MP469: Mathematical Methods — Part II

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An **ordinary** differential equation involves derivatives of a function $y(x)$ of a single independent variable x. A **linear** ordinary differential equations is one in which $y(x)$ and its derivatives appear linearly, i.e. there are no terms involving y^2, yy' or other non-linear combinations of y and its derivatives. In this course we shall deal exclusively with linear, second order differential equations, that is linear differential equations in which the highest derivative of y is its second derivative, y''.

Examples of second order, linear, ordinary differential equations are (in all of these equations λ is a constant):

- **Harmonic Oscillator Equation**

$$y'' = \lambda y, \qquad x \in (-\infty, \infty) \quad (\text{or some subset} \quad I \subseteq (-\infty, \infty))$$

- **Legendre's Equation**

$$(1 - x^2)y'' - 2xy' = \lambda y, \qquad x \in [-1, 1] \quad (x = \pm 1 \quad \text{are singular points})$$

- **Laguerre's equation**

$$xy'' + (1 - x)y' = \lambda y, \qquad x \in [0, \infty) \quad (x = 0 \quad \text{is a singular point})$$

- **Hermite's equation**

$$y'' - 2xy' = \lambda y \qquad x \in (-\infty, \infty) \quad (\text{or some subset} \quad I \subseteq (-\infty, \infty))$$

- **Bessel's equation**

$$x^2 y'' + xy' + x^2 y = \lambda y \qquad x \in [0, \infty) \quad (\text{or some subset} \quad I \subseteq [0, \infty)).$$

These equation are all of the form

$$\mathcal{L} y = \lambda y \tag{1}$$

where λ is a constant and

$$\mathcal{L} = a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x),$$

with $a_2(x)$, $a_1(x)$ and $a_0(x)$ functions of x. $\mathcal{L}$ is a second-order, linear, differential **operator** because it operates on everything to the right, not just by ordinary multiplication but also by the operation of differentiation. Thus for the example equations:

Harmonic Oscillator:	$a_2(x) = 1,$	$a_1(x) = a_0(x) = 0$	
Legendre:	$a_2(x) = 1 - x^2,$	$a_1(x) = -2x,$	$a_0(x) = 0$
Laguerre:	$a_2(x) = x,$	$a_1(x) = 1 - x,$	$a_0(x) = 0$
Hermite:	$a_2(x) = 1,$	$a_1(x) = -2x,$	$a_0(x) = 0$
Bessel:	$a_2(x) = x^2,$	$a_1(x) = x,$	$a_0(x) = x^2.$

Equations of the form

$$\mathcal{L}y = 0 \tag{2}$$

are called **homogeneous** differential equations. More generally we shall attempt to solve **inhomogeneous** equations of the form

$$\mathcal{L}y(x) = h(x)$$

where the right-hand side is some given function $h(x)$. Equation (1) will be a central to the analysis of both homogeneous and inhomogeneous equation, and we shall start by studying the former, equation (2) .

For each equation the interval I on which the equation is to be solved must be specified — the nature of the possible solutions can be different for different intervals. The operator $\mathcal{L}$ is called **normal** if $a_2(x) \neq 0$ in the range of x, $I \subseteq (-\infty, \infty)$. Points at which $a_2(x)$ vanishes are called **singular points** of the equation. For example Legendre's equation is normal in the open interval $I = (-1, 1)$ but not in the closed interval $I = [-1, 1]$, since $a_2(x) = 1 - x^2$ vanishes at the end points $x = \pm 1$, which are therefore singular points. In the following we shall always be dealing with ordinary, linear, second order equations which are normal in the interval of interest, with the possible exception of the end points at which some equations may not be normal. This will always be assumed unless otherwise stated.

In analogy with linear algebra (1) is called an **eigenvalue** equation, with λ an **eigenvalue** and y an **eigenfunction** of the differential operator $\mathcal{L}$. For a given $\mathcal{L}$ on an interval I there are in general many solutions of (1) with different eigenvalues. Even for a fixed eigenvalue there are many possible solutions but any solution can always be written as linear combination of only two different solutions. The two solutions chosen must be linearly independent in the sense that they are not constant multiples of each other (this is analogous to expressing a vector in two dimensions as a linear combination of two basis vectors — any two basis vectors will do, provided they are not parallel). Roughly speaking there are two independent solutions because solving a second order differential equation requires performing two integrations (to undo the two derivatives) and necessarily introduces two arbitrary integration constants. We need two further pieces of information to fix the two integration constants uniquely. This extra information, is usually given in terms of **boundary conditions**, that is information about the value of the function $y(x)$ and/or its derivatives at the end points of the interval of interest I.

For example if I is the interval $[a, b]$, then boundary conditions are of the form

$$\alpha_1 y(a) + \alpha_2 y(b) + \alpha_3 y'(a) + \alpha_4 y'(b) = \gamma_1$$

$$\beta_1 y(a) + \beta_2 y(b) + \beta_3 y'(a) + \beta_4 y'(b) = \gamma_2$$

where α_i, β_i and γ_i are constants. Common choices of boundary conditions are

1) $\alpha_2 = \alpha_4 = \beta_1 = \beta_3 = 0$, then the two boundary conditions specify the combination $\alpha_1 y(a) + \alpha_3 y'(a) = \gamma_1$ at one endpoint and $\beta_2 y(b) + \beta_4 y'(b) = \gamma_2$ at the other. These are called **unmixed boundary conditions**.
   A specific instance of unmixed boundary conditions is $\alpha_2 = \alpha_3 = \alpha_4 = \beta_1 = \beta_3 = \beta_4 = 0$, then the two boundary conditions specify y at the two endpoints, $\alpha_1 y(a) = \gamma_1$ and $\beta_2 y(b) = \gamma_2$. These are called **Dirichlet boundary conditions**.
   Another specific example of unmixed conditions is $\alpha_1 = \alpha_2 = \alpha_4 = \beta_1 = \beta_2 = \beta_3 = 0$, then the two boundary conditions specify y' at the two endpoints, $\alpha_3 y'(a) = \gamma_1$ and $\beta_4 y'(b) = \gamma_1$. These are called **Neumann boundary conditions**.
2) An important class of boundary conditions is $y(a) = y(b)$ and $y'(a) = y'(b)$, which requires $\alpha_3 = \alpha_4 = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 0$ and $\alpha_1 = -\alpha_2$, $\beta_3 = -\beta_4$. These are called **periodic boundary conditions**.
3) Another important class of boundary conditions is when $y(a)$ and $y'(a)$ are given while $y(b)$ and $y'(b)$ are left free. This corresponds to $\alpha_2 = \alpha_3 = \alpha_4 = 0$ and $\beta_1 = \beta_2 = \beta_4 = 0$ giving $\alpha_1 y(a) = \gamma_1$ and $\beta_3 y'(a) = \gamma_2$. Problems with these boundary conditions are called **initial value problems** and are very important in theoretical physics. An important theorem for such problems states that, with these boundary conditions, there exists a *unique* solution of any second order linear ordinary differential equation in an interval I in which the equation is normal (see Kreider, Kuller Ostberg and Perkins, "*An Introduction to Linear Analysis*", page 104).

Examples

Harmonic Oscillator: $y'' = \lambda y$

We shall take the closed interval $I = [0, \pi]$, and consider three cases in turn:

1) $\lambda > 0$: the general solution is $y(x) = A\mathrm{e}^{\sqrt{\lambda}x} + B\mathrm{e}^{-\sqrt{\lambda}x}$ with A and B constants. If we chose the Dirichlet boundary conditions, $y(0) = 0$, $y(\pi) = 1$ then $y(0) = A + B = 0 \Rightarrow A = -B$ while $y(\pi) = A(\mathrm{e}^{\sqrt{\lambda}\pi} - \mathrm{e}^{-\sqrt{\lambda}\pi}) = 1 \Rightarrow A = 1/(2\sinh(\sqrt{\lambda}\pi))$ and there is a unique solution for any $\lambda > 0$. On the other hand, if we choose as boundary conditions $y(0) = y(\pi) = 0$ then $y(0) = A + B = 0 \Rightarrow A = -B$ and $y(\pi) = A(\mathrm{e}^{\sqrt{\lambda}\pi} - \mathrm{e}^{-\sqrt{\lambda}\pi}) = 0 \Rightarrow A = 0$ and the only solution is the trivial one, $y = 0$.

2) $\lambda < 0$: the general solution is $y(x) = A\cos(\sqrt{-\lambda}x) + B\sin(\sqrt{-\lambda}x)$ with A and B constants. If we chose Dirichlet boundary conditions, $y(0) = 0$, $y(\pi) = 1$ then $y(0) = A = 0$ while $y(\pi) = B\sin(\sqrt{-\lambda}\pi) = 1 \Rightarrow B = 1/\sin(\sqrt{-\lambda}\pi)$ and a solution only exists if $\sqrt{-\lambda}$ is not an integer.
   On the other hand, if we choose as boundary conditions $y(0) = y(\pi) = 0$ then $y(0) = A = 0$ and $y(\pi) = B\sin(\sqrt{-\lambda}\pi) = 0$ and there is a non-trivial solution ($B \neq 0$) only if $\sqrt{-\lambda} = n$ where n is an integer, i.e. $\lambda = -n^2$, in which case B is

arbitrary.

3) $\lambda = 0$: The general solution is $y = A + Bx$. Choosing Dirichlet boundary conditions, $y(0) = 0$, $y(\pi) = 1$, gives $A = 0$ and $B = 1/\pi$, so the unique solution is $y = x/\pi$.

These examples are instructive because they show that the possible values of the eigenvalue λ are affected by the choice of boundary condition.

Legendre's equation: $(1 - x^2)y'' - 2xy' = \lambda y$

In general there are two linearly independent solutions on the interval $[-1, 1]$ for any given λ, but one of them diverges logarithmically at the end points, $x = \pm 1$, which are singular points of the equation and so is not allowed when I includes the end points. If we look for a series solution which terminates at a finite power of x, then the possible eigenvalues are constrained to be $\lambda = -n(n + 1)$, where n is a positive integer (or zero). The eigenfunctions are then the Legendre Polynomials $P_n(x)$ (or a constant multiple of them),

$$(1 - x^2)P_n'' - 2xP_n' + n(n + 1)P_n = 0.$$

By convention the Legendre polynomials are usually normalised so that $P_n(1) = 1$ and the first few are:

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{1}{2}(3x^2 - 1), \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

A general formula (known as Rodrigues' formula) for $P_n(x)$ is

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n$$

(the prefactor $\frac{1}{2^n n!}$ is to conform to the convention that $P_n(1) = 1$). Note that $P_n(x)$ is an even function, $P_n(-x) = P_n(x)$, if n is even and an odd function, $P_n(-x) = -P_n(x)$, if n is odd.

Laguerre's equation: $xy'' + (1 - x)y' = \lambda y$

Again of the two linearly independent solutions one diverges logarithmically, this time at the singular point $x = 0$, so this must be rejected on the interval $I = [0, \infty)$. The other one terminates in a series solution to a finite polynomial in x if and only if $\lambda = -n$, $n = 0, 1, 2, 3, \ldots$ These are the Laguerre polynomials, $L_n(x)$ satisfying

$$xL_n'' + (1 - x)L_n' + nL_n = 0.$$

The first few are:

$$L_0(x) = 1, \qquad L_1(x) = 1 - x, \qquad L_2(x) = \frac{x^2}{2} - 2x + 1,$$

$$L_3(x) = -\frac{x^3}{6} + \frac{3}{2}x^2 - 3x + 1$$

Hermite's equation: $y'' - 2xy' = \lambda y$

There are two linearly independent solutions, both perfectly well behaved on the whole real line $-\infty < x < \infty$. In physics we normally only need one of these, which can be constructed as a series solution which terminates if and only if $\lambda = -2n$, with $n = 0, 1, 2, 3, \ldots$ These are the Hermite polynomials satisfying

$$H_n'' - 2xH_n + 2nH_n = 0.$$

The first few are:

$$H_0(x) = 1, \qquad H_1(x) = 2x, \qquad H_2(x) = 4x^2 - 2, \qquad H_3(x) = 8x^3 - 12x.$$

The other linearly independent solution is not a finite polynomial, but an infinite series.