MP469: Differential Equations and Complex Analysis
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Ordinary Differential Equations

1. Definitions

An ordinary differential equation involves derivatives of a function \( y(x) \) of a single independent variable \( x \). A linear ordinary differential equations is one in which \( y(x) \) and its derivatives appear linearly, i.e. there are no terms involving \( y^2 \), \( yy' \) or other non-linear combinations of \( y \) and its derivatives. In this course we shall deal exclusively with linear, second order differential equations, that is linear differential equations in which the highest derivative of \( y \) is its second derivative, \( y'' \).

Examples of second order, linear, ordinary differential equations are (in all of these equations \( \lambda \) is a constant):

- **Harmonic Oscillator Equation**
  \[ y'' = \lambda y, \quad x \in (-\infty, \infty) \] (or some subset \( I \subseteq (-\infty, \infty) \))

- **Legendre’s Equation**
  \[ (1 - x^2)y'' - 2xy' = \lambda y, \quad x \in [-1, 1] \] (\( x = \pm 1 \) are singular points)

- **Laguerre’s equation**
  \[ xy'' + (1 - x)y' = \lambda y, \quad x \in [0, \infty) \] (\( x = 0 \) is a singular point)

- **Hermite’s equation**
  \[ y'' - 2xy' = \lambda y \quad x \in (-\infty, \infty) \] (or some subset \( I \subseteq (-\infty, \infty) \))

- **Bessel’s equation**
  \[ x^2y'' + xy' + x^2y = \lambda y \quad x \in [0, \infty) \] (or some subset \( I \subseteq [0, \infty) \)).

These equation are all of the form

\[ \mathcal{L}y = \lambda y \] (1)
where $\lambda$ is a constant and

$$L = a_2(x) \frac{d^2}{dx^2} + a_1(x) \frac{d}{dx} + a_0(x),$$

with $a_2(x)$, $a_1(x)$ and $a_0(x)$ functions of $x$. $L$ is a second-order, linear, differential operator because it operates on everything to the right, not just by ordinary multiplication but also by the operation of differentiation. Thus for the example equations:

<table>
<thead>
<tr>
<th>Equation</th>
<th>$a_2(x)$</th>
<th>$a_1(x)$</th>
<th>$a_0(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Harmonic Oscillator:</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>Legendre:</td>
<td>$1 - x^2$</td>
<td>$-2x$</td>
<td>$0$</td>
</tr>
<tr>
<td>Laguerre:</td>
<td>$x$</td>
<td>$1 - x$</td>
<td>$0$</td>
</tr>
<tr>
<td>Hermite:</td>
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<td>$-2x$</td>
<td>$0$</td>
</tr>
<tr>
<td>Bessel:</td>
<td>$x^2$</td>
<td>$x$</td>
<td>$x^2$</td>
</tr>
</tbody>
</table>

Equations of the form

$$Ly = 0$$

are called homogeneous differential equations. More generally we shall attempt to solve inhomogeneous equations of the form

$$Ly(x) = h(x)$$

where the right-hand side is some given function $h(x)$. Equation (1) will be a central to the analysis of both homogeneous and inhomogeneous equations, and we shall start by studying the former, equation (2).

For each equation the interval $I$ on which the equation is to be solved must be specified — the nature of the possible solutions can be different for different intervals. The operator $L$ is called normal if $a_2(x) \neq 0$ in the range of $x$, $I \subseteq (-\infty, \infty)$. Points at which $a_2(x)$ vanishes are called singular points of the equation. For example Legendre’s equation is normal in the open interval $I = (-1, 1)$ but not in the closed interval $I = [-1, 1]$, since $a_2(x) = 1 - x^2$ vanishes at the end points $x = \pm 1$, which are therefore singular points. In the following we shall always be dealing with ordinary, linear, second order equations which are normal in the interval of interest, with the possible exception of the end points at which some equations may not be normal. This will always be assumed unless otherwise stated.

In analogy with linear algebra (1) is called an eigenvalue equation, with $\lambda$ an eigenvalue and $y$ an eigenfunction of the differential operator $L$. For a given $L$ on an interval $I$ there are in general many solutions of (1) with different eigenvalues. Even for a fixed eigenvalue there are many possible solutions but any solution can always be written as linear combination of only two different solutions. The two solutions chosen must be linearly independent in the sense that they are not constant multiples of each other (this is analogous to expressing a vector in two dimensions as a linear combination of two basis vectors — any two basis vectors will do, provided they are not parallel). Roughly speaking there are two independent solutions because solving a second order differential equation
requires performing two integrations (to undo the two derivatives) and necessarily introduces two arbitrary integration constants. We need two further pieces of information to fix the two integration constants uniquely. This extra information is usually given in terms of boundary conditions, that is information about the value of the function \( y(x) \) and/or its derivatives at the end points of the interval of interest \( I \).

For example if \( I \) is the interval \([a, b]\), then boundary conditions are of the form

\[
\alpha_1 y(a) + \alpha_2 y(b) + \alpha_3 y'(a) + \alpha_4 y'(b) = \gamma_1 \\
\beta_1 y(a) + \beta_2 y(b) + \beta_3 y'(a) + \beta_4 y'(b) = \gamma_2
\]

where \( \alpha_i, \beta_i \) and \( \gamma_i \) are constants. Common choices of boundary conditions are

1) \( \alpha_2 = \alpha_4 = \beta_1 = \beta_3 = 0 \), then the two boundary conditions specify the combination \( \alpha_1 y(a) + \alpha_3 y'(a) = \gamma_1 \) at one endpoint and \( \beta_2 y(b) + \beta_4 y'(b) = \gamma_2 \) at the other. These are called **unmixed boundary conditions**.

A specific instance of unmixed boundary conditions is \( \alpha_2 = \alpha_3 = \alpha_4 = \beta_1 = \beta_3 = \beta_4 = 0 \), then the two boundary conditions specify \( y \) at the two endpoints, \( \alpha_1 y(a) = \gamma_1 \) and \( \beta_2 y(b) = \gamma_2 \). These are called **Dirichlet boundary conditions**.

Another specific example of unmixed conditions is \( \alpha_1 = \alpha_2 = \alpha_4 = \beta_1 = \beta_2 = \beta_3 = 0 \), then the two boundary conditions specify \( y' \) at the two endpoints, \( \alpha_3 y'(a) = \gamma_1 \) and \( \beta_4 y'(b) = \gamma_1 \). These are called **Neumann boundary conditions**.

2) An important class of boundary conditions is \( y(a) = y(b) \) and \( y'(a) = y'(b) \), which requires \( \alpha_3 = \alpha_4 = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 0 \) and \( \alpha_1 = -\alpha_2, \beta_3 = -\beta_4 \). These are called **periodic boundary conditions**.

3) Another important class of boundary conditions is when \( y(a) \) and \( y'(a) \) are given while \( y(b) \) and \( y'(b) \) are left free. This corresponds to \( \alpha_2 = \alpha_3 = \alpha_4 = 0 \) and \( \beta_1 = \beta_2 = \beta_4 = 0 \) giving \( \alpha_1 y(a) = \gamma_1 \) and \( \beta_3 y'(a) = \gamma_2 \). Problems with these boundary conditions are called **initial value problems** and are very important in theoretical physics. An important theorem for such problems states that, with these boundary conditions, there exists a unique solution of any second order linear ordinary differential equation in an interval \( I \) in which the equation is normal (see Kreider, Kuller Ostberg and Perkins, “An Introduction to Linear Analysis”, page 104).

**Examples**

**Harmonic Oscillator:** \( y'' = \lambda y \)

We shall take the closed interval \( I = [0, \pi] \), and consider three cases in turn:

1) \( \lambda > 0 \): the general solution is \( y(x) = A e^{\sqrt{\lambda} x} + B e^{-\sqrt{\lambda} x} \) with \( A \) and \( B \) constants.

If we chose the Dirichlet boundary conditions, \( y(0) = 0, y(\pi) = 1 \) then \( y(0) = A + B = 0 \Rightarrow A = -B \) while \( y(\pi) = A (e^{\sqrt{\lambda} \pi} - e^{-\sqrt{\lambda} \pi}) = 1 \Rightarrow A = 1/(2 \sinh(\sqrt{\lambda} \pi)) \) and there is a unique solution for any \( \lambda > 0 \). On the other hand, if we choose as boundary conditions \( y(0) = y(\pi) = 0 \) then \( y(0) = A + B = 0 \Rightarrow A = -B \) and \( y(\pi) = A (e^{\sqrt{\lambda} \pi} - e^{-\sqrt{\lambda} \pi}) = 0 \Rightarrow A = 0 \) and the only solution is the trivial one, \( y = 0 \).
2) $\lambda < 0$: the general solution is $y(x) = A\cos(\sqrt{-\lambda}x) + B\sin(\sqrt{-\lambda}x)$ with $A$ and $B$ constants. If we chose Dirichlet boundary conditions, $y(0) = 0$, $y(\pi) = 1$ then $y(0) = A = 0$ while $y(\pi) = B\sin(\sqrt{-\lambda}\pi) = 1 \Rightarrow B = 1/\sin(\sqrt{-\lambda}\pi)$ and a solution only exists if $\sqrt{-\lambda}$ is not an integer.

On the other hand, if we choose as boundary conditions $y(0) = y(\pi) = 0$ then $y(0) = A = 0$ and $y(\pi) = B\sin(\sqrt{-\lambda}\pi) = 0$ and there is a non-trivial solution ($B \neq 0$) only if $\sqrt{-\lambda} = n$ where $n$ is an integer, i.e. $\lambda = -n^2$, in which case $B$ is arbitrary.

3) $\lambda = 0$: The general solution is $y = A + Bx$. Choosing Dirichlet boundary conditions, $y(0) = 0$, $y(\pi) = 1$, gives $A = 0$ and $B = 1/\pi$, so the unique solution is $y = x/\pi$.

These examples are instructive because they show that the possible values of the eigenvalue $\lambda$ are affected by the choice of boundary condition.

**Legendre’s equation:** $(1 - x^2)y'' - 2xy' = \lambda y$

In general there are two linearly independent solutions on the interval $[-1, 1]$ for any given $\lambda$, but one of them diverges logarithmically at the end points, $x = \pm 1$, which are singular points of the equation and so is not allowed when $I$ includes the end points. If we look for a series solution which terminates at a finite power of $x$, then the possible eigenvalues are constrained to be $\lambda = -n(n+1)$, where $n$ is a positive integer (or zero). The eigenfunctions are then the Legendre Polynomials $P_n(x)$ (or a constant multiple of them),

$$(1 - x^2)P''_n - 2xP'_n + n(n+1)P_n = 0.$$  

By convention the Legendre polynomials are usually normalised so that $P_n(1) = 1$ and the first few are:

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

A general formula (known as Rodrigues’ formula) for $P_n(x)$ is

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$  \hspace{1cm} (3)

(the prefactor $\frac{1}{2^n n!}$ is to conform to the convention that $P_n(1) = 1$). Note that $P_n(x)$ is an even function, $P_n(-x) = P_n(x)$, if $n$ is even and an odd function, $P_n(-x) = -P_n(x)$, if $n$ is odd.
Laguerre’s equation: $xy'' + (1 - x)y' = \lambda y$

Again of the two linearly independent solutions one diverges logarithmically, this time at the singular point $x = 0$, so this must be rejected on the interval $I = [0, \infty)$. The other one terminates in a series solution to a finite polynomial in $x$ if and only if $\lambda = -n$, $n = 0, 1, 2, 3, \ldots$. These are the Laguerre polynomials, $L_n(x)$ satisfying

$$xL_n'' + (1 - x)L_n' + nL_n = 0.$$

The first few are:

$$L_0(x) = 1, \quad L_1(x) = 1 - x, \quad L_2(x) = \frac{x^2}{2} - 2x + 1,$$

$$L_3(x) = -\frac{x^3}{6} + \frac{3}{2}x^2 - 3x + 1$$

Hermite’s equation: $y'' - 2xy' = \lambda y$

There are two linearly independent solutions, both perfectly well behaved on the whole real line $-\infty < x < \infty$. In physics we normally only need one of these, which can be constructed as a series solution which terminates if and only if $\lambda = -2n$, with $n = 0, 1, 2, 3, \ldots$. These are the Hermite polynomials satisfying

$$H_n'' - 2xH_n' + 2nH_n = 0.$$

The first few are:

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \quad H_3(x) = 8x^3 - 12x.$$

The other linearly independent solution is not a finite polynomial, but an infinite series.
2. Linear Independence of Functions

For an a finite dimensional vector space a set of $k$ vectors $\{v_1, \ldots, v_k\}$ is **linearly independent** if the only solution of the equation

$$d_1v_1 + \cdots + d_kv_k = 0$$

is $d_1 = \cdots = d_k = 0$. If this equation has a solution where some or all of the $d_i$ are non-zero then the set of vectors is said to be **linearly dependent**.

Similarly a set of functions $\{y_1(x), \ldots, y_k(x)\}$ is said to be linearly independent on an interval $I$ if the only solution of the equation

$$d_1y_1(x) + \cdots + d_ky_k(x) = 0$$

on the interval is $d_1 = \cdots = d_k = 0$, when $d_1, \ldots, d_k$ are constants. If there exists a set of $d_i$’s which satisfy this equation in which some or all of the $d_i$’s is non zero, then the set of functions $\{y_1(x), \ldots, y_k(x)\}$ is said to be linearly dependent in the interval $I$.

Note that it is important to specify the interval $I$. The two functions $y_1(x) = x^3$ and $y_2(x) = |x|^3$ are linearly independent for $I = (-\infty, \infty)$, but they are linearly dependent for $I = [0, \infty)$ since $x^3 - |x|^3 = 0$ for positive $x$ and $d_1 = -d_2 = 1$ is a non-zero solution for the latter range.

There is a simple criterion for checking to see if a set of $k$ functions is linearly independent on an interval $I$, when the functions are $(k-1)$-times differentiable on that interval. We first derive a set of $k$-equations by differentiating (4):

$$d_1y_1(x) + \cdots + d_ky_k(x) = 0$$

$$d_1y_1^{(1)}(x) + \cdots + d_ky_k^{(1)}(x) = 0$$

$$\vdots$$

$$d_1y_1^{(k-1)}(x) + \cdots + d_ky_k^{(k-1)}(x) = 0$$

where $y_i^{(m)} = \frac{d^m y_i}{dx^m}$ is the $m$-th derivative of $y_i(x)$ (sometimes a prime will be used to denote differentiation; thus $y_i' = y_i^{(1)}$, $y_i'' = y_i^{(2)}$, etc.).

Equations (5) can be re-arranged as a $k \times k$ matrix equation

$$\begin{pmatrix}
y_1 & \cdots & y_k \\
y_1^{(1)} & \cdots & y_k^{(1)} \\
\vdots & \ddots & \vdots \\
y_1^{(k-1)} & \cdots & y_k^{(k-1)}
\end{pmatrix}
\begin{pmatrix}
d_1 \\
d_1^{(1)} \\
\vdots \\
d_k^{(k-1)}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}.$$  \hspace{1cm} (6)

Define a function $W(x)$ on $I$ (called the Wronskian) by $W(x) = \det \begin{pmatrix}
y_1 & \cdots & y_k \\
y_1^{(1)} & \cdots & y_k^{(1)} \\
\vdots & \ddots & \vdots \\
y_1^{(k-1)} & \cdots & y_k^{(k-1)}
\end{pmatrix}$.

If $W(x_0) \neq 0$ for any point $x_0 \in I$ then the inverse matrix exists at this point and we can
multiply both sides of (6) on the left by the inverse to conclude that \(d_1 = \cdots = d_k = 0\). Hence the set of functions \(\{y_1(x), \ldots, y_k(x)\}\) is linearly independent if the Wronskian is non-zero for any point \(x \in I\). Often we deal with the case \(k = 2\), when there are only two functions, when \(W(x) = y_1y_2' - y_1'y_2\).

**Example**

Consider the two functions \(y_1(x) = \cos(x)\) and \(y_2(x) = \sin(x)\) on the interval \(I = [-\pi, \pi]\), (here \(k = 2\)).

\[
W(x) = \text{det} \begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix} = \cos^2(x) + \sin^2(x) = 1
\]

so in this case \(W(x) \neq 0, \forall x \in I\) and we can conclude that the functions \(\cos(x)\) and \(\sin(x)\) are linearly independent on the whole real line \((-\infty, \infty)\).

It is important to note that the converse is not true. If \(W(x) = 0, \forall x \in I\) we cannot conclude that \(\{y_1(x), \ldots, y_k(x)\}\) are necessarily linearly dependent in \(I\). For example the two functions \(y_1(x) = x^3\) and \(y_2(x) = |x|^3\) have \(W(x) = \text{det} \begin{pmatrix} x^3 & |x|^3 \\ 3x^2 & 3x^3/|x| \end{pmatrix} = 0\) on the whole real line, but are linearly independent for \(I = (-\infty, \infty)\) as we have already seen.

We can however tighten the result to make it if and only if by imposing an extra constraint on the functions. For simplicity we shall just consider the case \(k = 2\), which the main case of interest in this course, but the result generalises to higher \(k\).

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**Theorem:** Two solutions of a normal, 2nd order, homogeneous, linear differential equation in an interval \(I\) are linearly independent if and only if the the Wronskian never vanishes on \(I\).

Proof: Let \(y_1(x)\) and \(y_2(x)\) be two solutions of a second order differential equation of the form specified in the statement of the theorem. If \(W(x) \neq 0\) for some \(x \in I\) then \(y_1\) and \(y_2\) are linearly independent as before.

To prove the converse suppose there exists a point \(x_0 \in I\) at which \(W(x_0) = 0\). Then there exists a non-zero solution \(\{d_1, d_2\}\) of the equations

\[
\begin{align*}
d_1y_1(x_0) + d_2y_2(x_0) &= 0 \\
d_1y'_1(x_0) + d_2y'_2(x_0) &= 0
\end{align*}
\]

because \(\text{det} \begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{pmatrix} = 0\). Using this solution define the function \(y(x) = d_1y_1(x) + d_2y_2(x),\) then \(y(x_0) = 0\) and \(y'(x_0) = 0\) by construction. Now we use the assumption of the theorem, that \(y_1\) and \(y_2\) are both solutions of a second order, linear, differential equation which is normal in the interval \(I\). We use \(y(x_0) = y'(x_0) = 0\) as initial conditions to solve the differential equation in the two halves of the interval \(I\) with \(x > x_0\).
and $x < x_0$. We now have an initial value problem and, as stated in the discussion on boundary conditions, the resulting solution is unique. Because the equation is assumed to be homogeneous, one solution with these boundary conditions is $y(x) = 0$, but the boundary conditions render the solution unique, so this is the only solution and we must have $y(x) = d_1y_1(x) + d_2y_2(x) = 0$, $\forall x \in I$. Since $d_1$ and $d_2$ are not both zero we conclude that $y_1(x)$ and $y_2(x)$ are linearly dependent.

We can use a similar result to show that there are at most two linearly independent solutions of any linear homogeneous, second order ordinary differential equation which is normal on an interval $I$. To see this let $y_1(x)$, $y_2(x)$ and $y_3(x)$ be three solutions of the equation

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0. \quad (7)$$

Then form the Wronskian

$$W(x) = \det \begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix} = \det \begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -\frac{1}{a_2}(a_1y_1' + a_0y_1) & -\frac{1}{a_2}(a_1y_2' + a_0y_2) & -\frac{1}{a_2}(a_1y_3' + a_0y_3) \end{pmatrix},$$

where, in the last equality, we have used equation (7) to re-write all the second derivatives in terms of lower derivatives. Now, for each $x$, we can add a multiple $a_0/a_2$ of the first row to the third row and a multiple $a_1/a_2$ times the second row to the last row — it is a property of determinants that this does not change the determinant — hence

$$W(x) = \det \begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ 0 & 0 & 0 \end{pmatrix} = 0$$

and the Wronskian necessarily vanishes for all $x \in I$. Choose one $x_0 \in I$, e.g. the smallest value of $x$ in $I$ if $I$ is closed, and we know that, since the determinant $W(x_0) = 0$ there exist three constants, $d_1$, $d_2$ and $d_3$, not all zero, for which $d_1y_1(x_0) + d_2y_2(x_0) + d_3y_3(x_0) = 0$ and $d_1y_1'(x_0) + d_2y_2'(x_0) + d_3y_3'(x_0) = 0$. So, as in the proof of the previous theorem, the function

$$y(x) = d_1y_1(x) + d_2y_2(x) + d_3y_3(x)$$

is a solution of the differential equation with the boundary conditions $y(x_0) = y'(x_0) = 0$, but the unique solution of the equation with these conditions is $y(x) = 0$. Hence

$$y(x) = d_1y_1(x) + d_2y_2(x) + d_3y_3(x) = 0,$$

with $d_1$, $d_2$ and $d_3$ not all zero, and so $y_1(x)$, $y_2(x)$ and $y_3(x)$ are necessarily linearly dependent. We have thus shown that there can be at most only two linearly independent solutions of any second order, linear, homogeneous, differential equation.

We can go further and say that, if $y_1$ and $y_2$ are two linearly independent solutions of (7), then any solution can be written as a linear combination of $y_1(x)$ and $y_2(x)$. This is a central result because it means that we only need to find two linearly independent solutions in order to find all solutions of the equation, so let’s repeat it with emphasis:
Theorem: If $y_1$ and $y_2$ are two linearly independent solutions of (7), then any solution can be written as a linear combination of $y_1(x)$ and $y_2(x)$, $y(x) = Ay_1(x) + By_2(x)$, with $A$ and $B$ constants.

A Second Solution

We have just seen that, if boundary conditions are not specified, a linear, homogeneous equation can have only two linearly independent solutions. If we know one solution in an interval $I$ in which the equation is normal then the Wronskian can be used to find a second solution. Suppose we know one solution $y_1(x)$, for example a series solution from the Frobenius method, so

$$y''_1 + \frac{a_1}{a_2} y'_1 + \frac{a_0}{a_2} y_1 = 0,$$  \hspace{1cm} (8)

where division by $a_2(x)$ is justified since the equation is normal in $I$. We can find a second linearly independent solution $y_2(x)$ by considering $W(x) = y_1 y'_2 - y'_1 y_2$, differentiating and then using (8):

$$W' = y_1 y''_2 - y'_1 y_2$$

$$= y_1 \left( -\frac{a_1}{a_2} y'_2 - \frac{a_0}{a_2} y_2 \right) - \left( -\frac{a_1}{a_2} y'_1 - \frac{a_0}{a_2} y_1 \right) y_2$$

$$= -\frac{a_1}{a_2} (y_1 y'_2 - y'_1 y_2) = -\frac{a_1}{a_2} W.$$

This gives a first order equation for $W(x)$ which is easily solved

$$\frac{dW(x)}{dx} = -\frac{a_1(x)}{a_2(x)} W(x) \Rightarrow \frac{dW}{W} = -\frac{a_1}{a_2} dx \Rightarrow W(x) = \exp \left( -\int x \frac{a_1(t)}{a_2(t)} dt \right)$$

(provided $W > 0$). Now $W = y_1 y'_2 - y'_1 y_2 = y_1^2 \frac{d}{dx} \left( \frac{y_2}{y_1} \right)$ so this gives $\frac{d}{dx} \left( \frac{y_2}{y_1} \right) = \frac{1}{y_1^2(x)} \exp \left( -\int x \frac{a_1(t)}{a_2(t)} dt \right)$ which can be integrated to

$$y_2(x) = y_1(x) \int x \left\{ \frac{\exp \left( -\int t' \frac{a_1(t)}{a_2(t)} dt \right)}{y_1^2(t')} \right\} dt',$$

provided $y_1(t') \neq 0$ in the region of the $t'$ integration.

We can use Wronskians to prove that there are at most two linearly independent solutions of a second order linear homogeneous ordinary differential equation

$$a_2 y'' + a_1 y' + a_0 y = 0.$$  \hspace{1cm} (9)

Suppose $y_1$, $y_2$ and $y_3$ are three solutions of the same equation,

$$a_2 y''_i + a_1 y'_i + a_0 y_i = 0$$

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with \( i = 1, 2, 3 \). The Wronskian associated with these three functions is

\[
W(x) = \det \begin{pmatrix}
  y_1 & y_2 & y_3 \\
  y'_1 & y'_2 & y'_3 \\
  y''_1 & y''_2 & y''_3
\end{pmatrix} = \det \begin{pmatrix}
  y_1 & y_2 & y_3 \\
  y'_1 & y'_2 & y'_3 \\
  y''_1 & y''_2 & y''_3 \\
  0 & 0 & 0
\end{pmatrix} = 0,
\forall x \in I,
\]

where we have used the fact that the last row is a linear combination of the first two and adding multiples of one row of a matrix to another does not change the determinant. Choose one point \( x_0 \in I \), then since the determinant vanishes at this point, \( W(x_0) = 0 \), and we can find three constants \( d_1, d_2 \) and \( d_3 \), not all zero, for which

\[
d_1 y_1(x_0) + d_2 y_2(x_0) + d_3 y_3(x_0) = 0
\]

and

\[
d_1 y'_1(x_0) + d_2 y'_2(x_0) + d_3 y'_3(x_0) = 0.
\]

Now construct the function \( y(x) = d_1 y_1(x) + d_2 y_2(x) + d_3 y_3(x) \). This satisfies the same equation

\[
a_2 y'' + a_1 y' + a_0 y = 0
\]

with boundary conditions \( y(x_0) = y'(x_0) = 0 \). But these boundary conditions suffice to specify the solution uniquely and \( y(x) = 0 \) also satisfies the differential equation with the same boundary conditions, hence

\[
y(x) = d_1 y_1(x) + d_2 y_2(x) + d_3 y_3(x) = 0
\]

for all \( x \) with \( d_1, d_2 \) and \( d_3 \) not all zero. Hence \( y_1(x) \), \( y_2(x) \) and \( y_3(x) \) must be linearly dependent and there are at most two linearly independent solutions of (9).
3. Orthogonal Function Expansions (Sturm-Liouville Theory)

You are familiar with Fourier series: expansions of functions in an interval \( I = [-L, L] \) in terms of trigonometric functions, \( \sin(\pi x / L) \) and \( \cos(\pi x / L) \). It is often useful to expand a function in terms of Legendre polynomials or some other special function. This can be done because these functions have some important properties that follow from the differential equations that they satisfy. To describe these properties we first give some definitions:

**Definition**

A second order, linear, differential operator \( \tilde{\mathcal{L}} \) defined on an interval \( I = (a, b) \) is said to be in **self-adjoint** form if it is written in the form

\[
\tilde{\mathcal{L}} = \frac{d}{dx} \left( p(x) \frac{d}{dx} \right) + q(x),
\]

where \( p(x) \) is a differentiable function that never vanishes in the interval \( I \) and \( q(x) \) is a continuous function in \( I \).

The definition of a self-adjoint operator is sufficiently general to include all normal, second order, linear differential operators on \( (a, b) \). To see this take such an operator

\[
\mathcal{L} = a_2(x) \frac{d^2}{dx^2} + a_1(x) \frac{d}{dx} + a_0(x)
\]

and multiply it by \( \frac{p(x)}{a_2(x)} \) (this is allowed because it is assumed that \( a_2(x) \neq 0, \forall x \in (a, b) \)). Then choosing \( p(x) \) so that \( p' = \left( \frac{a_1}{a_2} \right) p \) and \( q(x) \) to be \( q = \left( \frac{a_0}{a_2} \right) p \) ensures that

\[ \tilde{\mathcal{L}} := \left( \frac{p(x)}{a_2(x)} \right) \mathcal{L} = p(x) \frac{d^2}{dx^2} + p'(x) \frac{d}{dx} + q(x) = \frac{d}{dx} \left( p(x) \frac{d}{dx} \right) + q(x) \]

is in self-adjoint form. The homogeneous equations \( \mathcal{L}y = 0 \) and \( \tilde{\mathcal{L}}y = 0 \) have exactly the same solutions, solving one is completely equivalent to solving the other.

The function \( p(x) \) can be found from the condition \( p' = \left( \frac{a_1}{a_2} \right) p \),

\[
\frac{p'(x)}{p(x)} = \frac{a_1(x)}{a_2(x)} \Rightarrow p(x) = \exp \left( \int^x \frac{a_1(t)}{a_2(t)} dt \right).
\]
Examples

1) The harmonic oscillator equation is already self-adjoint with \( p(x) = 1 \) and \( q(x) = 0 \).

2) Legendre’s equations is already self-adjoint with \( p(x) = (1 - x^2) \), \( p'(x) = -2x \), and \( q(x) = 0 \) so
   \[
   \frac{d}{dx} \left[ (1 - x^2) \frac{dy}{dx} \right] = \lambda y.
   \]

3) Laguerre’s equation is not self-adjoint but can be made so by using
   \[
   p(x) = \exp \left\{ \int x \left( \frac{1-t}{t} \right) dt \right\} = xe^{-x}, \quad q(x) = 0
   \]
giving
   \[
   xe^{-x}y'' + (1 - x)e^{-x}y' = \lambda e^{-x}y \quad \Rightarrow \quad \frac{d}{dx} \left( xe^{-x} \frac{dy}{dx} \right) = \lambda e^{-x}y.
   \]

4) Hermite’s equation is not self-adjoint but can be made so by using
   \[
   p(x) = \exp \left\{ \int x (-2t)dt \right\} = e^{-x^2}, \quad q(x) = 0
   \]
giving
   \[
   e^{-x^2}y'' - 2xe^{-x^2}y' = \lambda e^{-x^2}y \quad \Rightarrow \quad \frac{d}{dx} \left( e^{-x^2} \frac{dy}{dx} \right) = \lambda e^{-x^2}y.
   \]

In the last two examples we have had to modify the eigenvalue equation in its original form, \( \mathcal{L}y = \lambda y \), to include a function on the right hand side,

\[
\tilde{\mathcal{L}}y = \lambda r(x)y,
\]
where \( r(x) = p(x)/a_2(x) \) is a continuous, non-negative function on \( I \). So for Laguerre’s equation \( r(x) = e^{-x} \) while for Hermite’s equation \( r(x) = e^{-x^2} \). In general \( r(x) \) is called the **weight function** for the differential equation.

The self-adjoint form is extremely important in mathematical physics because their eigenfunctions can be shown to have the following properties:

1) Eigenfunctions corresponding to distinct non-zero eigenvalues are linearly independent
2) Eigenfunctions corresponding to distinct eigenvalues are orthogonal (in the sense defined below).
3) The eigenfunctions of a self-adjoint operator form a complete set, i.e. they can be used to expand functions in the same way as \( \sin(nx) \) and \( \cos(nx) \) are used in Fourier series.
1) Linear Independence

Denote the eigenfunctions of a linear operator by $\phi_i$ and the corresponding eigenvalues by $\lambda_i$, where $i = 1, 2, \ldots$ labels the eigenvalues. In self-adjoint form the eigenvalue equation is

$$\tilde{L}\phi_i(x) = \lambda_i r(x) \phi_i(x).$$

We shall prove that $\{\phi_1, \ldots, \phi_k\}$ are linearly independent if $\{\lambda_1, \ldots, \lambda_k\}$ are a set of $k$ distinct non-zero eigenvalues. The proof is by induction.

i) Obviously a single non-zero eigenfunction $\phi_1$ is linearly independent, since $d_1 \phi_1 = 0$ if and only if $d_1 = 0$.

ii) Assume that eigenfunctions $\{\phi_1, \ldots, \phi_k\}$ with distinct non-zero eigenvalues $\{\lambda_1, \ldots, \lambda_k\}$ are linearly independent, i.e. the only way that

$$d_1 \phi_1 + \cdots + d_k \phi_k = 0 \quad (11)$$

can be satisfied with $d_i$ constants is to have $d_1 = \cdots = d_k = 0$.

iii) Let $\phi_{k+1}$ be another eigenfunction with non-zero eigenvalue, $\lambda_{k+1} \neq \lambda_i$, $1 \leq i \leq k$. We must show that the only way that the equation

$$d_1 \phi_1 + \cdots + d_k \phi_k + d_{k+1} \phi_{k+1} = 0 \quad (12)$$

can be satisfied is to have $d_1 = \cdots = d_k = d_{k+1} = 0$.

Applying $\tilde{L}$ to (12) gives

$$d_1 \tilde{L}\phi_1 + \cdots + d_k \tilde{L}\phi_k + d_{k+1} \tilde{L}\phi_{k+1} = 0$$

because $\tilde{L}$ is linear. Since $\tilde{L}\phi_i = \lambda_i r(x) \phi_i$ this means that

$$d_1 \lambda_1 r \phi_1 + \cdots + d_k \lambda_k r \phi_k + d_{k+1} \lambda_{k+1} r \phi_{k+1} = 0$$

$$\Leftrightarrow d_1 \lambda_1 \phi_1 + \cdots + d_k \lambda_k \phi_k + d_{k+1} \lambda_{k+1} \phi_{k+1} = 0. \quad (13)$$

Multiplying (12) by $\lambda_{k+1}$ and it subtracting the result from (13) gives

$$d_1 (\lambda_1 - \lambda_{k+1}) \phi_1 + \cdots + d_k (\lambda_k - \lambda_{k+1}) \phi_k = 0.$$

Since, by assumption, $\lambda_{k+1} \neq \lambda_i$ and $\phi_i$ are linearly independent for $1 \leq i \leq k$ this can only be true if $d_1 = \cdots = d_k = 0$. Equation (12) then implies that $d_{k+1} = 0$ as well so it must be the case that all of $d_1 = \cdots = d_k = d_{k+1} = 0$ for (12) to be true. Hence $\{\phi_1, \ldots, \phi_k, \phi_{k+1}\}$ are all linearly independent. \hfill $\Box$

2) Orthogonality

Let $\phi_1(x)$ and $\phi_2(x)$ be two eigenfunctions of a self-adjoint operator $\tilde{L}$ on an interval $[a, b]$. Associate with these two eigenfunctions the integral

$$\phi_1 \cdot \phi_2 := \int_a^b \phi_1(x) \phi_2(x) r(x) dx, \quad (14)$$

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where \( r(x) \) is the weight function for \( \widetilde{L} \). Note that \( \phi_1.\phi_2 \) is just a number, it is independent of \( x \). We shall call this number the \textit{inner product} of the two functions \( \phi_1(x) \) and \( \phi_2(x) \) — this is in analogy to the inner product of two vectors. When \( \phi_1.\phi_2 = 0 \) the two functions are said to be \textbf{orthogonal}. We shall show that, if \( \phi_1(x) \) and \( \phi_2(x) \) correspond to distinct eigenvalues \( \lambda_1 \) and \( \lambda_2 \) with \( \lambda_1 \neq \lambda_2 \), then \( \phi_1.\phi_2 = 0 \) (subject to some constraints on the boundary conditions which will be derived below). To see this consider

\[
\lambda_1 \phi_1 \phi_2 - \lambda_2 \phi_1 \phi_2 = \lambda_1 \int_a^b \phi_1(x) \phi_2(x) r(x) dx - \lambda_2 \int_a^b \phi_1(x) \phi_2(x) r(x) dx
\]

\[
= \int_a^b \{ \lambda_1 \phi_1(x) r(x) \} \phi_2(x) dx - \int_a^b \phi_1(x) \{ \lambda_2 \phi_2(x) r(x) \} dx
\]

\[
= \int_a^b \{ \tilde{L} \phi_1(x) \} \phi_2(x) dx - \int_a^b \phi_1(x) \{ \tilde{L} \phi_2(x) \} dx
\]

\[
= \int_a^b \left\{ \frac{d}{dx} \left[ p \frac{d\phi_1}{dx} \right] + q \phi_1 \right\} \phi_2 dx - \int_a^b \phi_1 \left\{ \frac{d}{dx} \left[ p \frac{d\phi_2}{dx} \right] + q \phi_2 \right\} dx.
\]

where \( p(x) \) and \( q(x) \) are the functions appearing in the self-adjoint form of \( \tilde{L} \) in (10). The two terms involving \( q(x) \) cancel and the remaining terms can be integrated by parts to give

\[
\lambda_1 \phi_1 \phi_2 - \lambda_2 \phi_1 \phi_2 = \left[ p \phi_1' \phi_2 \right]_a^b - \int_a^b p \frac{d\phi_1}{dx} \frac{d\phi_2}{dx} dx + \left[ p \phi_1 \phi_2' \right]_a^b - \int_a^b p \phi_1 \frac{d\phi_2}{dx} dx
\]

\[
= \left[ p(\phi_1' \phi_2 - \phi_1 \phi_2') \right]_a^b.
\]

Since \( \lambda_1 \neq \lambda_2 \) by assumption we can conclude that \( \phi_1.\phi_2 = 0 \) if and only if

\[
p(b) \{ \phi_1'(b) \phi_2(b) - \phi_1(b) \phi_2'(b) \} = p(a) \{ \phi_1'(a) \phi_2(a) - \phi_1(a) \phi_2'(a) \}.
\]

(15)

Three important cases in which these boundary conditions are satisfied are:

1) \textbf{Periodic boundary conditions:} if \( p(a) = p(b) \), then periodic boundary conditions, \( \phi_i(a) = \phi_i(b) \) and \( \phi_i'(a) = \phi_i'(b) \) (for \( i = 1, 2 \)), automatically satisfy (15). This is the case for the harmonic oscillator equation, for example.

2) \textbf{If } \( p(a) = p(b) = 0 \): then (15) is automatic. An example of this situation is Legendre’s equation with \( a = -1, b = 1 \) and \( p(x) = 1 - x^2 \).

3) \textbf{Unmixed boundary conditions:} if the boundary conditions are unmixed and of the form \( \alpha_1 \phi_1(a) + \alpha_3 \phi_1'(a) = 0 \) and \( \beta_2 \phi_1(b) + \beta_4 \phi_1'(b) = 0 \), for both \( i = 1 \) and \( i = 2 \), then both sides of (15) vanish identically.

In all three of these situations any two eigenfunctions corresponding to distinct eigenvalues are orthogonal, \( \phi_1.\phi_2 = 0 \). The term ‘orthogonal’ here is in analogy with ordinary
vector spaces: if $V$ is a finite dimensional vector space, of dimension $d$, with an inner product defined on it and $L : V \to V$ is a linear operator on $V$ then, in a given basis, $L$ can be represented as a matrix with components $L_a^b$, $a, b = 1, \ldots, d$. The matrix $L$ then "operates" on $V$ as a linear map of vectors to vectors — if $v$ is a vector with components $v_a$ in some basis then $L$ maps $v$ to $v'$ where, in matrix notation, $(v')_a = \sum_{b=1}^d L_a^b v_b$. The vector $v$ is an eigenvector of the matrix $L$ if

$$Lv = \lambda v,$$

where $\lambda$ is just a number, called the eigenvalue of $v$. Two vectors $v_1$ and $v_2$ are orthogonal if their inner product vanishes, $v_1 \cdot v_2 = 0$. If $v_1$ and $v_2$ are both eigenvectors of $L$ with distinct eigenvalues $\lambda_1 \neq \lambda_2$, then

$$v_1 \cdot (Lv_2) - (Lv_1) \cdot v_2 = v_1 \cdot (\lambda_2 v_2) - (\lambda_1 v_1) \cdot v_2 = (\lambda_1 - \lambda_2)v_1 \cdot v_2$$

so $v_1$ and $v_2$ are orthogonal if and only if

$$v_1 \cdot (Lv_2) - (Lv_1) \cdot v_2 = 0. \quad (16)$$

Any matrix $L$ which satisfies (16) for all vectors $v_1$ and $v_2$ is called a self-adjoint operator.

3) Completeness

For a finite dimensional vector space a basis is said to be complete if any vector can be written as a linear sum of the basis vectors. Similarly a set of functions is said to be complete if any (reasonably well behaved) function can be written as a linear combination of the functions in the set, with constant co-efficients. This is a generalisation of the concept of a Fourier series (expanding a periodic function in terms of sines and cosines). Label the eigenfunctions corresponding to linearly independent eigenfunctions of a self-adjoint differential operator in an interval $I = [a, b]$ by $\lambda_0, \lambda_1, \ldots$, in ascending order, and the corresponding eigenfunctions by $\phi_0(x), \phi_1(x), \ldots$. We assume here that the eigenvalues are discrete but they need not be distinct — two linearly independent eigenfunctions may have the same eigenvalue. The eigenfunctions are complete if any square integrable function* $h(x)$ in $I$ can be written as a sum

$$h(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) \quad (17)$$

where $c_n$ are constants. More precisely, the set $\{\phi_n(x)\}$ is a complete set if there exist constants $c_n$ such that

$$\lim_{N \to \infty} \int_a^b \left( h(x) - \sum_{n=0}^{N} c_n \phi_n(x) \right)^2 r(x) dx = 0.$$

* A function $h(x)$ is square integrable on an interval $[a, b]$ if $\int_{a}^{b} h^2 dx$ exists and is finite.
The square is to ensure that the integrand is positive everywhere, then there is no possibility that \( h(x) - \sum_{n=0}^{\infty} c_n \phi_n(x) \neq 0 \) but still integrates to zero because of oscillations — the integral of a positive integrand can only give zero if the integrand vanishes for all \( x \in (a, b) \).

The trigonometric functions, \( \sin(nx) \) and \( \cos(nx) \) give a complete set of functions on \( I = [-\pi, \pi] \). Other examples are:

- Legendre polynomials \( P_n(x) \) are a complete set on \( I = [-1, 1] \).
- The Laguerre polynomials \( L_n(x) \) are a complete set on \( I = [0, \infty) \).
- The Hermite polynomials are a complete set in \( I = (-\infty, \infty) \).

We shall not prove that these functions are complete on the stated intervals in this course, but merely state the fact.

When the eigenvalues of the eigenfunctions appearing in the expansion (17) are distinct we can use orthogonality of the eigenfunctions to determine the constants \( c_n \). Since

\[
\int_a^b h(x) \phi_m(x) r(x) dx = \sum_{n=0}^{\infty} c_n \int_a^b \phi_n(x) \phi_m(x) r(x) dx
\]

and the integral vanishes unless \( m = n \) by orthogonality, so only the term with \( n = m \) survives in the sum. This means that

\[
\int_a^b h(x) \phi_m(x) r(x) dx = c_m \int_a^b \{\phi_m(x)\}^2 r(x) dx
\]

so the \( c_n \) in (17) are given by

\[
c_n = \frac{\int_a^b h(x) \phi_n(x) r(x) dx}{\int_a^b \{\phi_n(x)\}^2 r(x) dx} \tag{18}
\]

and the \( c_n \) can be calculated either by evaluating the integrals analytically or numerically, if necessary.

**Examples**

1) Using harmonic functions \( \sin(nx) \) and \( \cos(nx) \), let \( h(x) = x, x \in (-\pi, \pi) \). Since \( h(x) \) is an odd function \( h(-x) = -h(x) \), we expect that only \( \sin(nx) \) will contribute. This expectation is confirmed by observing that \( \int_{-\pi}^{\pi} x \cos(nx) dx = 0 \). Then we only need to consider \( \phi_n(x) = \sin(nx) \) in equation (18) giving

\[
c_n = \frac{\int_{-\pi}^{\pi} x \sin(nx) dx}{\int_{-\pi}^{\pi} \sin^2(nx) dx} = \frac{\int_{-\pi}^{\pi} x \sin(nx) dx}{\pi}
\]

\[
= -\frac{1}{\pi n} \left[ x \cos(nx) \right]_{-\pi}^{\pi} + \frac{1}{n \pi} \int_{-\pi}^{\pi} \cos(nx) dx
\]

\[
= -\frac{1}{\pi n} \{\pi \cos n\pi - (-\pi) \cos(-n\pi)\}
\]

\[
= -\frac{2}{n} \cos n\pi = \frac{2}{n} (-1)^{n+1}.
\]
We conclude that

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx), \quad -\pi < x < \pi.$$  \tag{19}$$

Note that this formula is not correct for \(x \geq \pi\) or \(x \leq -\pi\), in particular the right-hand side is zero for \(x = \pm \pi\).

2) The function \(h(x) = x\) is even easier using, for example, Legendre polynomials. Since \(P_1(x) = x\) we can see right away, without having to do any integrals, that \(h(x) = P_1(x)\) in this case, so \(c_1 = 1\) and \(c_n = 0, \forall n \neq 1\). In fact this expansion is valid for all \(-\infty < x < \infty\) and not just \(x \in [-1, 1]\).

3) A slightly less trivial example is \(h(x) = (x - 2)^2\). Since this is a quadratic function we can get away with using only the first three Legendre polynomials, \(P_2(x) = \frac{1}{2}(3x^2 - 1)\), \(P_1(x) = x\) and \(P_0(x) = 1\). Then

$$h(x) = x^2 - 4x + 4 = \frac{2}{3}P_2(x) + \frac{1}{3} - 4x + 4 = \frac{2}{3}P_2(x) - 4P_1(x) + \frac{13}{3}P_0(x).$$

So \(c_0 = \frac{13}{3}, c_1 = -4, c_2 = \frac{2}{3}\) and, for \(n \geq 4, c_n = 0\).

4) Now consider the discontinuous function

$$h(x) = \begin{cases} V & 0 < x < 1 \\ -V & -1 < x < 0 \\ 0 & x = 0, \end{cases}$$

where \(V\) is a positive constant. This is an odd function, \(h(-x) = -h(x)\). If we expand this function using trigonometric functions then again we only need sine functions \(h(x) = \sum_{n=1}^{\infty} c_n \sin(nx)\) and equation (18) gives

$$c_n = V \left( \frac{\int_0^1 \sin(n\pi x)dx - \int_{-1}^{0} \sin(n\pi x)dx}{\int_{-1}^{1} \sin^2(n\pi x)dx} \right)$$

$$= V \left\{ \frac{1}{n} [-\cos(n\pi x)]_0^1 + \frac{1}{n} [\cos(n\pi x)]_{-1}^0 \right\}$$

$$= \frac{V}{\pi} \left\{ \frac{1}{n} (-1)^n - 1 + \frac{1}{n} (1 - (-1)^n) \right\}$$

$$= \frac{2V}{n\pi} (1 - (-1)^n)$$

$$= \begin{cases} \frac{4V}{\pi n} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

So

$$h(x) = \frac{4V}{\pi} \sum_{m=0}^{\infty} \frac{\sin((2m+1)x)}{2m+1}, \quad -1 < x < 1.$$

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We can also expand this function using Legendre polynomials for \( x \) in the range \((-1, 1)\). We already know that the Legendre polynomials are orthogonal and it is a standard conventions to normalise them so that

\[
\int_{-1}^{1} P_n(x)P_m(x)dx = \left( \frac{2}{2n + 1} \right) \delta_{n,m}.
\]  

(20)

Note also that \( P_n(x) \) is an even function when \( n \) is even and an odd function when \( n \) is odd, \( P_n(-x) = (-1)^n P_n(x) \), so we will only need odd \( n \) in an expansion of \( h(x) \). Using equation (18) again

\[
c_n = \frac{(2n + 1)V}{2} \left( \int_{0}^{1} P_n(x)dx - \int_{-1}^{0} P_n(x)dx \right)
\]

\[
= \begin{cases} (2n + 1)V \int_{0}^{1} P_n(x)dx & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}
\]

Of course \( \int_{0}^{1} P_n(x)dx \) is a number that depends on \( n \). In fact this can be evaluated using Rodrigues' formula (3):

\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n
\]

\[
\Rightarrow \int_{0}^{1} P_n(x)dx = \frac{1}{2^n n!} \left[ \frac{d^{n-1}}{dx^{n-1}}(x^2 - 1)^n \right]_0^1 = -\frac{1}{2^n n!} \left[ \frac{d^{n-1}}{dx^{n-1}}(x^2 - 1)^n \right]_{x=0}
\]

\[
= -\frac{1}{2^n n!} \sum_{j=0}^{n} \left( \begin{array}{c} n \\ j \end{array} \right) (-1)^{n-j} \left[ \frac{d^{n-1}}{dx^{n-1}}x^{2j} \right]_{x=0}.
\]

This vanishes if \( n \) is even, while if \( n \) is odd only the term in the sum with \( j = (n - 1)/2 \) is non-zero. So we consider \( n = 2m + 1 \) odd, in which case

\[
\int_{0}^{1} P_{2m+1}(x)dx = -\frac{1}{2^{2m+1}(2m+1)!} \binom{2m+1}{m} (-1)^{m+1}(2m)! = \frac{(-1)^m}{2^{2m+1}} \frac{(2m)!}{m!(m+1)!}.
\]

So

\[
c_{2m+1} = (-1)^m V \frac{(4m + 3)}{2^{2m+1}} \frac{(2m)!}{m!(m+1)!}
\]

and

\[
h(x) = V \sum_{m=0}^{\infty} (-1)^m \frac{(4m + 3)}{2^{2m+1}} \frac{(2m)!}{m!(m+1)!} P_{2m+1}(x)
\]

\[
= V \left( \frac{3}{2} P_1(x) - \frac{7}{8} P_3(x) + \frac{11}{16} P_5(x) - \ldots \right).
\]
Partial Differential Equations

1. Definition

A partial differential equation involves derivatives of a function $u(x, y, \ldots)$ of more than one independent variable. Partial differential equations necessarily involve partial derivatives such as $\frac{\partial u}{\partial x}$.

Examples of second order, linear, partial differential equations are:

- **Wave Equation**: $u(x, t)$
  \[
  \frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial t^2}
  \]

- **Heat Equation**: $u(x, t)$
  \[
  \frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}
  \]

- **Laplace’s Equation**: $u(x, y, z)$
  \[
  \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.
  \]

In all of these cases limits must be put on the range of $x$ and $t$ (or $x$, $y$ and $z$ in the case of Laplace’s equation) and, in order to ensure a unique solution of the equation, one must specify some properties of the function $u$ at the limit of these ranges. For example solving Laplace’s equation in some finite volume of 3-dimensional space requires specifying properties of $u(x, y, z)$ on the boundary surface of the volume of interest. If the value of $u$ is given at every point on the boundary then a unique solution exists (Dirichlet boundary conditions); alternatively a unique solution of Laplace’s equation exists if the derivative of $u$ in the direction normal to the boundary is specified at every point of the boundary (Neumann boundary conditions).

2. Separation of Variables

The method used for solving partial differential in this course will be that of separation of variables. Consider, for example, the wave equation

\[
\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial t^2}
\]

The method of solution by separation of variables requires first looking for solutions of the form $u(x, t) = T(t)X(x)$, i.e. solutions which factorise into a function $T(t)$ of $t$ only and a function $X(x)$ of $x$ only. This then reduces the partial differential equation (21) for $u(x, t)$ to ordinary differential equations for $T(t)$ and $X(x)$, which we can then try to solve using
the methods already studied. Of course not all solutions of the partial differential equation can be factorised, or separated, in this way but, for the equations that are studied in this course, a general solution can always be written as a linear combination of such separated solutions, so finding all possible separated solutions leads to the general solution.

Examples

Example 1): the wave equation

\[
\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial t^2}. \tag{22}
\]

This equation describes, for example, small lateral oscillations of a string or wire stretched under tension between two fixed points, with \(u(x, t)\) the lateral displacement of the wire at the point \(x\) at time \(t\) — a violin or a guitar string for instance. We shall not derive the equation in this course, we shall just treat the mathematical problem of finding solutions. Suppose the string has length \(L\) and the end points are fixed at \(x = 0\) and \(x = L\), this means that the displacement \(u(0, t) = u(L, t) = 0\) for all \(t\). Suppose we also know the initial profile of the string’s position and velocity, that is \(u(x, 0) = f(x)\) and \(\left(\frac{\partial u}{\partial t}\right)_{t=0} = v(x)\) where \(f(x)\) and \(v(x)\) are known functions. In the separation of variables method we try \(u = X(x)T(t)\) and look for a solution of (22) with this form. This implies that

\[
\left(\frac{d^2 X}{dx^2}\right) \frac{T}{X} = a^2 \left(\frac{d^2 T}{dt^2}\right),
\]

so the partial derivatives have disappeared and been replaced by ordinary derivatives. Dividing through by \(XT\) (assuming it is not zero) gives

\[
\frac{1}{X} \left(\frac{d^2 X}{dx^2}\right) = \frac{a^2}{T} \left(\frac{d^2 T}{dt^2}\right). \tag{23}
\]

Now notice that the left-hand side of this equation only depends on \(x\) while the right-hand side only depends on \(t\). Pick a value of \(t\) and keep it fixed, then the right-hand side is equal to a fixed number and

\[
\frac{1}{X} \frac{d^2 X}{dx^2} = \lambda,
\]

where \(\lambda\) is the value of \(\frac{a^2}{T} \left(\frac{d^2 T}{dt^2}\right)\) at our fixed choice of \(t\). The problem is now reduced to that of solving

\[
\frac{d^2 X}{dx^2} = \lambda X
\]

with \(\lambda\) a constant and this is just the harmonic oscillator equation, with the boundary conditions that \(X(0) = X(L) = 0\) (since \(u(0, t) = u(L, t) = 0\) for any \(t\)). With these boundary conditions \(\lambda < 0\) for a non-zero solution, more specifically \(\lambda = -\frac{n^2 \pi^2}{L^2}\) with \(n = 1, 2, 3, \ldots\), and

\[
X(x) = C_n \sin \left(\frac{n\pi x}{L}\right)
\]

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where \( C_n \) is a constant, for the moment arbitrary.

Now use this form for \( X(x) \) in (23) and we get an ordinary differential equation for \( T(t) \)

\[
\frac{a^2}{T} \left( \frac{d^2 T}{dt^2} \right) = \lambda \quad \Rightarrow \quad \left( \frac{d^2 T}{dt^2} \right) = -\left( \frac{n\pi}{aL} \right)^2 T,
\]

which is the harmonic oscillator equation for \( T(t) \). The most general solution of this equation is

\[
T(t) = A_n \cos \left( \frac{n\pi t}{aL} \right) + B_n \sin \left( \frac{n\pi t}{aL} \right),
\]

where \( A_n \) and \( B_n \) are constants.

So we have found an infinite number of solutions of (22) with the separated form \( u = TX \), with \( u(0, t) = u(L, t) = 0 \), one for each positive integer \( n \),

\[
u_n(x, t) = \left\{ A_n \cos \left( \frac{n\pi t}{aL} \right) + B_n \sin \left( \frac{n\pi t}{aL} \right) \right\} C_n \sin \left( \frac{n\pi x}{L} \right).
\]

Clearly the constants \( C_n \) are redundant here and we might as well write

\[
u_n(x, t) = \left\{ A_n \cos \left( \frac{n\pi t}{aL} \right) + B_n \sin \left( \frac{n\pi t}{aL} \right) \right\} \sin \left( \frac{n\pi x}{L} \right)
\]

by redefining \( A_n \) and \( B_n \) if necessary.

In fact, because the original equation is linear, any linear combination of the \( u_n \)

\[
u(x, t) = \sum_{n=1}^{\infty} \left\{ A_n \cos \left( \frac{n\pi t}{aL} \right) + B_n \sin \left( \frac{n\pi t}{aL} \right) \right\} \sin \left( \frac{n\pi x}{L} \right)
\]

is also a solution that satisfies the boundary conditions \( u(0, t) = u(L, t) = 0 \). Notice that this sum is not of separated form, in general. In fact (24) is the most general solution of (22) that has \( u(0, t) = u(L, t) = 0 \) (a proof of this is not given here).

The final step is to determine the constants \( A_n \) and \( B_n \) from the initial conditions \( u(x, 0) = f(x) \) and \( \left( \frac{\partial u}{\partial t} \right)_{t=0} = v(x) \). From (24)

\[
u(x, 0) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi x}{L} \right) = f(x)
\]

and

\[
\left( \frac{\partial u(x, t)}{\partial t} \right)_{t=0} = \left( \frac{\pi}{aL} \right) \sum_{n=1}^{\infty} nB_n \sin \left( \frac{n\pi x}{L} \right) = v(x).
\]

We can now extract \( A_n \) and \( B_n \) using

\[
\int_0^L \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{m\pi x}{L} \right) dx = \frac{L}{2} \delta_{mn},
\]

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for $m$ and $n$ positive integers, to get

$$A_m = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{\pi m x}{L} \right) dx, \quad B_m = \frac{2a}{\pi m} \int_0^L v(x) \sin \left( \frac{\pi m x}{L} \right) dx.$$  

Suppose, for example, that $v(x) = 0$ so the motion starts from rest at $t = 0$ with initial profile

$$f(x) = \begin{cases} \frac{2d}{L} x, & 0 \leq x \leq L/2 \\ \frac{2d}{2d \left( \frac{L-x}{L} \right)}, & L/2 \leq x \leq L. \end{cases} \quad (25)$$

So $B_n = 0$ and

$$A_m = \frac{4d}{L^2} \left\{ \int_0^{L/2} x \sin \left( \frac{m\pi x}{L} \right) dx + \int_{L/2}^L (L-x) \sin \left( \frac{m\pi x}{L} \right) dx \right\} = \frac{8d}{(\pi m)^2} \sin \left( \frac{m\pi}{2} \right), \quad (26)$$

Since $\sin \left( \frac{m\pi}{2} \right)$ is zero if $m$ is even and $(-1)^{(m-1)/2}$ if $m$ is odd the function (25) can be expressed as

$$f(x) = \frac{8d}{\pi^2} \sum_{n=1,3,5,...} (-1)^{(n-1)/2} \frac{1}{n^2} \sin \left( \frac{n\pi x}{L} \right).$$

Note in passing that, evaluating both sides of this expression at $x = L/2$ using (25), we have

$$d = \frac{8d}{\pi^2} \sum_{n=1,3,5,...} \frac{1}{n^2} \Rightarrow \sum_{n=1,3,5,...} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

We can put everything together now and write the full solution, with the initial condition (25), as

$$u(x, t) = 8 \frac{d}{\pi^2} \sum_{n=1,3,5,...} (-1)^{(n-1)/2} \frac{1}{n^2} \sin \left( \frac{n\pi x}{L} \right) \cos \left( \frac{n\pi t}{aL} \right).$$

Example 2): the heat equation

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}. \quad (27)$$

This equation describes, for example, the distribution of temperature as a function of time along a rod: $u(x, t)$ is the temperature at the point $x$ at time $t$. More generally (27) describes the diffusion of any physical quantity, not just heat, but we shall focus on heat as our example. Suppose, for instance, that the rod has length $L$ and is positioned with the left-hand end at $x = 0$ and the right-hand at $x = L$ and that the left-hand end has a fixed temperature $u(0, t) = 0$ while the right-hand end has its temperature held fixed at $u(L, t) = T$, where $T$ is a constant (in this example $T$ shall be used to denote a constant temperature and $S(t)$ will be used for the function of $t$ in the separated form: $u(x, t) = X(x)S(t)$). We shall suppose that the initial temperature profile of the rod at $t = 0$ is a known function of position, so $u(x, 0) = f(x)$ for some given function $f(x)$.  

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Using separation of variables we put \( u(x, t) = X(x)S(t) \) into the equation (27) gives
\[
\left( \frac{d^2 X}{dx^2} \right) S = a^2 X \left( \frac{dS}{dt} \right),
\]
so the partial derivatives have disappeared and been replaced by ordinary derivatives. Dividing through by \( XS \) (assuming it is not zero) gives
\[
\frac{1}{X} \left( \frac{d^2 X}{dx^2} \right) = \frac{a^2}{S} \left( \frac{dS}{dt} \right).
\]
(28)
Again the left-hand side of this equation only depends on \( x \) while the right-hand side only depends on \( t \). Pick a value of \( x \) and keep it fixed, then the left-hand side is equal to a fixed number and
\[
\frac{1}{X} \left( \frac{d^2 X}{dx^2} \right) = \frac{\lambda}{a^2},
\]
where \( \lambda \) is the value of \( \frac{1}{X} \left( \frac{d^2 X}{dx^2} \right) \) at our fixed choice of \( x \). The problem is now reduced to that of solving
\[
\frac{dS}{dt} = \left( \frac{\lambda}{a^2} \right) S
\]
with \( \lambda \) a constant. The solutions of this equation are exponential functions
\[
S(t) = C_\lambda e^{\lambda t / a^2},
\]
where \( C_\lambda \) are constants. If \( \lambda \) were positive then \( S(t) \), and so \( u(x, t) \), would grow exponentially for large \( t \) and, if \( u(x, t) \) has the physical interpretation of a temperature, this is not possible, so \( \lambda \leq 0 \). Writing \( \lambda = -\mu^2 \) equation (28) now gives an ordinary differential equation for \( X(x) \)
\[
\frac{d^2 X}{dx^2} = -\mu^2 X,
\]
the harmonic oscillator equation again. If \( \mu \neq 0 \) the most general solution is
\[
X(x) = A_\mu \cos(\mu x) + B_\mu \sin(\mu x)
\]
while for \( \mu = 0 \)
\[
X(x) = A_0 + B_0 x,
\]
where \( A_\mu \) and \( B_\mu \) are constants. Imposing the boundary condition that \( u(0, t) = 0 \), \( \forall t \), implies that \( X(0) = 0 \) which requires \( A_\mu = 0 \). The other boundary condition, that \( u(L, t) = T \), \( \forall t \), can easily be accommodated by taking \( B_0 = \frac{T}{L} \) and \( \mu = \frac{n\pi}{L} \) with \( n \) an integer (which we can always take to be positive) leading to the form
\[
u(x, t) = \frac{T x}{L} + \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{L} \right) \exp \left( -\left( \frac{n\pi}{aL} \right)^2 t \right)
\]
(29)
where the $C_\lambda$ have been absorbed into the $B_\mu$ and $B_{\bar{\mu}}$ have been relabelled as $B_n$.

The constants $B_n$ have still to be calculated from the initial temperature profile, $u(x,0) = f(x)$, but first observe that the late time behaviour is independent of $B_n$

$$\lim_{t \to \infty} u(x,t) = \frac{Tx}{L},$$

so at late times $u(x,t)$ becomes independent of $t$ and rises linearly with $x$ from 0 to $T$ as $x$ goes from 0 to $L$. This has the intuitive interpretation that, if a rod has one end fixed at temperature 0 and the other at temperature $T$ then one expects the equilibrium temperature to increase linearly from one end to the other. No matter what the initial temperature profile is it will always settle down to this equilibrium profile eventually and equation (29) gives us the wherewithal to follow this evolution through quantitatively. Note that the smaller the constant $a^2$ is the faster (29) relaxes to the equilibrium configuration (30). The constant $a^2$ in (27) is the inverse of the thermal conductivity of the rod — materials with a high thermal conductivity relax to their equilibrium temperature faster than materials with a low thermal conductivity.

Returning to the full solution as a function of time (29) consider the case when the initial temperature profile is constant along the rod and then jumps discontinuously to $T$ and the right-hand end:

$$u(x,0) = \begin{cases} 0, & 0 \leq x < L \\ T, & x = L. \end{cases}$$

A subtlety here is that $u(x,0)$ is not differentiable at $x = L$, but this is not an obstacle to solving the problem – we solve for $0 \leq x < L$, where $u(x,0)$ is differentiable, and impose the boundary condition that $\lim_{x \to L} u(x,0) = T$.

With the initial condition (31) setting $t = 0$ in (29) implies

$$\frac{Tx}{L} + \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{L} \right) = \begin{cases} 0, & 0 \leq x < L \\ T, & x = L. \end{cases}$$

This requires that the infinite sum cancels the linear term on the left-hand side for $0 \leq x < L$ (note that the sum vanishes for $x = L$, so the boundary condition at the right-hand end is automatically satisfied in this expression). We have already worked out the Fourier series for $x$, with $-\pi < x < \pi$, in (19) and it is a simple matter to extend this to $-L < x < L$, by rescaling $x \to \frac{\pi x}{L}$ in (19),

$$x = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \left( \frac{n\pi x}{L} \right), \quad -L < x < L.$$  

The constants $B_n$ can now be read off by putting (33) into (32),

$$B_n = -\frac{2T}{\pi} \frac{(-1)^{n+1}}{n},$$

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so the complete solution is, from (29),

\[ u(x, t) = \frac{T x}{L} + \frac{2T}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \left( \frac{n\pi x}{L} \right) \exp \left( -\left( \frac{n\pi}{aL} \right)^2 t \right). \]

**Example 3):** Laplace’s equation

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \]  \hspace{1cm} (34)

This equation is important for problems in electrostatics, where it describes the electric potential \( u(x, y, z) \) in a charge-free region of space. Generically we want to solve for \( u \) in a volume \( V \) that contains no electric charges, but there may be charges outside \( V \), or on its boundary, and these can produce an electric field \( E = -\nabla u \) inside \( V \). The net effect of these charges can be incorporated into boundary conditions on \( u \), i.e. by specifying \( u \) on the boundary of \( V \). If \( V \) is a cuboid then Cartesian co-ordinates are appropriate, with \( 0 < x < a, 0 < y < b \) and \( 0 < z < c \) for a cuboid of width \( a \), depth \( b \) and height \( c \). Suppose first of all that \( u \) vanishes on all except the top face and is given by a specified function \( u(x, y, c) = f(x, y) \) on the top face.

The technique of separation of variables requires looking for solutions of (34) of the form

\[ u(x, y, z) = X(x)Y(y)Z(z), \]

so,

\[ \left( \frac{d^2X}{dx^2} \right) YZ + X \left( \frac{d^2Y}{dy^2} \right) Z + XY \left( \frac{d^2Z}{dz^2} \right) = 0. \]

Dividing through by \( XYZ \) (assuming it is non-zero) we get

\[ \frac{1}{X} \frac{d^2X}{dx^2} + \frac{1}{Y} \frac{d^2Y}{dy^2} + \frac{1}{Z} \frac{d^2Z}{dz^2} = 0. \]

Since the second and third terms in this equation are independent of \( x \) we conclude that \( \frac{1}{X} \frac{d^2X}{dx^2} \) is a constant, independent of \( x \). The same argument allows to conclude that \( \frac{1}{Y} \frac{d^2Y}{dy^2} \) is independent of \( y \) and \( \frac{1}{Z} \frac{d^2Z}{dz^2} \) is independent of \( z \). So we are led to

\[ \frac{d^2X}{dx^2} = \lambda X, \quad \frac{d^2Y}{dy^2} = \mu Y, \quad \frac{d^2Z}{dz^2} = \nu Z \]

where \( \lambda, \mu \) and \( \nu \) are constants summing to zero,

\[ \lambda + \mu + \nu = 0. \]  \hspace{1cm} (35)

Looking for solutions of (34) that factorise has reduced the problem to that of three harmonic oscillator equations.
Since the boundary conditions require that $u$ should vanish for $x = 0, x = a, y = 0$ and $y = b$ it must be the case that $X(0) = X(a) = 0$ and $Y(0) = Y(b) = 0$. For non-zero solutions this is only possible if $\lambda$ and $\mu$ are negative, in particular we are forced to

$$X(x) = A_n \sin \left( \frac{n \pi x}{a} \right), \quad \lambda = -\left( \frac{n \pi}{a} \right)^2,$$

$$Y(y) = B_m \sin \left( \frac{m \pi y}{b} \right), \quad \mu = -\left( \frac{m \pi}{b} \right)^2,$$

where $n$ and $m$ are integers (which we can take to be positive) and $A_n$ and $B_m$ are constants.

This now fixes the constant $\nu$ in (35) to be

$$\nu = \left( \frac{n^2}{a^2} + \frac{m^2}{b^2} \right) \pi^2$$

so the final equation is

$$\frac{d^2 Z}{dz^2} = \nu Z, \quad \text{with} \quad \nu = \left( \frac{n^2}{a^2} + \frac{m^2}{b^2} \right) \pi^2 > 0.$$

This has solutions

$$Z = C_{nm} e^{\sqrt{\nu} z} + D_{nm} e^{-\sqrt{\nu} z}$$

where $C_{nm}$ and $D_{nm}$ are constants, which be be different for different pairs $(n, m)$. One of the boundary conditions was that $u(x, y, 0) = 0$ so we must impose $Z(0) = 0$ which requires $D_{nm} = -C_{nm}$ so

$$Z = C_{nm} \left( e^{\sqrt{\nu} z} - e^{-\sqrt{\nu} z} \right) = 2C_{nm} \sinh(\sqrt{\nu} z).$$

Combining this with the forms of $X$ and $Y$ obtained above gives the following solutions of Laplace’s equation, one for each pair of positive integers $m$ and $n$,

$$u_{nm}(x, y, z) = A_{nm} \sin \left( \frac{n \pi x}{a} \right) \sin \left( \frac{m \pi y}{b} \right) \sinh \left( \sqrt{\nu_{nm}} z \right),$$

where all the constants have been combined into one $A_{nm} = 2A_n B_m C_{nm}$. The different $\nu$’s are distinguished by subscripts: $\nu_{nm} = \left( \frac{n^2}{a^2} + \frac{m^2}{b^2} \right) \pi^2$. These are all solutions of Laplace’s equation which vanish on five faces of the cuboid, but not on the top face $z = c$. The most general solution satisfying these conditions is a linear combination of these, which can be represented by a double sum over $n$ and $m$:

$$u(x, y, z) = \sum_{n,m=1}^{\infty} A_{nm} \sin \left( \frac{n \pi x}{a} \right) \sin \left( \frac{m \pi y}{b} \right) \sinh \left( \sqrt{\nu_{nm}} z \right),$$

where $\nu_{nm}$ has been written in full.
To solve the problem uniquely we now need to specify \( u(x, y, z) \) on the top face, \( u(x, y, z) = f(x, y) \) for some given function \( f(x, y) \), and then the constants \( A_{nm} \) are determined by demanding that

\[
u(x, y, c) = f(x, y) = \sum_{n,m=1}^{\infty} A_{nm} \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{m\pi y}{b} \right) \sinh(\sqrt{\nu_{nm}}c).
\]

For example suppose \( u \) is constant on the top face, \( u(x, y, c) = C \). Then

\[
\sum_{n,m=1}^{\infty} A_{nm} \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{m\pi y}{b} \right) \sinh(\sqrt{\nu_{nm}}c) = C
\]

and the \( A_{nm} \) can extracted explicitly using

\[
\int_0^a \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{n'\pi x}{a} \right) \, dx = \frac{a}{2} \delta_{nn'} \quad \text{and} \quad \int_0^b \sin \left( \frac{m\pi y}{b} \right) \sin \left( \frac{m'\pi y}{b} \right) \, dx = \frac{b}{2} \delta_{mm'},
\]

for integers \( n, n', m \) and \( m' \). From this we can derive

\[
\left( \frac{A_{n'm'a'b}}{4} \right) \sinh(\sqrt{\nu_{n'm'c}}) = C \int_0^a \sin \left( \frac{n'\pi x}{a} \right) \, dx \int_0^b \sin \left( \frac{m'\pi y}{b} \right) \, dy.
\]

This vanishes if either \( n' \) or \( m' \) is even, and

\[
A_{n'm'} = \frac{16C}{\pi^2 n'm' \sinh(\sqrt{\nu_{n'm'c}})}
\]

if both \( n' \) and \( m' \) are odd. This gives

\[
f(x, y) = \frac{16C}{\pi^2} \sum_{n,m=1,3,5,...}^{\infty} \left( \frac{1}{nm} \right) \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{m\pi y}{b} \right) = \begin{cases} C, & \text{for } 0 < x < a; 0 < y < b \\ 0, & \text{for } x = 0, a; y = 0, b. \end{cases}
\]

The full solution, with \( u = C \) on the top face and zero on the other five faces, is

\[
u(x, y, z) = \frac{16C}{\pi^2} \sum_{n,m=1,3,5,...}^{\infty} \frac{\sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{m\pi y}{b} \right) \sinh \left( \pi z \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} \right)}{nm \sinh \left( \pi c \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} \right)}.
\]

This method of solving Laplace’s equation can easily be extended to the case where \( u \neq 0 \) on all six faces: because the equation is linear in \( u \) we can solve it for \( u = 0 \) on five faces and \( u \neq 0 \) on the remaining face; do this for \( u \neq 0 \) on each of the six faces in turn; then simply add the six resulting solutions. This will give the unique solution with \( u \neq 0 \) on all six faces.
The choice of Cartesian co-ordinates in the above example of Laplace’s equation is tailored to the shape of a cuboid, with boundary conditions on the surface. With other shapes different co-ordinates are often more useful. Consider for example the 2-dimensional Laplace equation
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{36}
\]
for a function \(u(x, y)\). Cartesian co-ordinates are appropriate here for a rectangular region, but if we want to solve the equation in the interior of a disc, with a circular boundary, 2-dimensional polar co-ordinates are more suitable. Let
\[
x = \rho \cos \phi, \quad y = \rho \sin \phi,
\]
where \(0 \leq \phi < 2\pi\) is a polar angle and \(0 \leq \rho < \infty\) a radial co-ordinate in two dimensions with \(\rho^2 = x^2 + y^2\). To solve (36) in the disc we need to think of \(u\) as a function of \(\rho\) and \(\phi\) and convert the partial derivatives to polar co-ordinates. The chain rule for differentiation gives
\[
\frac{\partial u}{\partial \rho} = (\frac{\partial x}{\partial \rho}) \left( \frac{\partial u}{\partial x} \right) + (\frac{\partial y}{\partial \rho}) \left( \frac{\partial u}{\partial y} \right) = \cos \phi \left( \frac{\partial u}{\partial x} \right) + \sin \phi \left( \frac{\partial u}{\partial y} \right),
\]
\[
\frac{\partial u}{\partial \phi} = (\frac{\partial x}{\partial \phi}) \left( \frac{\partial u}{\partial x} \right) + (\frac{\partial y}{\partial \phi}) \left( \frac{\partial u}{\partial y} \right) = -\rho \sin \phi \left( \frac{\partial u}{\partial x} \right) + \rho \cos \phi \left( \frac{\partial u}{\partial y} \right).
\]
Applying the chain rule a second time gives
\[
\frac{\partial^2 u}{\partial \rho^2} = \cos \phi \left\{ \cos \phi \left( \frac{\partial^2 u}{\partial x^2} \right) + \sin \phi \left( \frac{\partial^2 u}{\partial y \partial x} \right) \right\} + \sin \phi \left\{ \cos \phi \left( \frac{\partial^2 u}{\partial x \partial y} \right) + \sin \phi \left( \frac{\partial^2 u}{\partial y^2} \right) \right\}
\]
\[
= (\cos \phi)^2 \left( \frac{\partial^2 u}{\partial x^2} \right) + (\sin \phi)^2 \left( \frac{\partial^2 u}{\partial y^2} \right) + 2 \cos \phi \sin \phi \left( \frac{\partial^2 u}{\partial x \partial y} \right),
\]
\[
\frac{\partial^2 u}{\partial \phi^2} = -\rho \cos \phi \left( \frac{\partial u}{\partial x} \right) - \rho \sin \phi \left\{ -\rho \sin \phi \left( \frac{\partial^2 u}{\partial x^2} \right) + \rho \cos \phi \left( \frac{\partial^2 u}{\partial y \partial x} \right) \right\}
\]
\[
= \rho^2 (\sin \phi)^2 \left( \frac{\partial^2 u}{\partial x^2} \right) + \rho^2 (\cos \phi)^2 \left( \frac{\partial^2 u}{\partial y^2} \right) - 2 \rho^2 \cos \phi \sin \phi \left( \frac{\partial^2 u}{\partial x \partial y} \right) - \rho \left\{ \cos \phi \left( \frac{\partial u}{\partial x} \right) + \sin \phi \left( \frac{\partial u}{\partial y} \right) \right\}.
\]
By taking linear combinations of these equations one finds
\[
\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.
\]
Finally the 2-dimensional Laplace’s equation written in polar co-ordinates is:
\[
\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} = 0. \tag{37}
\]
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We shall now find solutions to this equation using separation of variables. First note that, if \( u(\rho, \phi) \) is to be a single valued function of position it must be periodic in the angular variable \( \phi \), \( u(\rho, \phi + 2\pi) = u(\rho, \phi) \). With this condition we seek separated solutions of the form

\[
u(\rho, \phi) = R(\rho)\Phi(\phi).
\]

Putting this form of \( u(\rho, \phi) \) into (37) we get

\[
\left( \frac{d^2R}{d\rho^2} \right) \Phi + \frac{1}{\rho} \left( \frac{dR}{d\rho} \right) \Phi + \frac{R}{\rho^2} \left( \frac{d^2\Phi}{d\phi^2} \right) = 0.
\]

Multiplying this by \( \frac{\rho^2}{R} \) we arrive at

\[
\frac{\rho^2}{R} \left( \frac{d^2R}{d\rho^2} \right) + \frac{\rho}{R} \left( \frac{dR}{d\rho} \right) + \frac{1}{\Phi} \left( \frac{d^2\Phi}{d\phi^2} \right) = 0.
\]

The by now familiar argument says that \( \frac{1}{\Phi} \left( \frac{d^2\Phi}{d\phi^2} \right) \) and \( \frac{\rho^2}{R} \left( \frac{d^2R}{d\rho^2} \right) + \frac{\rho}{R} \left( \frac{dR}{d\rho} \right) \) are constants. Since \( \Phi(\phi + 2\pi) = \Phi(\phi) \) is periodic we must have

\[
\Phi(\phi) = A_n \cos(n\phi) + B_n \sin(n\phi)
\]

were \( A_n \) and \( B_n \) are constants, \( n \) is an integer (which we can take to be non-negative) and

\[
\frac{d^2\Phi}{d\phi^2} = -n^2\Phi.
\]

There remains the equation

\[
\frac{\rho^2}{R} \left( \frac{d^2R}{d\rho^2} \right) + \frac{\rho}{R} \left( \frac{dR}{d\rho} \right) = n^2 \quad \Rightarrow \quad \rho^2 \left( \frac{d^2R}{d\rho^2} \right) + \rho \left( \frac{dR}{d\rho} \right) = n^2 R.
\]

For \( n \neq 0 \) this equation has linearly independent solutions \( R = \rho^n \) and \( R = \rho^{-n} \) so the general solution is

\[
R = C_n \rho^n + D_n \rho^{-n},
\]

where \( C_n \) and \( D_n \) are constants. For \( n = 0 \), \( \ln \rho \) is a solution (as well as \( R = \text{const} \)) so the general solution for \( n = 0 \) is

\[
R = C_0 + D_0 \ln \rho,
\]

with \( C_0 \) and \( D_0 \) again arbitrary constants.

Putting all this together, the general solution of (37) is

\[
u(\rho, \phi) = C_0 + D_0 \ln \rho + \sum_{n=1}^{\infty} \left( C_n \rho^n + D_n \rho^{-n} \right) \left( A_n \cos(n\phi) + B_n \sin(n\phi) \right).
\]
(For a given \( n \) we do not need all four of \( A_n, B_n, C_n, \) and \( D_n \): for instance, if \( C_n \neq 0 \), we can absorb it into a redefinition of \( A_n \) and \( B_n \) and \( D_n \).)

If we are to solve the equation in the interior of a disc, of radius \( a \) say, then \( D_n = 0, \forall n \), since otherwise \( u \) would diverge at \( \rho = 0 \). In this case the solution (38) reduces to

\[
    u(\rho, \phi) = C_0 + \sum_{n=1}^{\infty} \rho^n (A_n \cos(n\phi) + B_n \sin(n\phi)) \tag{39}
\]

where \( C_n \) has been absorbed into the constants \( A_n \) and \( B_n \). Finally \( A_n, B_n \) and \( C_0 \) must be determined by boundary conditions, such as specifying \( u(a, \phi) \) on the perimeter of the disc. For example if we are told that

\[
    u(a, \phi) = u_0 \cos(2\phi)
\]
on the perimeter, where \( u_0 \) is a constant, then

\[
    u(a, \phi) = C_0 + \sum_{n=1}^{\infty} a^n (A_n \cos(n\phi) + B_n \sin(n\phi)) = u_0 \cos(2\phi).
\]

Using orthogonality of the trigonometric functions

\[
    \int_0^{2\pi} \cos(m\phi) \cos(n\phi) d\phi = \pi \delta_{mn}
\]
\[
    \int_0^{2\pi} \cos(m\phi) \sin(n\phi) d\phi = 0,
\]

for positive integers \( m \) and \( n \), we derive

\[
    A_2 = \frac{u_0}{a^2}, \quad A_n = 0 \quad \text{for } n \neq 2, \quad C_0 = 0 \quad \text{and} \quad B_n = 0 \quad \text{for } \forall n.
\]

So the full solution with \( u(a, \phi) = u_0 \cos(2\phi) \) is

\[
    u(\rho, \phi) = u_0 \left( \frac{\rho}{a} \right)^2 \cos(2\phi)
\]

If the origin is not in the region in which (37) is to be solved, then the other terms in (38) must be taken into account. For example, suppose we want to solve (37) in an annulus centred on the origin with the inner ring of radius \( b \) and the outer ring of radius \( a \), with \( u(a, \phi) = u_0 \) and \( u(b, \phi) = u_1 \) constants, independent of \( \phi \). Then the angular part of \( u(\rho, \phi) \) is trivial, \( A_n = B_n = 0 \), and

\[
    u(\rho) = C_0 + D_0 \ln \rho.
\]

The \( \ln \rho \) term is now allowed because \( \rho = 0 \) has been excluded from the region of interest. The constants \( C_0 \) and \( D_0 \) are determined by

\[
    u(a) = C_0 + D_0 \ln a = u_0 \quad \text{and} \quad u(b) = C_0 + D_0 \ln b = u_1,
\]

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so
\[ D_0 (\ln a - \ln b) = u_0 - u_1 \quad \Rightarrow \quad D_0 = \frac{(u_0 - u_1)}{\ln(a/b)} \]
and
\[ C_0 = u_0 - \left( \frac{u_0 - u_1}{\ln(a/b)} \right) \ln a = \left( \frac{u_1 \ln a - u_0 \ln b}{\ln(a/b)} \right). \]

The solution in the interior of the annulus, \( b \leq \rho \leq a \), is now given by
\[ u(\rho) = \left( \frac{u_1 \ln a - u_0 \ln b}{\ln(a/b)} \right) + \left( \frac{u_0 - u_1}{\ln(a/b)} \right) \ln \rho = \frac{u_1 \ln(a/\rho) + u_0 \ln(\rho/b)}{\ln(a/b)}. \]

For many problems involving Laplace’s equation in 3-dimensions it is more convenient to use spherical polar co-ordinates \((r, \theta, \phi)\) rather than Cartesian co-ordinates \((x, y, z)\). These are related to each other in the usual way by
\[ x = r \cos \phi \sin \theta, \quad y = r \sin \phi \sin \theta, \quad z = r \cos \theta. \]

To translate (34) into a differential equation involving \((r, \theta, \phi)\) we need the following partial derivatives:
\[ \frac{\partial x}{\partial r} = \cos \phi \sin \theta, \quad \frac{\partial y}{\partial r} = \sin \phi \sin \theta, \quad \frac{\partial z}{\partial r} = \cos \theta, \]
\[ \frac{\partial x}{\partial \theta} = r \cos \phi \cos \theta, \quad \frac{\partial y}{\partial \theta} = r \sin \phi \cos \theta, \quad \frac{\partial z}{\partial \theta} = -r \sin \theta, \]
\[ \frac{\partial x}{\partial \phi} = -r \sin \phi \sin \theta, \quad \frac{\partial y}{\partial \phi} = r \cos \phi \sin \theta, \quad \frac{\partial z}{\partial \phi} = 0. \]

Using these the chain rule for differentiation implies that
\[ \frac{\partial}{\partial r} = \frac{\partial x}{\partial r} \frac{\partial}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial}{\partial y} + \frac{\partial z}{\partial r} \frac{\partial}{\partial z} = \cos \phi \sin \theta \frac{\partial}{\partial x} + \sin \phi \sin \theta \frac{\partial}{\partial y} + \cos \theta \frac{\partial}{\partial z}, \]
\[ \frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta} \frac{\partial}{\partial z} = r \cos \phi \cos \theta \frac{\partial}{\partial x} + r \sin \phi \cos \theta \frac{\partial}{\partial y} - r \sin \theta \frac{\partial}{\partial z}, \]
\[ \frac{\partial}{\partial \phi} = \frac{\partial x}{\partial \phi} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi} \frac{\partial}{\partial y} + \frac{\partial z}{\partial \phi} \frac{\partial}{\partial z} = -r \sin \phi \sin \theta \frac{\partial}{\partial x} + r \cos \phi \sin \theta \frac{\partial}{\partial y} + r \cos \phi \frac{\partial}{\partial \phi}. \]

These can be inverted, by taking linear combination with trigonometric function for example, to express partial derivatives of Cartesian co-ordinates in terms of polar co-ordinates:
\[ \frac{\partial}{\partial x} = \cos \phi \sin \theta \frac{\partial}{\partial r} + \frac{\cos \phi \cos \theta}{r} \frac{\partial}{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial}{\partial \phi}, \]
\[ \frac{\partial}{\partial y} = \sin \phi \sin \theta \frac{\partial}{\partial r} + \frac{\sin \phi \cos \theta}{r} \frac{\partial}{\partial \theta} + \frac{\cos \phi}{r \sin \theta} \frac{\partial}{\partial \phi}, \]
\[ \frac{\partial}{\partial z} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \phi}. \]
Using these, after a rather tedious calculation, the Laplace equation (34) can be expressed directly in terms of spherical polar co-ordinates:

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0.
\]

We shall now show how to solve Laplace’s equation in spherical polar co-ordinates

\[
\frac{1}{r} \frac{\partial^2 (ru)}{\partial r^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0
\tag{40}
\]

for a function \(u(r, \theta, \phi)\).

For simplicity let’s first tackle the problem when \(u\) is independent of \(\phi\) — later we shall look at the more general case when \(u\) also depends on \(\phi\). When \(u\) is independent of \(\phi\) equation (40) is

\[
\frac{1}{r} \frac{\partial^2 (ru)}{\partial r^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) = 0.
\]

As before we try a separation of variable form

\[
u = \frac{R(r)}{r} \Theta(\theta)
\]

(the \(1/r\) is just for convenience, it make the final differential equation for \(R\) simpler). Equation (40) then reduces to

\[
\frac{1}{r} \left( \frac{d^2 R}{dr^2} \right) \Theta + \frac{R}{r^3 \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = 0.
\]

Multiplying by \(r^3/(R \Theta)\) gives

\[
\frac{r^2}{R} \left( \frac{d^2 R}{dr^2} \right) + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = 0,
\]

and the first term in this equation depends only on \(r\), not on \(\theta\), while the second depends only on \(\theta\), not on \(r\) — each term must therefore be a constant. The equation can therefore be separated into two equations

\[
r^2 \frac{d^2 R}{dr^2} = -\lambda R
\]

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = \lambda \Theta,
\]

where \(\lambda\) is a constant. The first of these equations can be satisfied by the monomial \(R = r^{-n}\), where \(\lambda = -n(n+1)\) and by \(R = r^{n+1}\) with the same \(\lambda\). The general solution is therefore a linear combination of these

\[
R(r) = A_n r^{n+1} + B_n r^{-n}
\tag{41}
\]
with $A_n$ and $B_n$ constants. The second equation is now

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = -n(n+1)\Theta. \quad (42)$$

This is in fact Legendre’s equation in disguise, because we can change variables from $\theta$ to $x = \cos \theta$ (this is not the Cartesian co-ordinate $x$), with $-1 \leq x \leq 1$ for $0 \leq \theta \leq \pi$. Now

$$\frac{d}{d\theta} = \frac{d}{dx} \frac{dx}{d\theta} = -\sin \theta \frac{d}{dx} = -\sqrt{1 - x^2} \frac{d}{dx},$$

and (42) is the familiar

$$\frac{d^2}{dx^2} \left\{ (1 - x^2) \frac{d\Theta}{dx} \right\} = (1 - x^2) \frac{d^2\Theta}{dx^2} - 2x \frac{d\Theta}{dx} = -n(n+1)\Theta,$$

and the solutions are Legendre polynomials

$$\Theta = P_n(x) = P_n(\cos \theta)$$

with $n = 0, 1, 2, \ldots$. Combining this with (41) the general solution of (40), with $u$ independent of the polar angle $\phi$, is (remember $u = \frac{1}{r} R \Theta$)

$$u = \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-n-1}) P_n(\cos \theta).$$

Now consider the more general problem when $u$ depends on all three arguments $r$, $\theta$ and $\phi$. Observe first that, because increasing the polar angle $\phi$ by $2\pi$, $\phi \rightarrow \phi + 2\pi$, brings us back to the same point, $u$ must be a periodic function of $\phi$, $u(r, \theta, \phi + 2\pi) = u(r, \theta, \phi)$. Bearing this in mind, we look for a solution of (40) of the form

$$u(r, \theta, \phi) = \frac{R(r)}{r} \Theta(\theta) \Phi(\phi).$$

With this form the equation reduces to

$$\frac{1}{r} \left( \frac{d^2 R}{dr^2} \right) \Theta \Phi + \frac{R}{r^3 \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) \Phi + \frac{R\Theta}{r^3 \sin^2 \theta} \frac{d^2 \Phi}{d\phi^2} = 0.$$ 

Now multiplying by $(r^3 \sin^2 \theta)/(R\Theta \Phi)$ gives

$$\frac{r^2 \sin^2 \theta}{R} \left( \frac{d^2 R}{dr^2} \right) + \frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = 0. \quad (43)$$

The first two terms are independent of $\phi$, so the last term must be a constant, which we denote by $-m^2$, so

$$\frac{d^2 \Phi}{d\phi^2} = -m^2 \Phi.$$
Since $\Phi(\phi + 2\pi) = \Phi(\phi)$ we conclude that $m$ is an integer and

$$\Phi = C_m \cos(m\phi) + D_m \sin(m\phi)$$

with $C_m$ and $D_m$ constants. An alternative way of writing this is

$$\Phi = C_m e^{im\phi} + C_m^* e^{-im\phi},$$

where the $C_m$ are complex and $C_m = \frac{1}{2}(C_m - iD_m)$ with $C_m^* = \frac{1}{2}(C_m + iD_m)$. Either way, equation (43) is now

$$r^2 \left(\frac{d^2 R}{dr^2}\right) + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2 \theta} = 0.$$

The first term here is function of $r$ only while the second and third terms together give a function of $\theta$ only, and so must be a constant. Denoting this constant by $-n(n+1)$ as before we have

$$r^2 \left(\frac{d^2 R}{dr^2}\right) - n(n+1)R = 0$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta}\right) + n(n+1)\Theta - \frac{m^2\Theta}{\sin^2 \theta} = 0.$$

The equation for $R$ has the same solutions as before,

$$R(r) = A_n r^{n+1} + B_n r^{-n},$$

while the equation for $\Theta$ can be written in terms of $x = \cos \theta$ as

$$(1 - x^2) \frac{d^2 \Theta}{dx^2} - 2x \frac{d\Theta}{dx} + n(n+1)\Theta - \frac{m^2}{(1 - x^2)}\Theta = 0. \quad (44)$$

Solutions of this equation are called **Associated Legendre Functions** and are usually denoted by $P_n^m(x)$. For $m = 0$ the solutions are just the usual Legendre polynomials, $P_n^0(x) = P_n(x)$. The first few associated Legendre functions with $m \neq 0$ are:

- $P_1^1(x) = (1 - x^2)^{1/2} = \sin \theta$
- $P_2^1(x) = 3x(1 - x^2)^{1/2} = 3 \cos \theta \sin \theta$
- $P_2^0(x) = 3(1 - x^2) = 3 \sin^2 \theta$
- $P_3^1(x) = \frac{3}{2}(5x^2 - 1)(1 - x^2)^{1/2} = \frac{3}{2}(5 \cos^2 \theta - 1) \sin \theta$
- $P_3^2(x) = 15x(1 - x^2) = 15 \cos \theta \sin^2 \theta$
- $P_3^3(x) = 15(1 - x^2)^{3/2} = 15 \sin^3 \theta$.

An explicit formula for positive $m$, which is sometimes useful, is

$$P_n^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x).$$
Note that, since $P_n(x)$ is a polynomial of order $n$ in $x$, $P_n^m(x)$ vanishes for $m > n$, so we shall restrict to $m \leq n$. The sign of $m$ does not matter in (44) and, by convention, $P_{n-m}^m(x)$ are defined as being proportional to $P_n^m(x)$:

$$P_{n-m}^m(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x),$$

so $-n \leq m \leq n$ (the factor $(-1)^m \frac{(n-m)!}{(n+m)!}$ is just a standard convention). The associated Legendre functions are orthogonal for the same $m$ but different $n$:

$$\int_{-1}^{1} P_n^m(x) P_{n'}^m(x) dx = \left( \frac{2}{2n+1} \right) \frac{(n+m)!}{(n-m)!} \delta_{n'n},$$

where $C_{n,m}$ and $D_{n,m}$ are complex constants.

3. Some Other Partial Differential Equations

The techniques described above can be used to tackle other partial differential equations of a similar form. Examples are:

- the 2-dimensional heat equation for $u(x, y, t)$ in Cartesian co-ordinates

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = a^2 \frac{\partial u}{\partial t}$$

- the 2-dimensional heat equation for $u(\rho, \phi, t)$ in polar co-ordinates

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \phi} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} = a^2 \frac{\partial u}{\partial t}$$

- the 3-dimensional heat equation for $u(x, y, z, t)$ in Cartesian co-ordinates

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = a^2 \frac{\partial u}{\partial t}$$

- the 3-dimensional heat equation for $u(r, \theta, \phi, t)$ in spherical polar co-ordinates

$$\frac{1}{r} \frac{\partial^2 (ru)}{\partial r^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = a^2 \frac{\partial u}{\partial t}$$
• the 2-dimensional wave equation for $u(x, y, t)$ in Cartesian co-ordinates
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = a^2 \frac{\partial^2 u}{\partial t^2}
\]

• the 2-dimensional wave equation for $u(\rho, \phi, t)$ in polar co-ordinates
\[
\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \phi} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} = a^2 \frac{\partial^2 u}{\partial t^2}
\]

• the 3-dimensional wave equation for $u(x, y, z, t)$ in Cartesian co-ordinates
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = a^2 \frac{\partial^2 u}{\partial t^2}
\]

• the 3-dimensional wave equation for $u(r, \theta, \phi, t)$ in spherical polar co-ordinates
\[
\frac{1}{r} \frac{\partial^2 (ru)}{\partial r^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = a^2 \frac{\partial^2 u}{\partial t^2}.
\]

4. Spherical Harmonics

When a periodic function $f(\phi + 2\pi) = f(\phi)$ is expanded as a Fourier series, in terms of trigonometric functions $\sin(m\phi)$ and $\cos(m\phi)$, it can be viewed as a function on a circle parameterised by the angular variable $\phi$, i.e. each point $\phi$ on the circle is associated a unique real number $f(\phi)$. The expansion can be written in terms of sines and cosines
\[
f(\phi) = \sum_{m=0}^{\infty} \left( C_m \cos(m\phi) + D_m \sin(m\phi) \right),
\]
with $C_m$ and $D_m$ real constants, or in terms of complex exponentials
\[
f(\phi) = \sum_{m=0}^{\infty} \left( C_m e^{im\phi} + C_m^* e^{-im\phi} \right),
\]
where $C_m = \frac{1}{2}(C_m - iD_m)$ and $C_m^* = \frac{1}{2}(C_m + iD_m)$.

In a similar way one can expand functions on the surface of a sphere, parameterised by angular co-ordinates $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$, in terms of functions called Spherical Harmonics. These can be written in terms of the associated Legendre functions $P_n^m(\cos \theta)$ and the complex exponential $e^{im\phi}$. They are denoted $Y_n^m(\theta, \phi)$ and are defined by
\[
Y_n^m(\theta, \phi) = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} P_n^m(\cos \theta) e^{im\phi},
\]
(again the square root pre-factor is just a standard convention). A function on a spherical surface can then be written as

\[ f(\theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_{m,n} Y_n^m(\theta, \phi), \]

where \( C_{m,n} \) are constants. The \( Y_n^m \) are complex and

\[ (Y_n^m)^* = (-1)^m Y_n^m \]

so \( f(\theta, \phi) \) is real if \( C_{m,n}^* = (-1)^m C_{m,n} \).

The first few spherical harmonics are:

\[ Y_0^0 = \sqrt{\frac{1}{4\pi}}, \quad Y_0^1 = \sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}, \]

\[ Y_1^0 = \sqrt{\frac{5}{4\pi}}(3 \cos^2 \theta - 1), \quad Y_1^1 = \sqrt{\frac{15\pi}{8\pi}} \sin \theta \cos \theta e^{i\phi}, \quad Y_2^0 = \sqrt{\frac{15\pi}{2\pi}} \sin^2 \theta e^{2i\phi}, \]

The considerations of the previous section show that the \( Y_n^m(\theta, \phi) \) are eigenfunctions of the angular part of the Laplace operator (40)

\[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) Y_n^m + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} Y_n^m = -n(n+1)Y_n^m. \]

Spherical harmonics are orthogonal

\[ \int_{0}^{2\pi} d\phi \int_{0}^{\pi} \sin \theta d\theta \{ Y_{n'}^m(\theta, \phi) \}^* Y_n^m(\theta, \phi) = \delta_{n',n} \delta_{m',m}. \]

They also satisfy the **Addition Theorem for Spherical Harmonics**, which we shall state but not prove here. Let \((\theta, \phi)\) and \((\theta', \phi')\) label two points on the surface of a sphere with angular separation \( \gamma \), so

\[ \cos \gamma = \sin \theta \sin \theta' \cos(\phi - \phi') + \cos \theta \cos \theta', \]

then the addition theorem states that

\[ P_n(\cos \gamma) = \left( \frac{4\pi}{2n+1} \right) \sum_{m=-n}^{n} \{ Y_n^m(\theta', \phi') \}^* Y_n^m(\theta, \phi). \]

Spherical harmonics can be used to expand solutions of the 3-dimensional Laplace equation in spherical polar co-ordinates (this is just another way of writing equation (45)),

\[ u(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( A_{n,m} r^n + B_{n,m} r^{-(n+1)} \right) Y_n^m(\theta, \phi), \]

where the constants \( A_{n,m} \) and \( B_{n,m} \) must satisfy \( A_{n,-m} = (-1)^m A_{n,m}^* \) and \( B_{n,-m} = (-1)^m B_{n,m}^* \) in order to ensure that \( u(r, \theta, \phi) \) is a real function. In order to specify \( A_{n,m} \) and \( B_{n,m} \) uniquely boundary conditions on \( u \) must be given, usually by specifying the value of \( u \) on some sphere of fixed radius \( a \), \( u(a, \theta, \phi) = f(\theta, \phi) \) with \( f(\theta, \phi) \) given.
Complex Analysis

1. Complex Functions

Denote the set of all real numbers by \( \mathbb{R} \). A complex number, \( z \), is a linear combination of two real numbers, \( x \) and \( y \),

\[
z = x + iy,
\]

where \( i^2 = -1 \). Complex numbers can be thought of as pairs of real numbers \( (x, y) \) and so can be interpreted as points in a 2-dimensional plane, called the complex plane, with \( x \) and \( y \) Cartesian co-ordinates in the plane. But they are more than that, because two complex numbers can be multiplied together to give a third. The complex conjugate of a complex number is denoted by \( z^* \) with

\[
z^* = x - iy.
\]

We shall denote the set of all complex numbers by \( \mathbb{C} \).

A real function \( f(x) \) assigns a real number \( f(x) \) to every real number \( x \in \mathbb{R} \). A complex function \( f(z) \) assigns a complex number, \( f(z) \), to every \( z \in \mathbb{C} \). A complex function therefore has a real and an imaginary part,

\[
f(z) = u(x, y) + iv(x, y)
\]

where \( u(x, y) \) and \( v(x, y) \) are real functions. The function \( u \) is called the real part of \( f \), and is often denoted by \( \Re f \), while \( v \) is called the imaginary part and is denoted by \( \Im f \).

In general \( f(z) \) can depend on \( z \) and \( z^* \) independently, e.g.

\[
\begin{align*}
f(z, z^*) &= zz^* = x^2 + y^2 \quad \text{(in this case } v = 0) \\
f(z, z^*) &= z(z^* + 1) = x^2 + y^2 + x + iy,
\end{align*}
\]

but it may depend only on \( z \) (or only on \( z^* \)), e.g.

\[
\begin{align*}
f(z) &= z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy \\
f(z) &= e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y) \\
f(z) &= \cos z = \frac{1}{2} (e^{iz} + e^{-iz}) = \frac{1}{2} (e^{ix-y} + e^{-ix+y}) \\
&= \frac{1}{2} (e^{-y}(\cos x + i \sin x) + e^y(\cos x - i \sin x)) \\
&= \cosh y \cos x - i \sinh y \sin x.
\end{align*}
\]

In these last three examples \( f(z) \) can be written purely as a function of \( z \).

Sometimes it is convenient to use 2-dimensional polar co-ordinates, \( x = \rho \cos \phi \) and \( y = \rho \sin \phi \), so

\[
z = \rho (\cos \phi + i \sin \phi) = \rho e^{i\phi}.
\]
In particular, for $\rho = 1$ and $\phi = \pi$, we get Euler’s famous formula
\[ e^{i\pi} = -1. \]

Sometimes this polar decomposition is useful in functions, for example
\[ f(z) = \ln z = \ln(\rho e^{i\phi}) = \ln(\rho) + i\phi. \]

In particular, for $\phi = \pi$, we have $z = -\rho$ and, with $\rho > 0$,
\[ \ln(-\rho) = \ln(\rho) + i\pi \]
the logarithm of a negative number! There is a subtle ambiguity here though: since $e^{2\pi in} = 1$ for any integer $n$, we have $e^{i\phi} = e^{i(\phi + 2\pi n)}$ and
\[ \ln(z) = \ln(\rho e^{i(\phi + 2\pi n)}) = \ln(\rho) + i(\phi + 2\pi n). \]

We get a different value of $\log(z)$ for every integer $n$. This is really just a more extreme case of the familiar ambiguity $\sqrt{x^2} = \pm |x|$ associated with real functions. Indeed
\[ z^{\frac{1}{2}} = \sqrt{z} = \sqrt{\rho e^{i(\phi + 2\pi n)}} = \sqrt{\rho} e^{\frac{i\phi}{2}} e^{i\pi n} = \pm \sqrt{\rho} e^{\frac{i\phi}{2}}, \]
where $\sqrt{\rho}$ is taken to be the positive square root and the plus sign is for $n$ even, the minus sign for $n$ odd. The usual real case is $\phi = 0$. The complex function $z^{\frac{1}{2}}$ is a multi-valued function, it has two different values depending on whether $n$ is even or odd — these are called branches of the function. The function $f(z) = \ln(z)$ has an infinite number of branches, one for each integer $n$.

Many formula familiar from real analysis must be interpreted with care in complex analysis, for example
\[ 1^x = 1 \] (46)
for any real number $x$.\footnote{One way to prove this is to take logarithms, $\ln(1^x) = x \ln(1) = 0$ for all real $x$, but the only real number whose logarithm is zero is one, therefore $1^x = 1$, for all $-\infty < x < \infty$.}

Consider the complex expression, with $n$ an integer,
\[ 1^z = (e^{2\pi in})^{(x+iy)} = e^{2\pi inx} e^{-2\pi ny} \]
and again the result is different for every $n$, there is an infinite number of branches. So $1^z \neq 1$ for all $z$, unless $n = 0$, and this has implications for raising any number to a complex power. As another example take $e^z$, where $z = 1 + 2\pi ik$ with $k$ an integer. Then, since $e = e^{1+2\pi in}$,
\[ e^z = (e^{1+2\pi in})^{1+2\pi ik} = e^{(1+2\pi in)(1+2\pi ik)} = e^{1-4\pi^2 kn + 2\pi i(k+n)} = e^{1-4\pi^2 kn} \] (47)
where, in the last equation, we have used \( e^{2\pi i (k+n)} = 1 \) for any integers \( n \) and \( k \). So, for fixed \( k \), \( e^z \) has an infinite number of branches, one for each integer \( n \).

Now what about \( (e^z)^z \)? Well from (47)

\[
e^z = e^{1-4\pi^2 kn},
\]

so now

\[
(e^z)^z = \left(e^{1-4\pi^2 kn} e^{2\pi in'}\right)^z = \left(e^{1-4\pi^2 kn+2\pi in'}\right)^{1+2\pi ik}
\]

\[
= e^{1-4\pi^2 k(n+n')+2\pi i(n'+k-4\pi^2 k^2 n)}
\]

\[
= e^{1-4\pi^2 k(n+n')-8\pi^3 ik^2 n}
\]

(48)

where a second set of branches has been introduced, labelled by a new integer \( n' \). We now have a double infinity of branches, labelled by the two integers \( n \) and \( n' \). Compare this to

\[
e^{z^2} = \left(e^{1+2\pi in}\right)^{1-4\pi^2 k^2+4\pi ik}.
\]

Expanding the exponent

\[
e^{z^2} = \left(e^{1+2\pi in}\right)^{1-4\pi^2 k^2+4\pi ik}
\]

\[
= e^{1-4\pi^2 k(2n+k)+2\pi i(2k+n'-4\pi^2 nk^2)}
\]

\[
= e^{1-4\pi^2 k(2n+k)-8\pi^3 ink^2}.
\]

(49)

Comparing (48) and (49) we see that \( (e^z)^z = e^{z^2} \) if and only if

\[
n + n' = 2n + k \quad \Rightarrow \quad n' = n + k.
\]

This only makes sense, when \( z = 1 + 2\pi ik \), if we are careful how we choose the branches: there is only one set of branches, labelled by \( n \), and we must choose \( n' = n + k \) in (48) giving

\[
(e^z)^z = e^{z^2} = e^{1-4\pi^2 k(2n+k)-8\pi^3 ink^2 n}.
\]

Note that the choice \( n' = n + k \) is exactly what we would have found if we had never used \( e^{2\pi i (k+n)} = 1 \) in the last equation of (47), but instead had kept the phase \( 2\pi(n+k) \) in the exponent when calculating \( (e^z)^z \).

In fact it should be clear that we can iterate this to show that

\[
\underbrace{(\langle (e^z)^z \rangle \cdots)^z}_\text{\(N\) times} = e^{z^N}
\]

and the different branches can be consistently treated by replacing

\[
e \to e^{1+2\pi in}
\]
on both sides, provided the phase is never dropped, since then

\[
\left( \left( \left( e^z \right)^z \right)^z \ldots \right) = \left( \left( e^{1+2\pi in} \right)^{1+2\pi ik} \right)^{1+2\pi ik}
\]

\[
\underbrace{\left( e^{(1+2\pi in)(1+2\pi ik)} \right)}_{N \text{ times}} = \underbrace{\left( e^{1+2\pi in} \right)^{1+2\pi ik}}_{N \text{ times}} = e^{zN}
\]

is always true, for any integer \( n \).

Failure to choose the right branches can lead to errors, for example setting \( n = n' = 0 \) in (48) and (49) and assuming \((e^z)^z = e^{z^2}\) gives the equation

\[
e = e^{1-4\pi^2k^2},
\]

which is obviously only valid for \( k = 0 \). There is a whole class of conundrums in the theory of complex numbers based on this kind of example,* arguments that lead to apparent paradoxes, like (50) with \( k \neq 0 \), which are the result of choosing the wrong branch!

2. Differentiation

When \( f(z) \) is independent of \( z^* \) a natural definition of its derivative would be

\[
\frac{df}{dz} := \lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}
\]

but we must be careful to check that this definition is consistent. A potential problem arises because \( \delta z \) can be real or imaginary, or indeed can point in any direction in the complex plane. The limit \( \delta z \to 0 \) only makes sense if it is independent of this direction. To determine whether or not this is the case we write \( f = u + iv, \delta f = \delta u + i\delta v \) and \( \delta z = \delta x + i\delta y \) so

\[
\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}.
\]

Then, when \( \delta y = 0 \), we have

\[
\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta x \to 0} \left( \frac{\delta u + i\delta v}{\delta x} \right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}
\]

and, when \( \delta x = 0 \), we have

\[
\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta y \to 0} \left( \frac{\delta u + i\delta v}{i\delta y} \right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.
\]

* This known as Clausen’s puzzle.
Demanding that these are the same means that their real and imaginary parts must be the same, so

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.
\]

These conditions are not only necessary for the derivative of \( f(z) \) to exist, they are also sufficient, essentially because an arbitrary direction for \( \delta z \) is just a linear combination of \( \delta z = x \) and \( \delta z = iy \).

The conditions (51) that \( u(x, y) \) and \( v(x, y) \) must satisfy for \( \frac{df}{dz} \) to be well defined are called the Cauchy-Riemann conditions.

If a function \( f(z) \) is differentiable in some neighbourhood of a point \( z_0 \) then it is said to be analytic in that neighbourhood. An alternative name, synonymous with analytic, is holomorphic (from the Greek \( \alpha\lambda\omega\sigma \) ‘holos’, meaning whole, and \( \mu\rho\phi\eta \) ‘morphe’, meaning shape). If \( f(z) \) is analytic everywhere in the complex plane (except possibly for \( z \to \infty \)) then it is said to be an entire function.

Examples:

\[i) \quad f(z) = z^2 \quad \Rightarrow \quad u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy\]

\[
\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y}.
\]

The derivatives exist everywhere except for \( x \to \infty \) and \( y \to \infty \) so \( z^2 \) is an entire function.

\[ii) \quad f(z) = \frac{1}{z} = \frac{x - iy}{x^2 + y^2} \quad \Rightarrow \quad u(x, y) = \frac{x}{x^2 + y^2}, \quad v(x, y) = -\frac{y}{x^2 + y^2}\]

\[
\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{2yx}{(x^2 + y^2)^2} = -\frac{\partial v}{\partial x}.
\]

The derivatives exist everywhere except at the origin \( x = y = 0 \), so \( 1/z \) is analytic everywhere except at the origin.

 Functions that depend on \( z^* \) are not analytic

\[iii) \quad f(z^*) = z^* = x - iy \quad \Rightarrow \quad u(x, y) = x, \quad v(x, y) = -y\]

\[
\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = -1,
\]

so \( z^* \) does not satisfy the Cauchy-Riemann conditions.

\[iii) \quad f(z, z^*) = zz^* \quad \Rightarrow \quad u(x, y) = x^2 + y^2, \quad v(x, y) = 0\]
\[ \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0, \]

and again the Cauchy-Riemann conditions are not satisfied.

Analytic functions are related to the 2-dimensional Laplace equation:

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 u}{\partial y^2} \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

\[ \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial y^2} \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, \]

so both \( u(x, y) \) and \( v(x, y) \) satisfy the 2-dimensional Laplace equation.

3. Integration

An analytic function \( f(z) \) can be integrated along a curve connecting two fixed points, \( z_0 \) and \( z_0' \), by splitting the curve up into a large number of small segments with endpoints \( z_0 < z_1 < z_2 < \cdots < z_{N-1} < z_N = z_0' \) and then letting \( N \to \infty \). Let \( z_{j-1} < \zeta_j < z_j \) be fixed points in the \( j \)-th segment, as shown below,

then the integral can be defined as

\[ \int_{z_0}^{z_0'} f(z) \, dz = \lim_{N \to \infty} \sum_{j=1}^{N} f(\zeta_j)(z_j - z_{j-1}), \]

and is independent of the choices for \( \zeta_j \) in the limit.

Cauchy’s Integral Theorem

If \( f(z) \) is analytic within and on a closed curve \( C \), then

\[ \oint_C f(z) \, dz = 0. \]
The symbol \( \oint \) here indicates integration around a closed loop and a closed curve \( C \) is often called a \textit{contour} in complex integration.

\textbf{Proof:} first split \( f(z) \) and \( z \) up into real and imaginary parts,

\[
\oint_C f(z)dz = \oint_C (u + iv)(dx + iy) = \oint_C (udx - vdy) + i \oint_C (vdx + udy).
\]

Now apply Stokes’ theorem which states that, for any vector field \( \mathbf{V} = V_x \hat{x} + V_y \hat{y} \) with \( x \) and \( y \)-components \( V_x \) and \( V_y \),

\[
\oint_C (V_x dx + V_y dy) = \int_S \left( \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) dxdy,
\]

where \( S \) is the 2-dimensional region bounded by the curve \( C \). Applying this to the real part of (52), with \( V_x = u \) and \( V_y = -v \) gives

\[
\oint_C (udx - vdy) = \int_S \left( -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy,
\]

which vanishes from the Cauchy-Riemann conditions (51). Similarly applying Stokes’ theorem to the imaginary part of (52), with \( V_x = v \) and \( V_y = u \) gives

\[
\oint_C (vdx + udy) = \int_S \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy,
\]

which again vanishes from the Cauchy-Riemann conditions.

This shows that both the real and imaginary parts of \( \oint_C f(z)dz \) vanish if \( f(z) \) satisfies the Cauchy-Riemann conditions in the region \( S \) bounded by the curve \( C \).

An immediate consequence of Cauchy’s integral theorem is that, if a function is analytic in some region containing two points \( z_0 \) and \( z'_0 \), then the integral along a curve connecting \( z_0 \) and \( z'_0 \) is independent of the curve chosen, provided it does not stray out of the region in which \( f(z) \) is analytic.* This is because Cauchy’s integral theorem shows that integrating from \( z_0 \) to \( z'_0 \) along one curve and then integrating back from \( z'_0 \) to \( z_0 \) along a different curve gives opposite answers. Combining the two integrals into a single integral along a closed curve passing through \( z_0 \) and \( z'_0 \) is then zero.

\textbf{Cauchy’s Integral Formula}

* If the real and imaginary parts \( f(z) \) are thought of as potential energies for 2-dimensional problems in mechanics then this is analogous to the concept of conserved forces.
If \( f(z) \) is analytic within and on a closed curve \( C \), then

\[
\oint_C \frac{f(z)}{(z-z_0)}\,dz = 2\pi i f(z_0)
\]

where \( z_0 \) is an arbitrary point inside \( C \).

There is a convention here that the integral is performed in an anti-clockwise direction around the curve \( C \) — be careful of this: failure to adhere to this convention can introduce minus signs into some of the formulae!

**Proof:** although, by assumption, \( f(z) \) is analytic within \( C \), \( f(z)/(z-z_0) \) is not analytic at \( z = z_0 \). However, if we choose a different curve \( C' \) as shown below, which excludes \( z_0 \), then \( f(z)/(z-z_0) \) is analytic within and on \( C' \). Hence \( \oint_{C'} f(z)\,dz = 0 \) from Cauchy’s integral formula.

The curve \( C' \) can be decomposed as

\[ C' = C_{12} + C_{23} + C_{34} + C_{41} \]

and then we have

\[
\oint_{C'} \frac{f(z)}{(z-z_0)}\,dz = \int_{C_{12}} \frac{f(z)}{(z-z_0)}\,dz + \int_{C_{23}} \frac{f(z)}{(z-z_0)}\,dz + \int_{C_{34}} \frac{f(z)}{(z-z_0)}\,dz + \int_{C_{41}} \frac{f(z)}{(z-z_0)}\,dz = 0,
\]
where \( \int_{C_{12}} \frac{f(z)}{z - z_0} dz = \int_{z_1}^{z_2} \frac{f(z)}{(z - z_0)} dz \) along the segment \( C_{12} \), etc. In the limit \( z_4 \to z_1 \) and \( z_3 \to z_2 \), the segments \( C_{12} \) and \( C_{34} \) coincide, but the integrals are performed in opposite directions, so

\[
\int_{C_{12}} \frac{f(z)}{z - z_0} dz = -\int_{C_{34}} \frac{f(z)}{z - z_0} dz, \tag{53}
\]

leaving

\[
\oint_{C_{34}} \frac{f(z)}{z - z_0} dz = -\oint_{C_{23}} \frac{f(z)}{z - z_0} dz, \tag{54}
\]

where \( \lim_{z_4 \to z_1} C_{41} = C \) and \( C_{23} \) is now also a closed curve since we have set \( z_2 = z_3 \). Now we choose \( C_{23} \) to be a small circle of radius \( \epsilon \) centred on \( z_0 \) and parameterise the points on it by an angle \( \phi \), where

\[
z = z_0 + \epsilon e^{i\phi} \quad \text{so} \quad dz = i\epsilon e^{i\phi} d\phi,
\]

and \( \phi \) increases in an anti-clockwise direction. We must be careful of signs because the contour \( C_{23} \) is defined above as being traversed in the clockwise direction, so this introduces a minus sign.

\[
\oint_{C_{23}} \frac{f(z)}{(z - z_0)} dz = -\int_0^{2\pi} \frac{f(z_0 + \epsilon e^{i\phi})}{\epsilon e^{i\phi}} (i\epsilon e^{i\phi}) d\phi
\]

\[
= -i \int_0^{2\pi} f(z_0 + \epsilon e^{i\phi}) d\phi
\]

\[
\longrightarrow \epsilon \to 0 - i \int_0^{2\pi} f(z_0) d\phi
\]

\[
= -2\pi i f(z_0).
\]

where the limit \( \epsilon \to 0 \) has been taken in the third step. Using this in (54), with \( C_{41} = C \), now gives Cauchy’s integral formula:

\[
\oint_{C} \frac{f(z)}{(z - z_0)} dz = 2\pi i f(z_0). \tag{55}
\]

Differentiating Cauchy’s integral formula with respect to \( z_0 \) gives us

\[
f'(z_0) = \frac{1}{2\pi i} \oint_{C} \frac{f(z)}{(z - z_0)^2} dz.
\]

More generally, differentiating \( n \) times we get

\[
f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{C} \frac{f(z)}{(z - z_0)^{n+1}} dz, \tag{55}
\]

where \( f^{(n)}(z_0) := \frac{d^n}{dz_0^n} f(z_0) \).
4. Laurent Expansions and Complex Analyticity

A **Laurent expansion** of a complex function is a generalisation of Taylor expansions, familiar from real analysis and we shall start by describing Taylor expansions for complex functions.

**Taylor Expansions**

Suppose a function \( f(z) \) is analytic in some region containing a point \( z_0 \) and let \( z_1 \) be the nearest point to \( z_0 \) at which \( f(z) \) is non-analytic (\( z_1 \) might be at infinity, but it could also be a finite distance from \( z_0 \), as in the picture below). Let \( C \) be a contour which encloses \( z_0 \) but does not extend as far out as \( z_1 \).

\[
|z_1-z_0|
\]

\[
|z'-z_0|
\]

Then, from Cauchy’s integral formula, for any other point \( z \) inside \( C \)

\[
f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')dz'}{(z'-z)} = \frac{1}{2\pi i} \oint_C \frac{f(z')dz'}{(z'-z_0)} \left[ 1 - \frac{z-z_0}{z'-z_0} \right].
\]  \hspace{2cm} (56)

Now \( \left[ 1 - \left( \frac{z-z_0}{z'-z_0} \right) \right]^{-1} \) can be expanded in the usual way. Let \( w = \frac{z-z_0}{z'-z_0} \) and \( S_N = \sum_{n=0}^{N} w^n \). Then

\[
(1-w)S_N = \sum_{n=0}^{N} w^n - \sum_{n=1}^{N+1} w^n = 1 - w^{N+1}.
\]

Writing \( w = \rho e^{i\phi} \) we have \( \rho^{N+1} \to 0 \) as \( N \to \infty \), provided \( \rho < 1 \), hence \( w^{N+1} \to 0 \) as \( N \to \infty \) provided \( |w| < 1 \). So we see that

\[
(1-w)S_\infty = 1 \quad \Rightarrow \quad \frac{1}{1-w} = \sum_{n=0}^{\infty} w^n.
\]
This formula should be familiar for real \( w \) but it is also valid for complex \( w \), if \( |w| < 1 \). If the points \( z \) and \( z_0 \) are chosen so that \( |z - z_0| < |z' - z_0| < 1 \) for every point \( z' \) on the contour \( C \), then

\[
\frac{1}{1 - \frac{z - z_0}{z' - z_0}} = \sum_{n=0}^{\infty} \left( \frac{z - z_0}{z' - z_0} \right)^n
\]

and this can be used in (56) to give

\[
f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_C \frac{f(z')dz'}{(z' - z_0)^{n+1}}.
\]

Now, using (55), this can be re-expressed as

\[
f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} (z - z_0)^n f^{(n)}(z_0).
\]

(57)

This is exactly the same as the usual formula for the Taylor expansion for a real function. For complex functions we see that the formula works provided \( |z - z_0| < |z' - z_0| \), where \( z' \) lies on the contour used in (55). The only requirement that this contour has to satisfy is that \( f(z) \) is analytic within and on \( C \), so we may as well take \( C \) as large as we can. We can therefore take \( C \) to be a circle centred on \( z_0 \) and just inside \( z_1 \), the nearest point of non-analyticity of \( f(z) \) to \( z_0 \). The Taylor expansion (57) works for any point \( z \) that is closer to \( z_0 \) than \( z_1 \).

**Analytic Continuation**

Non-analytic points are extremely important. They are obstructions to extending series expansions of \( f(z) \) beyond certain limits. For example \( f(z) = \frac{1}{1+z} \) is non-analytic at \( z = -1 \). Taylor expanding about \( z_0 = 0 \),

\[
\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n z^n,
\]

is valid provided \( |z| < 1 \). The unit circle centred on the origin, \( |z| = 1 \), is called the **circle of convergence** for this expansion. The radius of this circle, unity in this case, is called the **radius of convergence**. Strictly speaking \( \frac{1}{1+z} \) and \( \sum_{n=0}^{\infty} (-1)^n z^n \) are different functions: they co-incide for \( |z| < 1 \) they but are not the same for \( |z| > 1 \), since the former is perfectly well behaved for \( |z| > 1 \) while the latter is not even defined in this region as the sum diverges. We might get a different radius of convergence if we Taylor expand about a different point though. For example expanding about \( z_0 = i \), we have

\[
f(z) = \frac{1}{1+z} = \frac{1}{(1+i)} \left( 1 + \frac{z-i}{1+i} \right)
\]

\[
= \frac{1}{(1+i)} \sum_{n=0}^{\infty} (-1)^n \left( \frac{z-i}{1+i} \right)^n,
\]
and this expansion converges provided \(|\frac{z - i}{1 + i}| < 1\), i.e. provided \(|z - i| < |1 + i| = \sqrt{2}\).

This circle of convergence has radius \(\sqrt{2}\) and is centred at \(z_0 = i\): it has a greater radius of convergence than the expansion about \(z_0 = 0\) and the two expansions are valid in different regions. This does not mean that this expansion is any better or worse than the previous one, they are just valid in different regions as shown in the following picture. The expansion about \(z_0 = 0\) is valid inside the solid circle and the expansion about \(z_0 = i\) inside the dashed circle, both expansions are valid in the region where the circles overlap.

Thus the sum \(\sum_{n=0}^{\infty} (-1)^n z^n\) is not defined outside \(|z| < 1\), but can be continued into the crescent shaped region above the semi-circular arch between 1 and \(-1\) by expanding about \(z_0 = i\) to give \(\frac{1}{1+i} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z - i}{1 + i}\right)^n\). In this manner the region in which the expansion is valid can be continued to different areas of the complex plane. By choosing many different points to expand about we can build up a patchwork covering the whole complex plane, excluding the point of non-analyticity \(z = -1\), giving a collection of infinite sums, one for each expansion point, all of which co-incide with the function \(\frac{1}{1+z}\) in the regions in which they converge. This process is called **analytic continuation** of the original expansion, \(\sum_{n=0}^{\infty} (-1)^n z^n\), outside of the region \(|z| < 1\).

Analytic continuation is very important in complex analysis since many functions can be defined through a series expansion, e.g.

\[
e^z = 1 + z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} z^n.
\]

This expansion actually converges for all \(|z| < \infty\), it has an infinite radius of convergence and is an example of an entire function.

**Laurent Expansions**

Let \(f(z)\) be analytic at \(z_0\) and let \(z_1\) be the nearest point of non-analyticity. Suppose there are no other points of non-analyticity out as far as another point \(z_2\). Let \(z\) be a point between \(z_1\) and \(z_2\), so

\[|z_1 - z_0| < |z - z_0| < |z_2 - z_0|.
\]
This means that \( f(z) \) is analytic in the region between the two curves \( C_1 \) and \( C_2 \) in the following figure:

We can combine \( C_1 \) and \( C_2 \) into one continuous curve, \( C' \), by cutting the annular region between them along the dotted lines indicated in the figure and including the dotted lines in the contour. The function \( f(z) \) is analytic in the region enclosed by \( C_1 \), \( C_2 \) and the dotted lines, so Cauchy’s integral formula (54) implies that

\[
f(z) = \frac{1}{2\pi i} \oint_{C'} \frac{f(z')dz'}{(z' - z)} = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')dz'}{(z' - z)} - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z')dz'}{(z' - z)}. \tag{58}\]

In the second equation above we have taken the limit of the two dotted line segments coinciding, so that the integrals along them exactly cancel, and we have chosen to integrate in an anti-clockwise direction along \( C_1 \), which then requires a clockwise integration around \( C_2 \) — hence the minus sign in the latter integral.

Expanded the denominators of the integrands in the same way as we did for the Taylor expansion previously we have, for \( C_1 \),

\[
\frac{1}{2\pi i} \oint_{C_1} \frac{f(z')dz'}{(z' - z)} = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')dz'}{(z' - z_0)[1 - (\frac{z - z_0}{z' - z_0})]} = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_{C_1} \frac{f(z')dz'}{(z' - z_0)^{n+1}}. \tag{59}\]

The sum converges since, by construction, \( \frac{z - z_0}{z' - z_0} \) is less than 1 for \( z' \) on \( C_1 \) (see the figure above).

While, for \( C_2 \),

\[
\frac{1}{2\pi i} \oint_{C_2} \frac{f(z')dz'}{(z' - z)} = -\frac{1}{2\pi i} \oint_{C_2} \frac{f(z')dz'}{(z - z_0)[1 - (\frac{z' - z_0}{z - z_0})]}.
\]
\[ f(z) = \frac{1}{2\pi i} \left\{ \sum_{n=0}^{\infty} (z - z_0)^n \oint_{C_1} \frac{f(z')dz'}{(z' - z_0)^{n+1}} + \sum_{n=1}^{\infty} (z - z_0)^{-n} \oint_{C_2} \frac{f(z')dz'}{(z' - z_0)^{-n+1}} \right\} \]

\[ = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_{C_1} \frac{f(z')dz'}{(z' - z_0)^{n+1}} + \sum_{n=-1}^{\infty} (z - z_0)^n \oint_{C_2} \frac{f(z')dz'}{(z' - z_0)^{n+1}} \]

\[ = \sum_{n=-\infty}^{\infty} a_n(z_0)(z - z_0)^n, \]

where

\[ a_n(z_0) = \begin{cases} \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')dz'}{(z' - z_0)^{n+1}} & \text{for } n \geq 0 \\ \frac{1}{2\pi i} \oint_{C_2} \frac{f(z')dz'}{(z - z_0)^{n+1}} & \text{for } n < 0. \end{cases} \]

Now observe that the integrals involved in calculating \( a_n(z_0) \) are completely independent of \( z \), they depend only on \( z_0 \), and \( f(z')/(z' - z_0)^{n+1} \) is analytic in the annulus between \( z_2 \) and \( z_1 \). Due to Cauchy’s integral theorem, we can distort \( C_1 \) and \( C_2 \) until they lie on top of each other, giving a single curve \( C = C_1 = C_2 \), without changing the value of \( a_n(z_0) \) (provided neither \( C_1 \) nor \( C_2 \) passes through the points of non-analyticity, \( z_1 \) and \( z_2 \), in the process). Hence we finally arrive at

\[ f(z) = \sum_{n=-\infty}^{\infty} a_n(z_0)(z - z_0)^n \quad \text{with} \quad a_n(z_0) = \frac{1}{2\pi i} \oint_{C} \frac{f(z')dz'}{(z' - z_0)^{n+1}}, \quad (61) \]

where \( C \) is any curve threading between the two points of non-analyticity nearest to \( z_0 \), \( z_1 \) and \( z_2 \), as shown below:
Equation (61) is called the **Laurent expansion** of \( f(z) \) about \( z_0 \). Note that it differs from the Taylor expansion in that the summation contains negative powers of \((z - z_0)\), this is an inevitable consequence of the existence of the point of non-analyticity, \( z_1 \), lying between \( z \) and \( z_0 \). The summation may truncate at a finite negative power if there exists some \( N > 0 \) such that \( a_{-n}(z_0) = 0, \forall n > N \).

As a final point, before going on to give an example of a Laurent expansion, note that nothing stops us taking the limit \( z_1 \to z_0 \) so that \( z_0 \) becomes a non-analytic point and \( z_2 \) is the next closest non-analytic point to \( z_0 \). This limit allows us to discuss expansions about non-analytic points.

**Example**

\[
 f(z) = \frac{1}{z(1 - z)}
\]

This function has two points of non-analyticity, \( z = 0 \) and \( z = 1 \). Now perform a Laurent expansion around the point \( z_0 = 0 \). The next nearest point of non-analyticity to \( z = 0 \) is \( z = 1 \), so we can use the expression for \( a_n \) in (61) with the curve \( C \) any loop around the origin which has no point greater than a unit distance from the origin, i.e. \(|z'| < 1\) for all points \( z' \) on \( C \). Equation (61) then gives

\[
 a_n(0) = \frac{1}{2\pi i} \oint_C \frac{dz'}{(z')^{n+1}z'(1 - z')}
 = \frac{1}{2\pi i} \oint_C \frac{dz'}{(z')^{n+2}(1 - z')}
 = \frac{1}{2\pi i} \oint_C \frac{dz'}{(z')^{n+2} \sum_{m=0}^{\infty} (z')^m}
 = \frac{1}{2\pi i} \sum_{m=0}^{\infty} \oint_C (z')^{m-n-2} dz'.
\]

By Cauchy’s integral theorem the integrals are independent of the curve chosen, provided \( C \) is restricted to lie between \(|z| = 1\) and \( z = 0 \), so we can evaluate the integrals by choosing \( C \) to be a circle of radius \( a < 1 \), centred on the origin. On this circle \( z' = ae^{i\phi'} \), so \( dz' = aie^{i\phi'} d\phi' \), and

\[
 a_n(0) = \frac{i}{2\pi i} \sum_{m=0}^{\infty} a^{m-n-1} \int_0^{2\pi} e^{i(m-n-1)\phi'} d\phi'.
\]

Now

\[
 \int_0^{2\pi} e^{i(m-n-1)\phi'} d\phi' = \int_0^{2\pi} \cos((m - n - 1)\phi')d\phi' + i \int_0^{2\pi} \sin((m - n - 1)\phi')d\phi'
 = 2\pi \delta_{m,n+1},
\]

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so
\[ a_n(0) = \sum_{m=0}^{\infty} a^{m-n-1}\delta_{m,n+1} = \begin{cases} 1, & n \geq -1 \\ 0, & n < -1 \end{cases} \text{ independent of } a. \]

Using this in (61) gives
\[ f(z) = \frac{1}{z} + 1 + z + z^2 + z^3 \cdots = \sum_{n=-1}^{\infty} z^n. \]

This example has been used to illustrate the application of the integral form of \( a_n(z_0) \) in (61), but in this particular case there is a quicker way of getting the Laurent expansion. Since the function \( f(z) = \frac{1}{z(1-z)} \) has two factors, \( 1/z \) and \( 1/(1-z) \), we can Taylor expand \( 1/(1-z) \) around \( z_0 = 0 \) to obtain
\[ \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \]
(for \( |z| < 1 \)) and then multiply by \( 1/z \) to get
\[ f(z) = \frac{1}{z} \sum_{n=0}^{\infty} z^n = \sum_{n=-1}^{\infty} z^n \]
as before. Sometimes this can be a useful trick to obtain the Laurent expansion of a function without having to do the integrals in (61).

5. Poles and Branch Cuts

In this section we introduce a classification of the singularities (points of non-analyticity) that a function \( f(z) \) might have. If \( f(z) \) is non-analytic at a point \( z_0 \), but is analytic at all neighbouring points, then \( z_0 \) is called an isolated singularity of \( f(z) \). All the non-analytic behaviour we have encountered so far (at least for functions that depend only on \( z \) and not both \( z \) and \( z^* \)) has been of this form, for example \( f(z) = 1/z \) has an isolated singularity at \( z = 0 \). In a Laurent expansion about an isolated singularity \( z_0 \),
\[ f(z) = \sum_{n=-\infty}^{\infty} a_n(z_0)(z-z_0)^n, \]
if \( a_n(z_0) = 0 \) for all \( n < -N \) and \( a_{-N}(z_0) \neq 0 \), with \( N > 0 \), then \( f(z) \) is said to have a pole of order \( N \) at \( z_0 \). A pole of order one is called a simple pole. A function which is analytic everywhere except for isolated poles of finite order is called a meromorphic function (from the Greek \( \mu\epsilon\rho\omicron\sigma \) ‘meros’, meaning part).
Examples

i) \( f(z) = \frac{1}{z(1-z)} = \sum_{n=-1}^{\infty} z^n \)

has \( a_{-1}(0) = 1 \) and \( a_n(0) = 0 \) for all \( n < -1 \), so \( N = 1 \) in this case, and \( z = 0 \) is a simple pole. The other non-analytic point, at \( z = 1 \), is also a simple pole.

ii) \( f(z) = \frac{1}{z^2(1-z)} = \sum_{n=-2}^{\infty} z^n \)

has a pole of order two (sometimes called a double pole) at \( z = 0 \) and a simple pole at \( z = 1 \),

iii) \( f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1}{z} \right)^n = \sum_{n=-\infty}^{0} \frac{1}{|n|!} z^n. \)

In this example the Laurent series does not terminate at any finite \( N \), but extends all the way down to \( N = -\infty \). A singularity of this form is called an essential singularity.

Branch Points

Consider the function \( f(z) = z^\alpha \) when \( \alpha \) is not an integer. For example if \( \alpha = 1/2 \) then we can use the polar decomposition \( z = \rho e^{i\phi} \) to express \( f(z) \) as

\[
\begin{align*}
   f(z) &= e^{1/2} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1}{z} \right)^n = \sum_{n=-\infty}^{0} \frac{1}{|n|!} z^n.
\end{align*}
\]

A little thought reveals that this is actually not a very well defined function. Suppose we start at \( z = 1 \) and take \( z \) around the unit circle centred on \( z = 0, z = e^{i\phi} \). Writing \( f = e^{i\phi/2} \) as a function of \( \phi \) we have \( f(\phi = 0) = 1 \) and as we go round from \( \phi = 0 \) to \( \phi = 2\pi \) we return to \( z = 1 \) but we find that \( f(\phi = 2\pi) = e^{i\pi} = -1 \). Thus \( f = \pm 1 \) at \( z = 1 \), reflecting the usual sign ambiguity in a square root. A similar ambiguity in sign is present for every value of \( z \), except \( z = 0 \), so \( f(z) = z^{1/2} \) really has two values for all \( z \neq 0 \) — it is said to be multi-valued. The function \( z^{1/2} \) is said to have two branches and the point where the two branches coincide, \( z = 0 \), is called a branch point.

We can however construct a single-valued function, \( f_s(z) \), from \( f(z) = z^{1/2} \) at the expense of introducing a discontinuity. We first of all demand that on the positive real axis, with \( \phi = 0 \) and \( z = \rho = x \), \( f_s(x) = \sqrt{x} \), with a plus sign. We then demand that, for \( 0 \leq \phi < 2\pi \), the phase of \( f_s(z) \) is \( \phi/2 \). This means that, when we go all the way round the origin to just below the positive real axis, \( f_s(z) \) has changed sign, but it jumps back to the positive sign on the positive real axis. We now have a single-valued function, but it is discontinuous (and so not differentiable) across the positive real axis. Actually we
could have chosen any line emanating from the origin and extending out to infinity to get a single-valued, discontinuous function in this way, but it would be a different function for different choices of line. We could even have chosen a curved line, it does not have to be straight. The line chosen in order to construct a discontinuous single-valued function associated with a continuous multi-valued function is called a cut-line for the function.

\[ f(z) = e^{2\pi i \rho /2} \]

\[ A \text{ cut-line for the function } f(z) = z^\alpha, \text{ with } \alpha \notin \mathbb{Z}. \]

In the example chosen above, \( f(z) = z^{1/2} \), the derivative of \( f, f' = \frac{1}{2} z^{-1/2} \), is singular at the branch-point \( z = 0 \), but this need not be the case. For example \( f(z) = z^{3/2} \) also has a branch-point at \( z = 0 \) but its first derivative is perfectly finite there (though its second derivative is singular at \( z = 0 \)).

6. Louiville’s Theorem

A function that is everywhere finite and analytic is a constant.

**Proof:** By assumption \( f(z) \) is everywhere finite, so \( \exists K \) such that \( |f(z)| < K, \forall z \). Cauchy’s integral formula then implies that, for any two points \( z_1 \) and \( z_2 \),

\[
\left| f(z_1) - f(z_2) \right| = \frac{1}{2\pi i} \oint_C \left\{ \frac{1}{z' - z_1} - \frac{1}{z' - z_2} \right\} f(z')dz' \]

with \( C \) any closed curve encircling both \( z_1 \) and \( z_2 \). From this we deduce that

\[
\left| f(z_1) - f(z_2) \right| < \frac{K}{2\pi} \left| \oint_C \left\{ \frac{1}{z' - z_1} - \frac{1}{z' - z_2} \right\} dz' \right|. \]

We are free to chose \( C \) to be a circle of radius \( R \) centred on \( z_1 \) large enough to enclose \( z_2 \). Parameterise this circle by \( z' = z_1 + R e^{i\phi'} \), so that \( dz' = R i e^{i\phi'} d\phi' \), and let \( R > 2|z_1 - z_2| \). Then

\[
\left| \oint_C \left\{ \frac{1}{z' - z_1} - \frac{1}{z' - z_2} \right\} dz' \right| \leq \left| \int_0^{2\pi} \frac{(z_1 - z_2)}{(z' - z_1)(z' - z_2)} d\phi' \right| \leq \int_0^{2\pi} \frac{|z_1 - z_2|}{|z' - z_2|} d\phi', \quad \text{since } |z' - z_1| = R
\]

\[
\leq \int_0^{2\pi} \frac{|z_1 - z_2|}{|z' - z_2|} d\phi', \quad \text{since } \left| \int f(z')dz' \right| \leq \int |f(z')dz'|. \]
Choosing $R > 2|z_2 - z_1|$ implies that $|z' - z_2| > R/2$, so

$$|f(z_1) - f(z_2)| < \frac{K |z_1 - z_2|}{2\pi (R/2)} \int_0^{2\pi} d\phi' = \frac{2K}{R} |z_2 - z_1| \xrightarrow{R \to \infty} 0.$$ 

Since $|f(z_1) - f(z_2)|$ cannot have any explicit dependence on $R$ we conclude that

$$|f(z_1) - f(z_2)| = 0 \quad \Rightarrow \quad f(z_1) = f(z_2) \quad \forall z_1, z_2.$$

## 7. Calculus of Residues

Cauchy’s integral formula is extremely useful for calculating definite integrals. Consider the Laurent expansion of a function $f(z)$ around an isolated singular point $z_0$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z_0)(z - z_0)^n.$$

Take any contour $C$ encircling $z_0$, with no other singularities inside $C$, and let $\tilde{z}$ be any point on $C$, then

$$\oint_C a_n(z_0)(z - z_0)^n dz = a_n(z_0) \left. \frac{(z - z_0)^{n+1}}{(n+1)} \right|_{\tilde{z}} = 0, \quad \text{provided } n \neq -1.$$

The case $n = -1$ must be treated carefully. Since there are no singularities other than $z_0$ inside $C$, the contour can always be distorted to a circle of radius $r$ centered on $z_0$ without changing the value of the integral around $C$. So

$$\oint_C a_{-1}(z_0) \frac{dz}{(z - z_0)} = a_{-1}(z_0) \int_0^{2\pi} \frac{re^{i\phi} d\phi}{re^{i\phi}} = 2\pi i a_{-1}(z_0),$$

where the circular contour is parameterised by $z = z_0 + re^{i\phi}$. This means that

$$\frac{1}{2\pi i} \oint_C f(z) dz = a_{-1}(z_0). \quad (62)$$

This is a very powerful result, it means that we only need to know $a_{-1}(z_0)$ in order to calculate the integral — none of the other $a_n$’s is relevant! Note that this is true for any contour $C$, provided only that there are no singularities inside $C$ other than $z_0$. The co-efficient $a_{-1}(z_0)$ is called the **residue** because, like the grin of the Cheshire cat, it is all that remains of $f(z)$ after the integral is performed.
If $C$ encloses a finite set of isolated singularities, say $k + 1$ of them, like this
\[ z_0 \times z_1 \times \cdots \times z_k \]
then we can still do the integral. First deform $C$ to $C'$ as shown below,

Because $C'$ does not enclose any singularities we have
\[ \oint_{C'} f(z) \, dz = 0, \]
by Cauchy’s integral theorem. Now, when the straight line segments are taken infinitesimally close to one another, the integrals along opposing edges cancel and the perimeter becomes $C$, as in the proof of Cauchy’s integral formula given earlier. So
\[ \oint_{C'} f(z) \, dz = \oint_C f(z) \, dz + \oint_{C_0} f(z) \, dz + \oint_{C_1} f(z) \, dz + \cdots + \oint_{C_k} f(z) \, dz = 0. \]
This can be re-written as
\[ \oint_C f(z) \, dz = - \sum_{j=0}^{k} \oint_{C_j} f(z) \, dz. \quad (63) \]
Now each little counter $C_j$ encloses only one singularity, so we can use the result (62), taking note of the fact that the $C_j$ above are being traversed in a clockwise direction and (62) was derived by integrating around $C$ in an anti-clockwise direction, to deduce that
\[ \oint_{C_j} f(z) \, dz = -2\pi i a_{-1}(z_j). \]
Equation (63) now reads
\[ \oint_C f(z) \, dz = 2\pi i \{ a_{-1}(z_0) + a_{-1}(z_1) + \cdots + a_{-1}(z_k) \}, \]
leading to

**The Residue Theorem:**

\[
\oint_C f(z) \, dz = 2\pi i \sum_{j=0}^{k} a_{-1}(z_j).
\]

As an example of the application of the residue theorem consider the definite integral

\[
\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}.
\]

Treating the real line \(-\infty < x < \infty\) as the horizontal axis in the complex \(z\)-plane, with \(z = x + iy\), we can extend this to

\[
\int_C \frac{dz}{1 + z^2}
\]

where \(C\) is some contour that includes the real line. For example we can take \(C\) to be a large semi-circle above the real axis, centred on \(z = 0\) and radius \(R\), with its diameter lying along the real axis. Then, on the semi-circular arch, \(z = Re^{i\phi}\) and \(dz = Rie^{i\phi}d\phi\) with \(0 \leq \phi \leq \pi\), so

\[
\int_C \frac{dz}{1 + z^2} = Ri \int_0^{\pi} \frac{e^{i\phi} d\phi}{1 + R^2 e^{2i\phi}} + \int_{-R}^{R} \frac{dx}{1 + x^2}.
\]

As \(R \to \infty\) the first integral on the right-hand side vanishes so, taking the semi-circle to be of infinite radius,

\[
\int_C \frac{dz}{1 + z^2} = \int_{-\infty}^{\infty} \frac{dx}{1 + x^2}.
\]

Now we can use the residue theorem to evaluate the left-hand side. The integrand

\[
\frac{1}{1 + z^2} = \frac{1}{(1 + iz)(1 - iz)} = \frac{1}{(z + i)(z - i)}
\]

has two simple poles, one at \(z = i\) and one at \(z = -i\). The pole at \(z = -i\) lies below the real axis and is outside the contour \(C\), but the one at \(z = i\) is inside \(C\), and is the only point of non-analyticity inside \(C\). Expanding about \(z_0 = i\)

\[
\frac{1}{z + i} = \frac{1}{2i} \left[1 + \left(\frac{z-i}{2i}\right)\right] = \frac{1}{2i} \sum_{n=0}^{\infty} \left(-1\right)^n \left(\frac{z-i}{2i}\right)^n
\]

so

\[
\frac{1}{1 + z^2} = -\frac{1}{4} \sum_{n=-1}^{\infty} \left(-1\right)^{n+1} \left(\frac{z-i}{2i}\right)^n.
\]
The residue of the simple pole at \( z_0 = i \) is the \( n = -1 \) term:

\[
a_{-1}(z_0 = i) = \frac{1}{2i}.
\]

There is only one pole, and hence only one residue, within \( C \) so

\[
\int_C \frac{dz}{1 + z^2} = 2\pi i a_{-1}(z_0 = i) = \pi.
\]

Thus we have used the residue theorem to evaluate

\[
\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \pi.
\]

Evaluating definite integrals using the residue theorem in this way is called the **calculus of residues**.

8. Fourier Transforms

You are familiar with Fourier series for a function on an interval \(-T/2 < t < T/2\).

\[
f(t) = \sum_{n=0}^{\infty} A_n \cos \left( \frac{2\pi nt}{T} \right) + \sum_{n=1}^{\infty} B_n \sin \left( \frac{2\pi nt}{T} \right).
\]

where the constants \( A_n \) and \( B_n \) are determined, using orthogonality of the trigonometric functions for positive integers \( n \) and \( n' \)

\[
\int_{-\pi}^{\pi} \cos(n\theta) \cos(n'\theta) d\theta = \pi \delta_{nn'}, \quad \int_{-\pi}^{\pi} \sin(n\theta) \sin(n'\theta) d\theta = \pi \delta_{nn'}, \quad \int_{-\pi}^{\pi} \sin(n\theta) \cos(n'\theta) d\theta = 0,
\]

to be

\[
A_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos \left( \frac{2\pi nt}{T} \right) dt, \quad B_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin \left( \frac{2\pi nt}{T} \right) dt
\]

for positive \( n \), while

\[
A_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(t) dt.
\]

In the Fourier series \( \frac{n\pi}{T} \) is a frequency,

\[
\omega_n = \frac{n\pi}{T},
\]
and for large $T$ the $\omega_n$ are close together for successive $n$, approaching a continuous variable $\omega$ as $T \to \infty$. As $T \to \infty$, the sums over $n$ go over to Riemann integrals over angular frequency, $\omega$, with infinitesimal frequency interval

$$d\omega = \frac{2\pi}{T}.$$ 

If $\int_{-\infty}^{\infty} f(t) \cos(\omega_n) dt$ and $\int_{-\infty}^{\infty} f(t) \sin(\omega_n) dt$ are finite $A_n$ and $B_n$ will vanish as $T \to \infty$ so we define

$$a(\omega_n) := \frac{T}{2\pi} A_n = \frac{1}{\pi} \int_{-T/2}^{T/2} f(t) \cos(\omega_n t) dt \quad \text{as} \quad T \to \infty \quad a(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos(\omega t) dt$$

$$b(\omega_n) := \frac{T}{2\pi} B_n = \frac{1}{\pi} \int_{-T/2}^{T/2} f(t) \sin(\omega_n t) dt \quad \text{as} \quad T \to \infty \quad b(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin(\omega t) dt,$$

then

$$f(t) = \int_{-\infty}^{\infty} \left( a(\omega) \cos(\omega t) + b(\omega) \sin(\omega t) \right) d\omega.$$ 

It is conventional to define the cosine transform $\tilde{f}_c(\omega)$ and the sine transform $\tilde{f}_s(\omega)$ of $f(t)$ as

$$\tilde{f}_c(\omega) := \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \cos(\omega t) dt$$

$$\tilde{f}_s(\omega) := \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \sin(\omega t) dt.$$

These integrals certainly exist if $\int_{0}^{\infty} |f(t)| dt$ exists and is finite.

If $f(-t) = f(t)$ is an even function then $\tilde{f}_c(\omega) = \sqrt{\frac{1}{\pi}} a(\omega)$ and if $f(-t) = f(t)$ is an odd function then $\tilde{f}_s(\omega) = \sqrt{\frac{1}{\pi}} b(\omega)$.

Another type of integral transform that is very useful in physics when periodic phenomena are under consideration is the Fourier transform,

$$\hat{f}(\omega) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt. \quad (64)$$

Specifying $\hat{f}(\omega)$ is completely equivalent to specifying $f(t)$ because

$$f(t) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega t} d\omega, \quad (65)$$

as we shall now show.

Using the definition of $\hat{f}(\omega)$ in the right hand side of (65) gives

$$\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(t') e^{i\omega t'} dt' \right) e^{-i\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{i\omega(t'-t)} d\omega \right) f(t') dt'. \quad (66)$$
We now argue \( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t'-t)} d\omega \) can be interpreted as an integral representation of the Dirac delta-function. First suppose that \( t \neq t' \) in the integral. The integral can be computed by using complex integration in the complex \( \omega \)-plane. Consider \( t < t' \) and \( t > t' \) separately. When \( t < t' \) \( e^{i\omega(t'-t)} \) is complex analytic everywhere in the upper-half complex-\( \omega \) plane, \( \text{Im}(\omega) > 0 \), so we choose the contour \( C^+ \) below — a large semi-circle in the upper-half plane centred on the origin. As the radius of the semi-circle becomes infinite

\[
\frac{1}{2\pi} \int_{C^+} e^{i\omega(t'-t)} d\omega \to \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t'-t)} d\omega = 0
\]

from Cauchy’s integral formula. When \( t > t' \) use the contour \( C^- \) below and the same reasoning gives

\[
\frac{1}{2\pi} \int_{C^-} e^{i\omega(t'-t)} d\omega \to \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t'-t)} d\omega = 0.
\]

So \( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t'-t)} d\omega = 0 \) when \( t \neq t' \).

When \( t = t' \) the integral is divergent. Consider

\[
\int_{-\infty}^{\infty} e^{i\omega\tau} d\omega = 2 \int_{\infty}^{0} \frac{\sin(\omega T)}{\omega} d\omega = 4 \int_{0}^{\infty} \frac{\sin(\omega T)}{\omega} d\omega = 2\pi,
\]

which is finite \( \forall T \), in particular for \( T \to \infty \)

\[
\int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{i\omega\tau} d\omega \right) d\tau = 2\pi.
\]

We have shown that \( \int_{-\infty}^{\infty} e^{i\omega \tau} d\omega \) is zero for \( \tau \neq 0 \), infinite for \( \tau = 0 \) and its integral is finite and equal to \( 2\pi \). From the defining properties of the Dirac delta-function we conclude that

\[
\delta(t - t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-t')} d\omega.
\]

Using this in (66)
\[
\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{i\omega (t' - t)} d\omega \right) f(t') dt' \\
= \int_{-\infty}^{\infty} \delta(t' - t) f(t') dt' = f(t)
\]
as claimed.

To summarise the Fourier transform and the inverse transform are

\[
\tilde{f}(\omega) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt, \quad f(t) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t} d\omega.
\]

The Fourier, sine and cosine transforms are related. If \( f(t) \) is an even function the imaginary part of \( \tilde{f}(\omega) \) vanishes and

\[
\tilde{f}_c(\omega) = \tilde{f}(\omega),
\]
while if \( f(t) \) is an odd function the \( \tilde{f}(\omega) \) is pure imaginary and

\[
\tilde{f}_s(\omega) = -i \tilde{f}(\omega).
\]

These integrals are examples of a class of functions called integral transforms where, given a function \( f(t) \), we construct a new function \( \tilde{f}(\omega) \) as an integral

\[
\tilde{f}(\omega) := \int_{-\infty}^{\infty} f(t) K(\omega, t) dt
\]
where \( K(\omega, t) \) is called the kernel of the integral transform.

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* Note: some texts use a slightly different convention for the definition of the Fourier transform, namely

\[
\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt, \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t} d\omega.
\]

This is more natural for physics applications, where \( \nu = 2\pi \omega \) with \( \nu \) a frequency in Hertz, hence \( d\nu = \frac{d\omega}{2\pi} \). This convention is used in the particle physics module MP466.