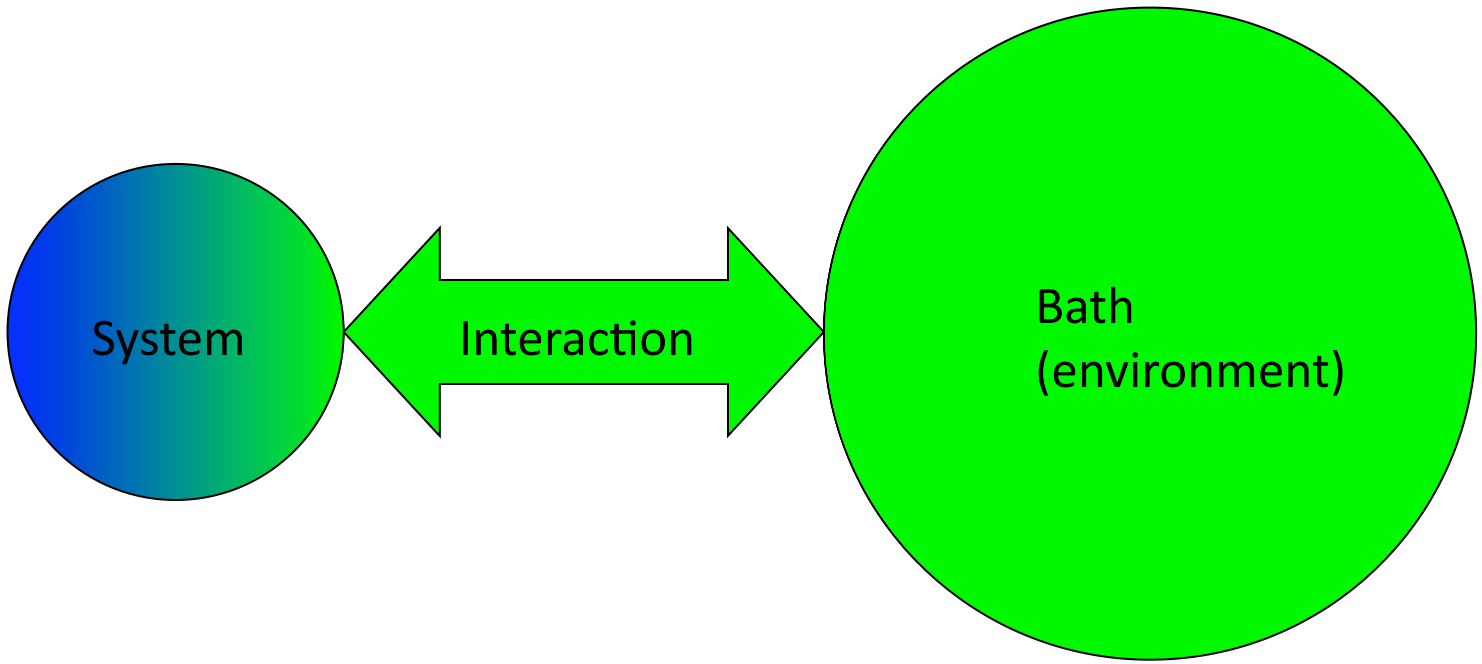
We can consider that the system S and its environment E form a closed universe that evolves under unitary dynamics generated by some Hamiltonian

$$H = H_S + H_E + H_{SE}$$

where H_S is the Hamiltonian of the system only, H_E represents the environment and H_{SE} is the interaction between the system and the environment.

The system only evolves as an open quantum system under a reduced dynamics that is NOT unitary. The effect of the environment appears as noise onto the system's intrinsic dynamics. Quantum states of the system and the environment interact and become entangled. They loose their purity and become mixed.

$$H = H_S + H_{SB} + H_B$$

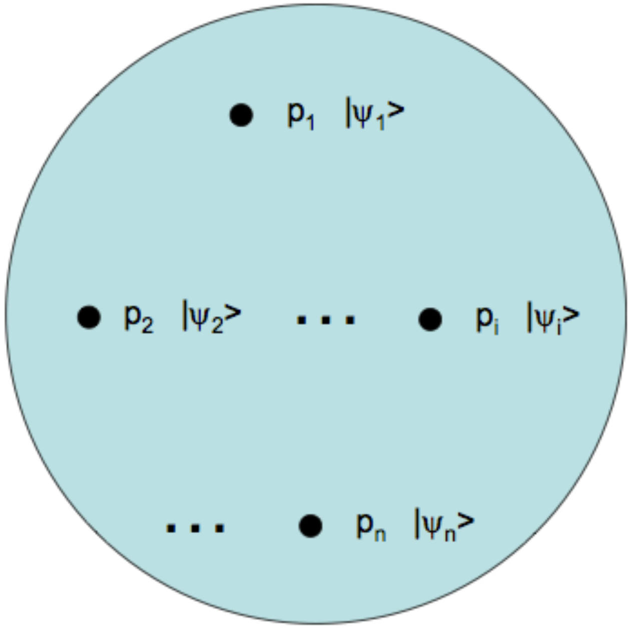


Ensemble of quantum pure states

Suppose a quantum system is in one of a number of pure states $|\psi_i\rangle$ with a probability p_i. We call the set an ensemble of pure states.

The state of the ensemble is described by the density operator

$$\rho = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|$$



Density operator

The postulates of quantum mechanics can be reformulated using the concept of density operator:

Examples:

Quantum evolution of an initial state ρ_0 and action of other quantum operators

$$|\psi(0)\rangle \quad \rightarrow \quad |\psi(t)\rangle = U_t(H)|\psi(0)\rangle = U_t|\psi(0)\rangle$$

$$\rho_0 = \sum_i p_i \, |\psi_i(0)\rangle\langle\psi_i(0)| \quad \rightarrow \quad \rho(t) = U\rho_0 U^\dagger = \sum_i p_i \, U_t \, |\psi_i(0)\rangle\langle\psi_i(0)| \, U_t^\dagger$$

Measurement

- when the measurement described by the operator M_m is performed on the pure state $|\psi_i\rangle$, the result m is obtained with the probability

$$\begin{aligned} p(m|i) &= \langle\psi_i|M_m^\dagger M_m|\psi_i\rangle = \sum_k \langle\psi_i|k\rangle\langle k|M_m^\dagger M_m|\psi_i\rangle = \sum_k \langle k|M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|k\rangle \\ &= \mathrm{tr}\left(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|\right) \end{aligned}$$

- the probability of the result m to be measured on the ensemble described by $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ is then

$$p(m) = \sum_i p_i \,\mathrm{tr}\left(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|\right) = \sum_i p_i \,\mathrm{tr}\left(M_m^\dagger M_m\,\rho_i\right) = \mathrm{tr}\left(M_m^\dagger M_m\,\rho\right)$$

- and the state immediately after the measurement is

$$\rho_m = \frac{M_m\,\rho\,M_m^\dagger}{\mathrm{tr}\left(M_m^\dagger M_m\,\rho\right)}$$

Density operator

An operator $\rho = \sum_i \; p_i \, |\psi_i\rangle\langle\psi_i|$ is the density operator associated to some ensemble $\{p_i, |\psi_i\rangle\}$ iff it satisfies the conditions:

1. **Trace condition**

$$\operatorname{tr} \rho = 1$$

2. **Positivity**

$$\rho \text{ is a positive operator}$$

Proof:

1.

$$\operatorname{tr}\rho = \sum_i p_i \operatorname{tr}|\psi_i\rangle\langle\psi_i| = \sum_i p_i = 1$$

2. Suppose $|\phi\rangle$ is an arbitrary vector in a state space

$$\langle\phi|\rho|\phi\rangle = \sum_i p_i \langle\phi|\psi_i\rangle\langle\psi_i|\phi\rangle = \sum_i p_i \, |\langle\phi|\psi_i\rangle|^2 \geq 0$$

Conversely, suppose ρ is any operator satisfying both conditions above. Since ρ is positive, it must have a spectral decomposition $\rho = \sum_j \lambda_j |j\rangle\langle j|$ where the vectors $|j\rangle$ are orthogonal, and λ_j are nonnegative eigenvalues of ρ. From the trace condition we have $\sum_j \lambda_j = 1$. Therefore a system in a state $|j\rangle$ with the probability λ_j will have the density operator ρ.

Is a quantum state mixed or pure?

Purity $\mathrm{tr}\,\rho^2$

pure states: $\mathrm{tr}\,\rho^2 = \mathrm{tr}\,(|\psi\rangle\langle\psi|\psi\rangle\langle\psi|) = \mathrm{tr}\,(|\psi\rangle\langle\psi|) = 1$

mixed states: $\mathrm{tr}\,\rho^2 < 1$

Von Neumann entropy $S = -\mathrm{tr}\,(\rho \log \rho)$

pure states: $S = 0$

mixed states: $S > 0$

Bloch representation of mixed states

An arbitrary single qubit density matrix

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}$$

can be written as

$$\rho = \frac{1}{2}(1 + \vec{r} \,.\, \vec{\sigma})$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices, and $\vec{r} = (r_x, r_y, r_z)$ is the Bloch vector of the components

$$\begin{aligned} r_x &= 2\, Re\, \rho_{10} \\ r_y &= 2\, Im\, \rho_{10} \\ r_z &= \rho_{00} - \rho_{11} \end{aligned}$$

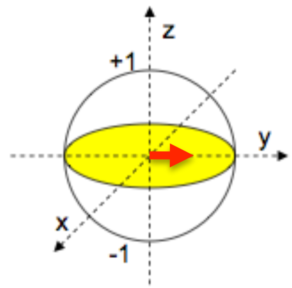
whose length for mixed states is $\|\vec{r}\| = \sqrt{r_x^2 + r_y^2 + r_z^2} < 1$.

Example:
1.

$$\rho = \frac{3}{4}\,|\phi_1\rangle\langle\phi_1| + \frac{1}{4}\,|\phi_2\rangle\langle\phi_2|$$

where $|\phi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$ and $|\phi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$. In the matrix representation, we get

$$\rho = \frac{3}{4}\frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} + \frac{1}{4}\frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -\frac{i}{2} \\ \frac{i}{2} & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 + r_z & r_x - i\,r_y \\ r_x + i\,r_y & 1 - r_z \end{pmatrix}$$

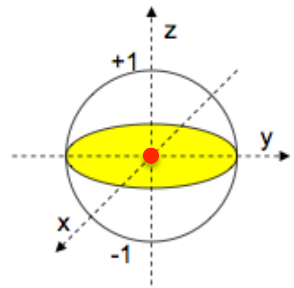


where $\vec{r} = \left(0, \frac{1}{2}, 0\right)$. The length of the Bloch vector is $\|\vec{r}\| = \frac{1}{2}$.

2. Maximally mixed state:

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

the Bloch vector is null: $\vec{r} = (0, 0, 0)$.



Reduced density operator

Suppose we have a physical system A and B whose state is described by the density matrix ρ^{AB}. The reduced density operator for system A is

$$\rho_A = \mathrm{tr}_B\, \rho^{AB}$$

where tr_B is an operator map known as *partial trace* over system B. It is defined as

$$\rho_A = \mathrm{tr}_B\, (|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) = |a_1\rangle\langle a_2| \quad \mathrm{tr}\, (|b_1\rangle\langle b_2|)$$

where $|a_1\rangle$ and $|a_2\rangle$ are any two vectors in A, and $|b_1\rangle$ and $|b_2\rangle$ are any two vectors in B. $\mathrm{tr}\, (|b_1\rangle\langle b_2|)$ is the usual trace, so, using the completeness relation, we get

$$\mathrm{tr}\, (|b_1\rangle\langle b_2|) = \sum_k \langle k|b_1\rangle\langle b_2|k\rangle = \sum_k \langle b_2|k\rangle\langle k|b_1\rangle = \langle b_2| \left(\sum_k |k\rangle\langle k| \right) |b_1\rangle = \langle b_2|b_1\rangle$$

Reduced density operator for independent subsystems

A state of the composite system AB consisting of independent subsystems is a product state described by the density operator

$$\rho^{AB} = \rho \otimes \sigma$$

The reduced density operators for the composite system in a product state is calculated as

$$\begin{aligned} \rho_A &= \mathrm{tr}_B \rho^{AB} = \mathrm{tr}_B (\rho \otimes \sigma) = \rho \, (\mathrm{tr}\, \sigma) = \rho \\ \rho_B &= \mathrm{tr}_A \rho^{AB} = \mathrm{tr}_A (\rho \otimes \sigma) = (\mathrm{tr}\, \rho) \, \sigma = \sigma \end{aligned}$$

Reduced density matrix of entangled subsystems

Example: $\psi = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$$\rho = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)$$

The partial trace over the second qubit is

$$\begin{aligned}
\rho_1 &= \mathrm{tr}_2\left[\frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)\right] \\
&= \frac{1}{2}\left(|0\rangle\langle 0|\langle 0|0\rangle + |0\rangle\langle 1|\langle 1|0\rangle + |1\rangle\langle 0|\langle 0|1\rangle + |1\rangle\langle 1|\langle 1|1\rangle\right) \\
&= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)
\end{aligned}$$

Does the density operator

$$\rho_1 = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{\mathbb{I}}{2}$$

described a mixed state?

$$\operatorname{tr}\rho_1^2 = \operatorname{tr}\left(\frac{\mathbb{I}^2}{4}\right) = \frac{1}{2} < 1$$

The state of the joint system of two qubits ρ above is a pure state, however the state of each qubit individually is mixed.

Reduced density matrix and the Schmidt decomposition

Suppose $|\psi\rangle$ is a pure state of a bipartite composite system AB. The Schmidt decomposition is given as

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle|i_B\rangle$$

where $|i_A\rangle$ and $|i_B\rangle$ form orthonormal sets and the Schmidt coefficients λ_i are real and satisfy $\sum_i \lambda_i^2 = 1$.

The density matrix of the system is

$$\rho = |\psi\rangle\langle\psi| \quad = \quad \sum_i \lambda_i^2 \, |i_A\rangle\langle i_A| \otimes |i_B\rangle\langle i_B|$$

$$\rho = |\psi\rangle\langle\psi| \quad = \quad \sum_i \lambda_i^2 \, |i_A\rangle\langle i_A| \otimes |i_B\rangle\langle i_B|$$

The reduced density matrices are

$$\rho^A = \sum_i \lambda_i^2 \, |i_A\rangle\langle i_A|$$
$$\rho^B = \sum_i \lambda_i^2 \, |i_B\rangle\langle i_B|$$

Note that the eigenvalues of ρ^A and ρ^B are identical and are equal to λ_i^2.