

MP469: Differential Equations and Complex Analysis

Problem Sheet 3

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(1) Evaluate the real and imaginary parts of the following complex functions, where a is a real constant:

(i) z^3

(ii) $\frac{1}{(z-a)(z+a)}$

(iii) $\sinh z$

(iv) e^{-az}

(v) $\ln z$

(vi) z^a

where a is a real constant.

(2) Show that the complex functions

$$f(z) = e^z \quad \text{and} \quad f(z) = \cos(z)$$

satisfy the Cauchy-Riemann conditions for $|z| < \infty$, and are thus analytic entire functions.

(3) Determine the points of non-analyticity of the functions in question (1).

(4) Develop Taylor expansions of the following functions about the given points, z_0 :

(i) $\frac{1}{(z-i)}, \quad z_0 = 0$

(ii) $\frac{1}{(z-i)}, \quad z_0 = 1$

(iii) $\ln(1+z), \quad z_0 = 0$

Determine the radii of convergence in examples (i) and (ii).

(5) Determine the Laurent expansions of the following functions about the points z_0 , in the regions indicated:

(i) $\frac{1}{(z-i)}, \quad z_0 = i \quad \text{over the whole complex plane except the point } z = i$

(ii) $\frac{1}{(z-1)(z+1)}, \quad z_0 = 1 \quad 0 < |z-1| < 2$

(iii) $\frac{1}{z^2(z-1)}, \quad z_0 = 0 \quad 0 < |z| < 1$

(iv) $\frac{\sin z}{z^2}, \quad z_0 = 0 \quad 0 < |z| < \infty$

(6) Find all the poles of the functions in question (5), determine their order, and calculate their residues.

(7) Assuming that $f(z)$ is analytic within and on the contour C , show that

$$\oint_C \frac{f'(z)dz}{(z - z_0)} = \oint_C \frac{f(z)dz}{(z - z_0)^2}.$$

(8) Show that

$$\oint_C \frac{dz}{(z - a)(z + a)} = 0,$$

where C is a contour which encloses both the points $z = \pm a$, where a is real. Hence deduce that

$$\oint_C \frac{dz}{z^2} = 0$$

for any contour C that encloses the origin.

(9) Determine the nature of the singularities of each of the following functions and evaluate the residues in (i)-(iii) (in all cases ($a > 0$)):

(i) $\frac{1}{z^2 + a^2}$

(ii) $\frac{1}{(z^2 + a^2)^2}$

(iii) $\frac{e^{iz}}{z^2 - a^2}$

(iv) $\frac{\sin(1/z)}{z^2 + a^2}$

(10) Using the calculus of residues, establish the following definite integrals:

(i) $\int_0^{2\pi} \frac{d\theta}{1 + a \cos \theta} = \frac{2\pi}{\sqrt{1 - a^2}}$ where $-1 < a < 1$

(ii) $\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx = \frac{\pi}{a} e^{-a}$ a real and positive

(iii) $\int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$ *Hint:* $\sin^2 x = (1/2)(1 - \cos 2x)$

(iv) $\int_0^{\infty} \frac{x^{a-1}}{1 + x^2} dx = \frac{\pi}{2 \sin(\pi a/2)}$ where $0 < a < 1$.