# MP469: Laplace's Equation in Spherical Polar Co-ordinates

For many problems involving Laplace's equation in 3-dimensions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{1}$$

it is more convenient to use spherical polar co-ordinates $(r, \theta, \phi)$ rather than Cartesian co-ordinates $(x, y, z)$. These are related to each other in the usual way by

$$\begin{aligned} x &= r\cos\phi\sin\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\theta. \end{aligned}$$

To translate (1) into a differential equation involving $(r, \theta, \phi)$ we need the following partial derivatives:

$$\begin{aligned} &\frac{\partial x}{\partial r} = \cos\phi\sin\theta, \qquad \frac{\partial y}{\partial r} = \sin\phi\sin\theta, \qquad \frac{\partial z}{\partial r} = \cos\theta, \\ &\frac{\partial x}{\partial \theta} = r\cos\phi\cos\theta, \qquad \frac{\partial y}{\partial \theta} = r\sin\phi\cos\theta, \qquad \frac{\partial z}{\partial \theta} = -r\sin\theta, \\ &\frac{\partial x}{\partial \phi} = -r\sin\phi\sin\theta, \qquad \frac{\partial y}{\partial \phi} = r\cos\phi\sin\theta, \qquad \frac{\partial z}{\partial \phi} = 0. \end{aligned}$$

Using these the chain rule for differentiation implies that

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} + \frac{\partial z}{\partial r}\frac{\partial}{\partial z} = \cos\phi\sin\theta\frac{\partial}{\partial x} + \sin\phi\sin\theta\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial z}, \\ \frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z} = r\cos\phi\cos\theta\frac{\partial}{\partial x} + r\sin\phi\cos\theta\frac{\partial}{\partial y} - r\sin\theta\frac{\partial}{\partial z}, \\ \frac{\partial}{\partial \phi} &= \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \phi}\frac{\partial}{\partial z} = -r\sin\phi\sin\theta\frac{\partial}{\partial x} + r\cos\phi\sin\theta\frac{\partial}{\partial y}. \end{aligned}$$

These can be inverted, by taking linear combination with trigonometric function for example, to express partial derivatives of Cartesian co-ordinates in terms of polar co-ordinates:

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\phi\sin\theta\frac{\partial}{\partial r} + \frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial y} &= \sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{\sin\phi\cos\theta}{r}\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z} &= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}. \end{aligned} \tag{2}$$

One way of deriving the Laplacian in 3-dimensional spherical polars is is to expand the unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ in the directions of increasing $r$, $\theta$ and $\phi$ respectively, in terms of the Cartesian unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$:

$$\begin{aligned} \hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}, \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}, \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}. \end{aligned} \tag{3}$$

These can be inverted to give

$$\begin{aligned}
\hat{\mathbf{x}} &= \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\boldsymbol{\phi}},\\
\hat{\mathbf{y}} &= \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\boldsymbol{\phi}},\\
\hat{\mathbf{z}} &= \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\theta}.
\end{aligned} \tag{4}$$

Then define the vector differential operator

$$\nabla := \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \tag{5}$$

and the Laplacian can be written as $\nabla^2 = \nabla.\nabla$, using the vector dot product. Putting (2) and (4) in (5), and using standard trigonometric identities, gives

$$\nabla := \hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}. \tag{6}$$

Now to calculate $\nabla^2 = \nabla.\nabla$ in spherical polars we must be careful since the polar unit vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ are not constant. From (3)

$$\begin{array}{lll}
\dfrac{\partial}{\partial r}\,\hat{\boldsymbol{r}} = 0, & \dfrac{\partial}{\partial r}\,\hat{\boldsymbol{\theta}} = 0, & \dfrac{\partial}{\partial r}\,\hat{\boldsymbol{\phi}} = 0,\\
\dfrac{\partial}{\partial\theta}\,\hat{\boldsymbol{r}} = \ \hat{\boldsymbol{\theta}}, & \dfrac{\partial}{\partial\theta}\,\hat{\boldsymbol{\theta}} = -\,\hat{\boldsymbol{r}}, & \dfrac{\partial}{\partial\theta}\,\hat{\boldsymbol{\phi}} = 0,\\
\dfrac{\partial}{\partial\phi}\,\hat{\boldsymbol{r}} = -\sin\theta\,\hat{\boldsymbol{\phi}}, & \dfrac{\partial}{\partial\phi}\,\hat{\boldsymbol{\theta}} = \cos\theta\,\hat{\boldsymbol{\phi}}, & \dfrac{\partial}{\partial\phi}\,\hat{\boldsymbol{\phi}} = -\sin\theta\,\hat{\boldsymbol{r}} - \cos\theta\,\hat{\boldsymbol{\theta}}.
\end{array}$$

Using these in (6) the Laplace differential operator in equation (1) can be expressed directly in terms of spherical polar co-ordinates:

$$\begin{aligned}
\nabla^2 u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\\
&= \frac{1}{r}\frac{\partial^2(ru)}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}.
\end{aligned}$$