MP467: Cosmology

Degeneracy Pressure

Relevant for very high densities of particles with intrinsic spin of $\frac{1}{2}\hbar$. Such particles obey the Pauli exclusion principle and are called Fermions. If we try to put a lot of Fermions (e.g. electrons or neutrons) into a small box, the exclusion principle generates a pressure which resists the addition of more particles. This is called the *degeneracy pressure* because, as we shall see, it depends on the fact that quantum mechanical energy levels have finite degeneracy. It is partly responsible (together with electrostatic repulsion) for the force on the soles of your feet that stops you falling through the floor

Consider N particles confined in a cubical volume $V = a^3$. Quantum states are superpositions of standing waves, with wave-length λ. Define the *wave-number*, k, to be $k = \frac{2\pi}{\lambda}$. Quantum mechanics only allows discrete wave-vectors $\vec{k} = (m_x, m_y, m_z)\frac{2\pi}{a}$, where m_x, m_y and m_z are integers, because a whole number of wavelengths must fit in the box of width a.

The number of quantum states in a volume $d^3k = dk_x dk_y dk_z$ of wave-vector space is $dm_x dm_y dm_z = \left(\frac{a}{2\pi}\right)^3 d^3k$ at least for large integers m_x, m_y, m_z.

Now the number of quantum states in a spherical shell of thickness dk and radius k in k-space is

$$\left(\frac{a}{2\pi}\right)^3 4\pi k^2 dk = \frac{V}{(2\pi)^3} 4\pi k^2 dk$$

In quantum mechanics the de Broglie relation is $p = \frac{h}{\lambda} = \hbar k$, so the number of quantum states for particles with momentum in the range $p \to p + dp$ is

$$\frac{V}{(2\pi)^3} 4\pi \left(\frac{p}{\hbar}\right)^2 d\left(\frac{p}{\hbar}\right) = \frac{4\pi V}{h^3} p^2 dp =: g(p)dp$$

$g(p)$ is called degeneracy factor. Because of electron (or neutron) spin, there can be two states with the same p, spin up and spin down $\Rightarrow$ for electrons and neutrons

$$g(p) = \frac{8\pi V}{h^3} p^2. \tag{1}$$

Possible values of a particle's energy, $\epsilon(p)$ are also discrete (when confined to a box).

Let the lowest possible energy be ϵ_0. If N electrons go into the box, all possible energy levels will be filled sequentially up to some top energy $\epsilon_F = \epsilon(p_F)$. ϵ_F (the Fermi-energy) and p_F (the Fermi-momentum) depend on N and are given, when N is large, by approximating the sum over energy levels, or equivalently momenta, by an integral

$$N = \int_0^{p_F} g(p)dp = \frac{8\pi V}{h^3} \int_0^{p_F} p^2 dp = \frac{8\pi V}{3h^3} p_F^3 \qquad \Rightarrow$$

$$p_F = h\left(\frac{3}{8\pi}\frac{N}{V}\right)^{1/3} = h\left(\frac{3n}{8\pi}\right)^{1/3} \tag{2}$$

where $n := N/V$ is the number of particles per unit volume.

We want to calculate $P(\rho, T)$, but the calculation is simplified when T is "small" (by which is meant $k_B T \ll$ energy level differences, $\Delta\epsilon$), in which case the Fermi gas is said to be *degenerate*. The internal energy is then

$$E = \int_0^{p_F} \epsilon(p) g(p) dp = \frac{8\pi}{h^3} V \int_0^{p_F} \epsilon(p) p^2 dp \tag{3}$$

with $\epsilon(p) = \frac{p_F^2}{2m}$ for non-relativistic momenta and $\epsilon(p) = \sqrt{m^2c^4 + p^2c^2}$ for relativistic momenta (the relativistic form is used below as the non-relativistic form can then follows by taking $c \to \infty$).

The first Law of Thermodynamics, for an adiabatic process, is[1]

$$dE = -PdV + TdS + \mu dN.$$

For $T \approx 0$ this reduces to $dE = -PdV + \mu dN$ so, using (3) and noting that $\left(\frac{\partial g(p)}{\partial V}\right)_N = \frac{g(p)}{V}$ from (1),

$$\begin{aligned} P &= -\left(\frac{\partial E}{\partial V}\right)_N = -\frac{E}{V} - \frac{8\pi V}{h^3} \epsilon(p_F) p_F^2 \left(\frac{\partial p_F}{\partial V}\right)_N \\ \mu &= \left(\frac{\partial E}{\partial N}\right)_V = \frac{8\pi V}{h^3} \epsilon(p_F) p_F^2 \left(\frac{\partial p_F}{\partial N}\right)_V . \end{aligned}$$

Now from (2)

$$\left(\frac{\partial p_F}{\partial V}\right)_N = -\left(\frac{1}{3V}\right) p_F, \qquad \left(\frac{\partial p_F}{\partial N}\right)_V = \left(\frac{1}{3N}\right) p_F$$

hence

$$\begin{aligned} P &= -\frac{E}{V} + \frac{8\pi V}{h^3} \epsilon(p_F) p_F^2 \frac{p_F}{3V} = -\frac{E}{V} + n\epsilon(p_F) \\ \mu &= \frac{8\pi V}{h^3} \epsilon(p_F) p_F^2 \frac{p_F}{3N} = \frac{8\pi V}{3h^3 n} \epsilon(p_F) p_F^3 = \epsilon(p_F). \end{aligned}$$

$$\Rightarrow \qquad P = \frac{\epsilon_F N - E}{V}, \qquad \mu = \epsilon_F. \tag{4}$$

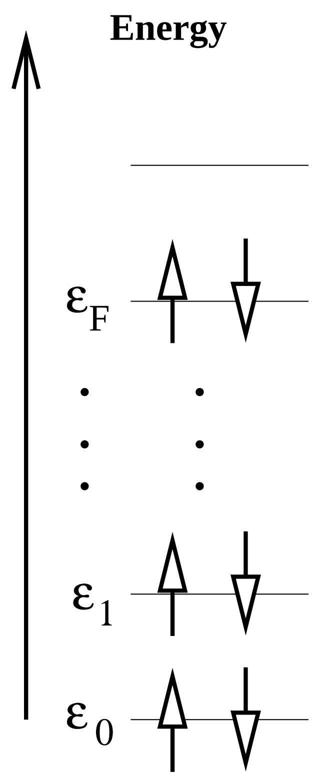
Now

$$\begin{aligned} P &= \epsilon_F n - \frac{E}{V} = \\ &= mc^2 \sqrt{1 + \frac{p_F^2}{m^2c^2}} \frac{8\pi}{3} \left(\frac{p_F}{h}\right)^3 - \frac{8\pi mc^2}{h^3} \int_0^{p_F} \sqrt{1 + \frac{p^2}{m^2c^2}} p^2 dp \\ &= \frac{8\pi mc^2}{h^3} (mc)^3 \left\{ \frac{1}{3} \sqrt{1 + z_F^2} z_F^3 - \int_0^{z_F} \sqrt{1 + z^2} z^2 dz \right\} \quad \left(z = \frac{p}{mc},\ dp = mcdz\right) \\ &= \frac{8\pi m^4 c^5}{h^3} \int_0^{z_F} \frac{z^4}{3\sqrt{1+z^2}} dz \quad (z = \sinh\alpha,\ \sqrt{1+z^2} = \cosh\alpha,\ dz = \cosh\alpha d\alpha) \\ &= \frac{\pi m^4 c^5}{h^3} \left\{ z_F \left(\frac{2}{3} z_F^2 - 1\right) \sqrt{1 + z_F^2} + \ln\left(z_F + \sqrt{1 + z_F^2}\right) \right\} \\ &\longrightarrow \begin{cases} \frac{8\pi}{15} \frac{m^4c^5}{h^3} z_F^5, & z_F \ll 1,\ p_F \ll mc \text{ (non-relativistic)} \\ \frac{2\pi m^4 c^5}{3h^3} z_F^4, & z_F \gg 1,\ p_F \gg mc \text{ (relativistic)} \end{cases} \end{aligned}$$

[1] μ is the chemical potential, i.e. the energy required to add one extra particle — we shall prove momentarily that $\mu = \epsilon_F$.

$$P = \begin{cases} \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{5m} n^{5/3} & \text{non-relativistic} \\ \left(\frac{3}{8\pi}\right)^{1/3} \frac{ch}{4} n^{4/3} & \text{relativistic.} \end{cases} \tag{5}$$

It has been assumed that the temperature is negligible, these expressions are valid, provided $k_B T \ll \epsilon_F$. This is the equation of state for a "‘cold"’ degenerate gas of electrons or neutrons but note that, when n is large, ϵ_F can be so large that T is negligible even for temperatures of many thousands of degrees.



Discrete energy levels, filled according to the exclusion principle.