

3 Cosmology and Expansion of the Universe

On very large length scales ($> 100Mpc$) the distribution of galaxies appears to be isotropic. If we assume that all points in space are equivalent, *i.e.* we are not at a special point (this is called the *Copernican Principle*), then the distribution must be isotropic about all other points and this actually implies that distribution of galaxies is homogeneous on very large scales, *i.e.* it is uniform. Therefore mass density ρ is independent of position on large enough length scales.

3.1 Cosmological Constant

Consider a spherical shell of matter, radius R and thickness δR , of mass $\delta m = 4\pi R^2 \delta R \rho$ around a mass $M(R) = \frac{4\pi R^3 \rho}{3}$. In Newtonian Gravity, the dynamics of the shell is the same as it would be if the central mass were concentrated at a point in the centre and any mass outside the shell doesn't affect it at all if it is spherically distributed (consequence of the inverse square force). The Newtonian equation of motion for the shell is

$$\delta m \ddot{R} = -\frac{G\delta m M}{R^2}. \quad (3.1)$$

We shall assume the following:

1. $\dot{R} \ll c$ (non-relativistic velocities);
2. $\frac{2GM}{c^2 R} = \frac{8\pi G}{3c^2} R^2 \rho \ll 1$ (weak gravitational field, Newtonian Gravity is valid);
3. $R \gg$ galactic separation, so we approximate ρ by a smooth function $\rho(R)$.

It might be thought that 1) and 2) are related by the virial theorem $\Rightarrow \frac{\dot{R}^2}{c^2} \approx \frac{2GM}{c^2 R}$, but the virial theorem only applies to stable gravitationally bound systems and is not applicable to galaxies that are so far apart that they are not bound to each other by gravitational forces.

2) and 3) \Rightarrow *galactic separation* $\ll R \ll \sqrt{\frac{3}{8\pi G\rho}}c$. Provided R lies in this range we shall assume we can use equation (3.1) and immediately get the first integral

$$\begin{aligned} \frac{1}{2}\delta m \dot{R}^2 - \frac{G\delta m M}{R} &= E \\ \frac{1}{2}\dot{R}^2 - \frac{GM}{R} &= \frac{E}{\delta m} = \frac{E}{4\pi R^2 \delta R \rho} := \epsilon. \end{aligned}$$

where E is the total energy of the shell and ϵ the energy of the shell per unit mass, which is a constant if δm is constant and has dimensions of *velocity*². Assuming uniform density M can be eliminated in favour of ρ ,

$$\frac{1}{2}\dot{R}^2 - \frac{4\pi}{3}GR^2\rho = \epsilon. \quad (3.2)$$

This equation was derived using Newtonian gravity and non-relativistic physics. In relativistic physics Newton's $1/R^2$ force is invalid (it's only valid for small velocities). We should use the full power of Einstein's General theory to determine the dynamics of the Universe and this lies outside the scope of this course. Remarkably General Relativity gives the same equation but with a different interpretation. In General Relativity R is not labelling a radial position, it is a length scale determining the physical size of lengths in 3-dimensional space. If $R = R(t)$ then when $\dot{R} > 0$ space is said to be expanding while when $\dot{R} < 0$ space is contracting. We can interpret R as the physical distance between any two fixed galaxies, provided their separation is of the order of $100Mpc$, or more. This distinction between the meaning of R in a Cosmological context, as discussed here, and in the context of Newtonian gravity, as discussed earlier in the section on stellar equilibrium, is very important. For example if the mass M inside a sphere of radius R , $M = \frac{4\pi}{3}\rho R^3$, is independent of R then $\rho(R) \propto \frac{1}{R^3}$. If R labelled positions in space then ρ would not be independent of position, contradicting the Copernican Principle. If instead R describes cosmological length scales rather than position then it is perfectly consistent with the Cosmological Principle for ρ to depend on R , ρ is then independent of position but it depends on the cosmological length scale. $\rho \propto \frac{1}{R^3}$ then means that the average density goes down as R increases (the Universe expands), but ρ is the same everywhere at any one time.

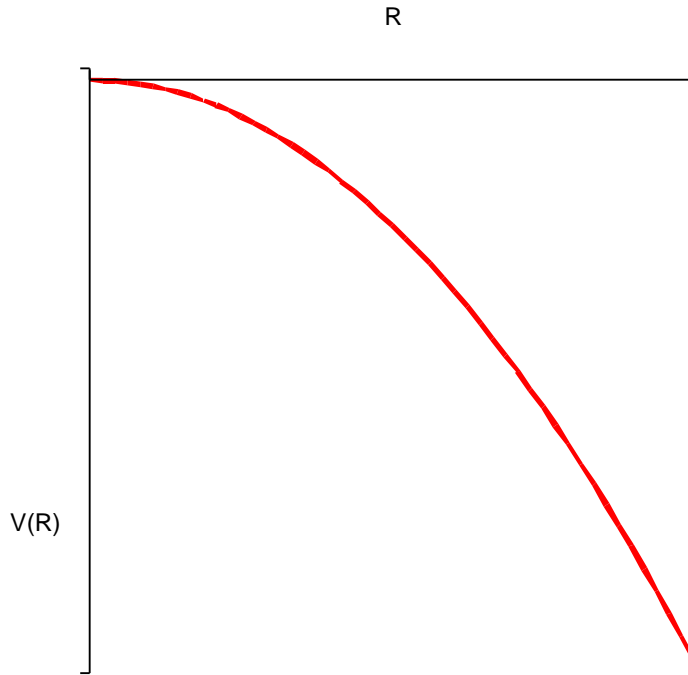
Equation (3.2) is the same as the energy of a point particle of unit mass moving on the half-line $R > 0$ in a quadratic potential, like a harmonic oscillator equation but with a *negative* co-efficient. Suppose $\rho = \rho_0$ is independent of R . Define the constant

$$\Lambda := \frac{8\pi G\rho_0}{c^2},$$

which has dimensions of $(length)^2$, then

$$\frac{1}{2}\dot{R}^2 - \frac{\Lambda}{6}R^2 = \epsilon \quad (3.3)$$

(the $1/6$ is a standard convention in cosmology). We can get a qualitative understanding of the behaviour simply by plotting the potential $V(R) = -\frac{\Lambda}{6}R^2$,



If the energy $\epsilon > 0$ then R can have any value in the range $0 < R < \infty$, with $|\dot{R}|$ increasing with R ; if $\epsilon < 0$ then there is a repulsive barrier and R cannot reach zero. Neither of these cases allows for a static solution with $\dot{R} = 0$. A static solution is only possible when $\epsilon = 0$, which allows for $R = \dot{R} = 0$, but this is clearly unstable — the slightest deviation away from $R = 0$ and R will start to grow, eventually reaching infinity. We conclude from this that $\rho = \text{const}$ does not allow for a stable static solution: on sufficiently large scales the distances between galaxies cannot be constant, so ρ must change with time, the Universe is either expanding or contracting — observationally it is expanding and $R(t)$ is growing as a function of time.

Consider two galaxies a distance R apart with M being the total amount of mass in a sphere of radius R centred on one of the galaxies. Even if R is changing with time it is reasonable to assume, if mass cannot be created or destroyed, that M is constant. In that case

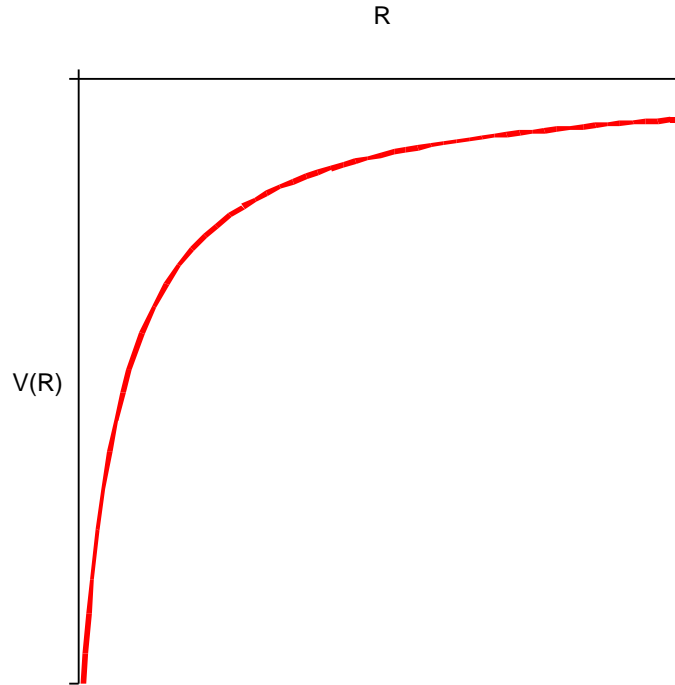
$$M = \frac{4\pi\rho}{3}R^3 \quad \Rightarrow \quad \rho(R) = \frac{3M}{4\pi} \frac{1}{R^3}.$$

Note that, with the interpretation of R as being the physical distance between the galaxies, $\rho \propto 1/R^3$ does not mean that ρ depends on position — ρ is independent of position but is a decreasing function of time if $R(t)$ is an increasing function of time.

For notational convenience let $A = 2GM$, then equation (3.2) gives

$$\frac{1}{2}\dot{R}^2 - \frac{A}{2R} = \epsilon. \quad (3.4)$$

This equation has the same mathematical form as that of the energy of a projectile with unit mass thrown vertically upward from the surface of the Earth, moving in a potential $V(R) = -\frac{A}{2R}$,

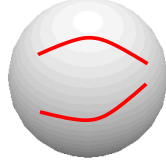


There is an attractive force towards $R = 0$: if $\dot{R} > 0$ initially the late time behaviour depends on the sign of ϵ . ϵ has dimensions of $(velocity)^2$ so define a dimensionless parameter K by $\epsilon = -\frac{1}{2}Kc^2$. Then there are three types of behaviour:

- | | |
|---------|--|
| $K < 0$ | R increases indefinitely and $\dot{R} > 0$ always; |
| $K > 0$ | R reaches a maximum value and decreases again; |
| $K = 0$ | R increases indefinitely but $\dot{R} \rightarrow 0$ as $t \rightarrow \infty$. |

$K = 0$ is a critical value, corresponding to the notion of escape velocity in Newtonian dynamics but the physical interpretation of K in General Relativity is very different. K is a measure of the *curvature* of 3-dimensional space.

- $K = 0$ is flat Euclidean space;
- $K > 0$ is a 3-dimensional analogue of the surface of a 2-dimensional sphere. In a space with positive K , the trajectories of two projectiles will bend toward each other, rather like the paths of two ships following great circles on the surface of the Earth.



Mathematically a 3-dimensional sphere of radius r can be described by imposing the constraint $v^2 + x^2 + y^2 + z^2 = r^2$ on Cartesian co-ordinates (v, x, y, z) in flat 4-dimensional Euclidean space. This is a natural extension of the geometry of a circle in 2-dimensional Euclidean space with Cartesian co-ordinates (y, z) , $y^2 + z^2 = r^2$ (a “1-dimensional sphere”), and the usual 2-dimensional sphere, $x^2 + y^2 + z^2 = r^2$;

- $K < 0$ is a 3-dimensional space in which trajectories *diverge*. It is a 3-dimensional analogue of a hyperbola $y^2 - z^2 = r^2$ in 2-dimensional Euclidean space, and can be described by imposing the constraint $v^2 + x^2 + y^2 - z^2 = r^2$ on Cartesian co-ordinates (v, x, y, z) in flat 4-dimensional Euclidean space.

The full story appears to be that, at the present time, the mass density of the Universe at length scales of $100Mpc$ and greater appears to have two components, one with $\rho = const$ corresponding to a constant mass density, and one with $\rho \propto 1/R^3$ corresponding to a constant amount of mass in a sphere of radius $R(t)$. Denoting the latter by ρ_{Mat} a combination of these two possibilities can produce a static solution. Let

$$\rho = \rho_{Mat} + \rho_{\Lambda} = \frac{3A}{8\pi GR^3} + \Lambda c^2/8\pi G$$

where ρ_{Mat} is the density of ordinary matter. Then

$$\begin{aligned} \dot{R}^2 &= \frac{8\pi G}{3} \rho_{Mat} R^2 - Kc^2 + \frac{\Lambda c^2 R^2}{3} \\ \Rightarrow \dot{R}^2 &= \frac{A}{R} - Kc^2 + \frac{\Lambda c^2 R^2}{3} \\ \Rightarrow 2\ddot{R} &= -\frac{A}{R^2} + \frac{2\Lambda c^2 R}{3} \end{aligned}$$

Choosing $\Lambda = \frac{3A}{2c^2 R^3}$ (a repulsive force) gives a solution with no acceleration and

$$\dot{R}^2 = \frac{A}{R} - Kc^2 + \frac{A}{2R},$$

so $K = \frac{3A}{2c^2 R} > 0$ gives a static solution with R a constant. (called Einstein’s static universe, in which space is a 3-dimensional sphere with finite volume). This is static, but unfortunately unstable. The *cosmological constant* Λ was introduced by Einstein in order to obtain static solutions because he did not know at the time that R was changing and he assumed that it should be constant. At the present day $R(t)$ appears to be increasing

but only changes very slowly, on cosmological time-scales of the order of billions of years. Nevertheless there is by now considerable observational evidence that Λ is positive.

We have arrived at the *Friedmann-Equation*

$$\boxed{\left(\frac{\dot{R}}{R}\right)^2 + \frac{c^2 K}{R^2} = \frac{8\pi G \rho_{Mat}}{3} + \frac{\Lambda c^2}{3}} \quad (3.5)$$

This is a dynamical equation that determines the behaviour of the cosmological length scale $R(t)$ for a given mass density ρ_{Mat} and constants K and Λ . In the General Theory of Relativity it is valid for relativistic velocities $\dot{R} \approx c$ and for strong gravitational fields $\frac{8\pi G}{3c^2} R^2 \rho_{Mat} \approx 1$, but still requires that $R \gg$ galactic separations.

3.2 Redshift-distance relation

Consider two galaxies A and B with coordinates (r_A, θ_A, ϕ_A) and (r_B, θ_B, ϕ_B) . Choose co-ordinates so that $\theta_A = \theta_B = \frac{\pi}{2}$ and $\phi_A = \phi_B = 0$. Take $r_A > r_B$, fixed for each galaxy (the coordinates are fixed to the galaxies and are called co-moving co-ordinates, analogous to Lagrangian coordinates). The distance between the galaxies is $R(t)(r_A - r_B)$ where $R(t)$ is the cosmological scale factor.

Consider a beam of light passing between A and B. It travels a distance $R(t)\delta r$ in the time $\delta t = \frac{R(t)\delta r}{c}$. Suppose light leaves A at time t_1 and arrives at B at time t_2

$$\int_{t_1}^{t_2} \frac{dt}{R(t)} = \frac{1}{c} \int_{r_A}^{r_B} dr. \quad (3.6)$$

At a later time light leaves A at $t_1 + \Delta t_1$, light would reach B at $t_2 + \Delta t_2$

$$\begin{aligned} \int_{t_1 + \Delta t_1}^{t_2 + \Delta t_2} \frac{dt}{R(t)} &= \frac{1}{c} \int_{r_A}^{r_B} dr = \int_{t_1}^{t_2} \frac{dt}{R(t)} \\ \Rightarrow \int_{t_2}^{t_2 + \Delta t_2} \frac{dt}{R(t)} &= \int_{t_1}^{t_1 + \Delta t_1} \frac{dt}{R(t)}. \end{aligned}$$

Take Δt_1 and Δt_2 to be the inverse of optical frequency and assume $R(t) = R(t_1)$ is essentially constant between t_1 and $t_1 + \Delta t_1$ and $R(t) = R(t_2)$ between t_2 and $t_2 + \Delta t_2$ (this always is an extremely good approximation: for optical frequencies $\Delta t \approx 10^{-15} s$ while the scale factor R only changes appreciably on time scales of order $10^9 yrs \approx 3 \times 10^{16} s$), then

$$\begin{aligned} \frac{1}{R(t_1)} \Delta t_1 &= \frac{1}{R(t_2)} \Delta t_2 \\ \frac{R(t_1)}{R(t_2)} &= \frac{\Delta t_1}{\Delta t_2} = \frac{\nu_2}{\nu_1} \end{aligned} \quad (3.7)$$

If $R(t_2) > R(t_1)$ then $\nu_2 < \nu_1$, *i.e.* the light is redshifted.

Assuming $R(t)$ is a slowly varying analytic function of time it can be Taylor expanded around t_2 as

$$R(t_1) = R(t_2)\{1 - (t_2 - t_1)H + \dots\} \quad H = \left. \frac{\dot{R}}{R} \right|_{t_2},$$

with $(t_2 - t_1)H \ll 1$. Let $s_{AB} = c(t_2 - t_1)$ (time-of-flight distance) then

$$\frac{R(t_2)}{R(t_1)} \approx 1 + (t_2 - t_1)H = 1 + \frac{s_{AB}H}{c}.$$

If $t_2 = t_0$ is the present day then $H_0 := \left. \frac{\dot{R}}{R} \right|_{t_0}$ is called the Hubble constant. Suppose light left a distance galaxy at a time $t = t_1$ and arrives at our telescope at the present day $t_0 = t_2$. Then, assuming $t_0 - t \ll H_0^{-1}$, we have

$$z := \frac{\delta\nu}{\nu_2} = \frac{\nu_1 - \nu_2}{\nu_2} = \frac{\nu_1}{\nu_2} - 1 = \frac{R(t_0)}{R(t)} - 1 = \frac{s_{AB}H_0}{c}.$$

z is called the **redshift** of the galaxy, it is a measure of the amount by which light from a distant galaxy is shifted toward the red end of the spectrum, *i.e.* towards longer wave-lengths. So

$$z = \frac{H_0}{c}s_{AB} \propto s_{AB} \quad (3.8)$$

giving the **redshift distance relation**

$$\boxed{s_{AB} = \frac{c}{H_0}z.} \quad (3.9)$$

H_0 has dimensions of $(\text{time})^{-1}$ but is usually quoted as $\text{kms}^{-1}\text{Mpc}^{-1}$. The current best estimate of H_0 from observations is

$$H_0 = 70.4 \pm 1.4 \text{ km s}^{-1} \text{ Mpc}^{-1} \\ := h \times 100 \text{ km s}^{-1} \text{ Mpc}^{-1} \quad \text{where} \quad h = 0.704 \pm 0.014.$$

This means that a galaxy at a distance of 1 Mpc exhibits a redshift corresponding to a velocity of about 70 km s^{-1} . The linear relation (3.9) is only valid for "small" $(t_2 - t_1)$, *i.e.* $\frac{\delta\nu}{\nu} \ll 1$.

For example there is a cluster of galaxies in the constellation of Virgo (≈ 2500 galaxies) with an average redshift $c\frac{\delta\nu}{\nu} = 1150 \text{ km s}^{-1} \Rightarrow \frac{\delta\nu}{\nu} = 0.00383$ giving $s_{AB} = \frac{c}{H_0}\frac{\delta\nu}{\nu} = 16 \text{ Mpc} = 50 \text{ Mlyrs}$. The current redshift record is $\frac{\delta\nu}{\nu} \approx 10$ which gives a naïve redshift distance relation of $s_{AB} \approx 4.5 \times 10^4 \text{ Mpc}$, but this is not the true physical distance because $\frac{\delta\nu}{\nu} > 1$ is not small.

$\frac{1}{H_0}$ has dimensions of time: $\frac{1}{H_0} = \frac{1}{h} (3 \times 10^{17} \text{ s}) = \frac{1}{h} 10^{10} \text{ yr} \approx 14$ billion years which is the approximate age of the universe (more accurate modelling gives $t_0 = 13.7 \times 10^9 \text{ years}$). If $R(t)$ were linear then R would have been zero 14 billion years ago, but this is only a rough approximation. In this approximation $R_0 = \frac{1}{H_0}c = \frac{1}{h} \times 10^{26} \text{ m}$ is the approximate size of the observable universe.

3.3 Friedmann Equation

Hubble discovered the linear redshift-distance relation in 1929. In recent years the observational data have become good enough to go beyond a linear approximation for $R(t)$.

The Friedmann equation (3.5) can be written as

$$H^2 + \frac{c^2 K}{R^2} = \frac{8\pi G}{3} \rho_{Mat} + \frac{\Lambda c^2}{3}$$

where ρ_{Mat} is the mass density of matter, as in stars, gas and dust in galaxies.

When $\Lambda = K = 0$,

$$\rho_{Mat}^{\Lambda=K=0} = \frac{3}{8\pi G} H^2.$$

Using present day values

$$\frac{3}{8\pi G} H_0^2 = h^2 \times (1.32 \times 10^{-26} \text{kgm}^{-3}) = 9.38 \times 10^{-27} \text{kgm}^{-3}.$$

Observationally, counting the visible luminous galaxies gives $\rho_{Luminous} \approx 3 \times 10^{-29} \text{kgm}^{-3}$. Using observations of orbital dynamics of galaxies in clusters (Kepler's Law) $\rho_{Mat} \approx 10^{-27} \text{kgm}^{-3}$ (equivalent to about 1 proton per cubic metre). The most accurate value we have to date comes from the dynamics of galaxies on cosmological scales and is $\rho_{Mat} = 2.5 \times 10^{-27} \text{kgm}^{-3}$. So $\rho_{Mat} < \rho_{Mat}^{\Lambda=K=0}$. Returning to the more general case with $\Lambda \neq 0$

$$\begin{aligned} H^2 &= \frac{8\pi G \rho_{Mat}}{3} - \frac{c^2 K}{R^2} + \frac{\Lambda c^2}{3} \\ \Rightarrow 1 &= \frac{8\pi G \rho_{Mat}}{3H^2} - \frac{c^2 K}{H^2 R^2} + \frac{\Lambda c^2}{3H^2} \end{aligned}$$

Define the following 3 constants using present day values H_0 and $R_0 = R(t_0)$

$$\Omega_M := \frac{8\pi G \rho_{Mat}}{3H_0^2} = \frac{A}{H_0^2 R_0^3}, \quad \Omega_K := -\frac{c^2 K}{H_0^2 R_0^2} \quad \text{and} \quad \Omega_\Lambda := \frac{\Lambda c^2}{3H_0^2}.$$

Then only two of these are independent, since

$$\Omega_M + \Omega_K + \Omega_\Lambda = 1.$$

These can be related to a Taylor expansion of $R(t)$,

$$\begin{aligned} R(t) &= R_0 + (t - t_0) \dot{R}_0 + \frac{1}{2} (t - t_0)^2 \ddot{R}_0 + \dots \\ &= R_0 \left(1 + H_0 (t - t_0) + \frac{1}{2} (t - t_0)^2 \frac{\ddot{R}_0}{R_0} + \dots \right), \end{aligned}$$

where, in what I hope is an obvious notation, $\dot{R}_0 = \dot{R}(t_0)$ and $\ddot{R}_0 = \ddot{R}(t_0)$. Define the *deceleration parameter*, $q_0 = -\frac{1}{2} \frac{\ddot{R}_0}{H_0^2 R_0}$, then

$$\begin{aligned}
\dot{R}^2 &= \frac{8\pi G \rho_{Mat} R^2}{3} - c^2 K + \frac{\Lambda c^2 R^2}{3}, & \rho_{Mat} &= \frac{3}{8\pi G} \frac{A}{R^3} \\
\Rightarrow & & & = \frac{A}{R} - c^2 K + \frac{\Lambda c^2 R^2}{3} \\
\Rightarrow & 2\dot{R}\ddot{R} &= -\frac{A\dot{R}}{R^2} + \frac{2\Lambda c^2 R\dot{R}}{3} \\
\Rightarrow & \frac{\ddot{R}}{R} &= -\frac{A}{2R^3} + \frac{\Lambda c^2}{3} = -\frac{4\pi G \rho_{Mat}}{3} + \frac{\Lambda c^2}{3} \\
\Rightarrow & \frac{\ddot{R}}{H^2 R} &= -\frac{A}{2H^2 R^3} + \frac{\Lambda c^2}{3H^2} \\
\Rightarrow & 2q_0 &= \frac{1}{2} \Omega_M - \Omega_\Lambda.
\end{aligned}$$

So Ω_Λ and Ω_M are directly related to the parameters in a Taylor expansion of $R(t)$.

What we actually measure is the redshift z , but we can convert from t to z . From (3.7), with $t_2 = t_0$ and $t_1 = t$, and the definition $z = \frac{\nu_1}{\nu_2} - 1$ we have $\frac{R_0}{R(t)} = 1 + z$

$$\Rightarrow \quad dz = -\frac{R_0}{R^2} dR \quad \Rightarrow \quad \frac{dz}{1+z} = -\frac{dR}{R} \quad \Rightarrow \quad \frac{\dot{R}}{R} = -\frac{\dot{z}}{1+z}.$$

Now use this to write the Friedmann equation as

$$\begin{aligned}
\left(\frac{\dot{R}}{R}\right)^2 &= \frac{A}{R^3} - \frac{c^2 K}{R^2} + \frac{\Lambda c^2}{3} = \Omega_M H_0^2 \left(\frac{R_0}{R}\right)^3 + \Omega_K H_0^2 \left(\frac{R_0}{R}\right)^2 + \Omega_\Lambda H_0^2 \\
\dot{z}^2 &= (1+z)^2 \left\{ (1+z)^3 H_0^2 \Omega_M + (1+z)^2 H_0^2 \Omega_K + H_0^2 \Omega_\Lambda \right\}.
\end{aligned}$$

This is a non-linear differential equation (no approximations) for $z(t)$ in terms of the constants H_0 , Ω_M , Ω_K and Ω_Λ . Now we invoke (3.6) with B being our Galaxy, so we set $R_B = 0$, to write the distance d_A to a galaxy A at redshift z as

$$\begin{aligned}
d_A &= R_0 r_A = c R_0 \int_t^{t_0} \frac{dt}{R} = c \int_t^{t_0} (1+z) dt = -c \int_t^{t_0} \frac{(1+z) dz}{\dot{z}} \\
\Rightarrow \quad d_A &= c \int_0^z \frac{dz}{\sqrt{(1+z)^3 H_0^2 \Omega_M + (1+z)^2 H_0^2 \Omega_K + H_0^2 \Omega_\Lambda}}. \quad (3.10)
\end{aligned}$$

This is an exact non-linear redshift-distance relation.

For example

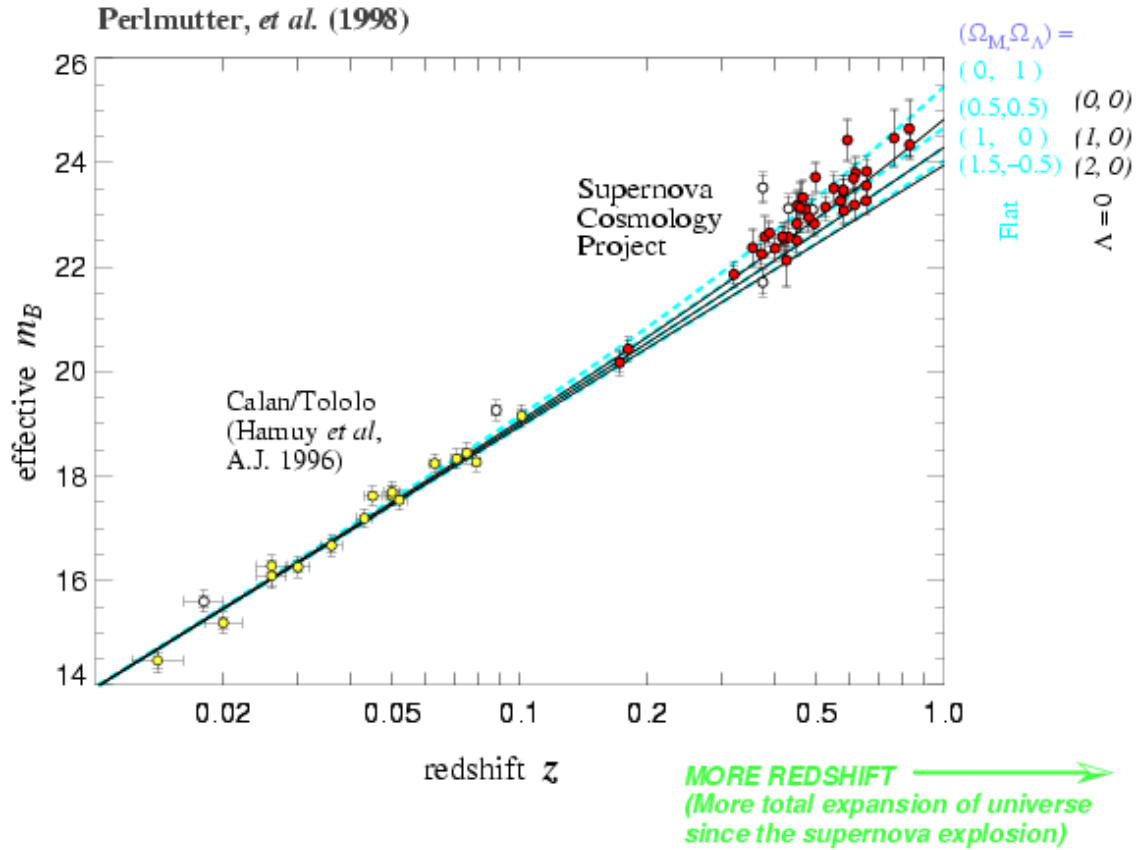
$$d_A(z) = \frac{cz}{H_0}$$

if $\Omega_\Lambda = 1$, $\Omega_M = \Omega_K = 0$, while

$$d_A(z) = \frac{2c}{H_0} \left(1 - \frac{1}{\sqrt{1+z}} \right)$$

if $\Omega_M = 1, \Omega_\Lambda = \Omega_K = 0$

Although the case $\Omega_\Lambda = 1, \Omega_M = \Omega_K = 0$ looks very like (3.9) they are not exactly the same since the distance $d_A = R_0 r_A$ is not the same as the time-of-flight distance $s_{AB} = c(t_0 - t)$ (with $t_A = t$ and $t_B = t_0$), though their difference is negligible at small $z \ll 1$.



Observational data plotting galactic distance against redshift (the vertical axis is the magnitude, which is essentially the logarithm of the distance).

Taken from <http://supernova.lbl.gov/>

Observationally

$$\Omega_M = 0.273 \pm 0.014 \quad \begin{cases} 0.0456 \pm 0.0016 & \text{"ordinary matter" (neutrons, protons)} \\ 0.227 \pm 0.014 & \text{"Dark matter" (unknown, not yet detected in the lab)} \end{cases}$$

$$\Omega_\Lambda = 0.728 \pm 0.016 \quad \text{often called "Dark Energy".}$$

$$1 = \Omega_M + \Omega_K + \Omega_\Lambda \Rightarrow \Omega_K = -0.001 \pm 0.014$$

$\Omega_K = 0 \Rightarrow 1 = \Omega_M + \Omega_\Lambda \Rightarrow q = \frac{1}{2} \left[\frac{1}{2}(1 - \Omega_\Lambda) - \Omega_\Lambda \right] = \frac{1}{4} - \frac{3}{4}\Omega_\Lambda = -0.296 \pm 0.012 < 0$
implying that the expansion rate of the Universe is accelerating.

A value of 0.27 for Ω_M means that the density in matter at the present time is

$$\rho_{Mat} = 0.27 \times \frac{3H_0^2}{8\pi G} = 2.5 \times 10^{-27} \text{kgm}^{-3},$$

which is equivalent to about 1.5 protons per cubic metre. We shall see later that this cannot all be due to ordinary matter with which we are familiar, protons and neutrons. Only about 1/7 can be protons and neutrons, the other 6/7 appears to be some new form of matter about which very little is known – this is known as “Dark Matter” because it is not visible. The mass density in protons and neutrons, known as baryonic matter because protons and neutrons are classified as ‘baryons’ by particle physicists, is then $\rho_b = \frac{1}{7}\rho_{Mat} = 3.6 \times 10^{-28} \text{kgm}^{-3}$. This is still ten times the amount of visible matter in the Universe, in the form of luminous stars and galaxies, but this is not inconsistent — it just means that a lot of the ordinary matter is not hot enough to be emitting light. There are other observations however that imply that the whole of the $2.6 \times 10^{-27} \text{kgm}^{-3}$ in matter cannot be protons and neutrons, this is covered later when the nuclear synthesis of light elements in the very early Universe and the ratio of proton to neutron density is discussed.

3.4 The Friedmann Models

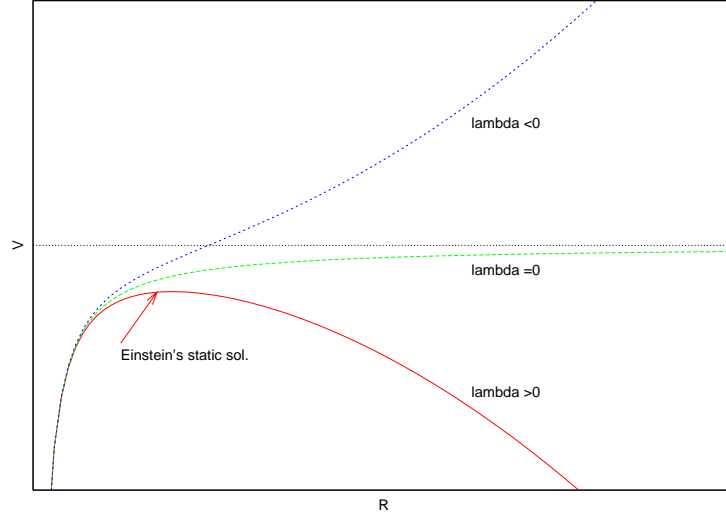
The Friedmann equation is

$$\dot{R}^2 = \frac{A}{R} - c^2 K + \frac{\Lambda c^2 R^2}{3}.$$

Pursuing the analogy with 1-dimensional particle mechanics:

$$\underbrace{\frac{1}{2}\dot{R}^2}_{\text{kinetic energy}} - \underbrace{\frac{1}{2}\left[\frac{\Lambda c^2 R^2}{3} + \frac{A}{R}\right]}_{\text{potential energy}} = \underbrace{-\frac{c^2 K}{2}}_{\text{total energy}}$$

Think of $V(R) = -\frac{A}{2R} - \frac{\Lambda c^2 R^2}{6}$ as the potential energy per unit mass of a particle moving in one dimension.



The “potential” $V(R)$ plotted as a function of R .

The behaviour of solutions depends, among other things, on the sign of K . Rescale R to set $K = \pm 1$ (or zero). Look for solutions with $\dot{R} > 0$ and $R(0)$ a non-negative constant (possibly zero). The most general case, with the three constants K , A and Λ all non-zero requires numerical solution. Analytic solutions can be found in various special cases, by setting one or other of the constants to zero.

i) Empty Models: $A = 0 \Rightarrow \dot{R} = c\sqrt{\frac{\Lambda R^2}{3} - K}$

$$\frac{dR}{\sqrt{\frac{\Lambda R^2}{3} - K}} = c dt \quad \Rightarrow \quad \int \frac{dR}{\sqrt{\frac{\Lambda R^2}{3} - K}} = ct$$

- a) $\Lambda = 0, K = 0 \Rightarrow R = \text{const}$
- b) $\Lambda = 0, K = -1 \Rightarrow R = ct$, **Milne-universe** ($R(0) = 0$)
- c) $\Lambda > 0, K = 0 \Rightarrow R(t) = R(0) \exp\left(\sqrt{\frac{\Lambda}{3}} ct\right)$
- d) $\Lambda > 0, K = 1 \Rightarrow R(t) = \sqrt{\frac{3}{\Lambda}} \cosh\left(\sqrt{\frac{\Lambda}{3}} ct\right)$ **de Sitter space** ($R(0) = \sqrt{\frac{3}{\Lambda}}$)
- e) $\Lambda > 0, K = -1 \Rightarrow R(t) = \sqrt{\frac{3}{\Lambda}} \sinh\left(\sqrt{\frac{\Lambda}{3}} ct\right)$

f) $\Lambda < 0, K = -1 \Rightarrow R(t) = \sqrt{\frac{3}{\Lambda}} \sin\left(\sqrt{\frac{\Lambda}{3}} ct\right)$ **Oscillating universe**

ii) $A \neq 0, \Lambda = 0$

$$\dot{R} = \sqrt{\frac{A}{R} - c^2 K} \quad \Rightarrow \quad dt = \frac{dR}{\sqrt{\frac{A}{R} - c^2 K}}$$

a) $K = 0, dt = \sqrt{\frac{R}{A}} dR \Rightarrow t = \frac{2}{3\sqrt{A}} (R(t)^{3/2} - R(0)^{3/2});$
with initial condition $R(0) = 0$ (universe "started" at $t = 0$ with zero size)

$$t = \frac{2}{3\sqrt{A}} R(t)^{3/2}$$

$$R(t) = \left(\frac{9A}{4}\right)^{1/3} t^{2/3}.$$

b) $K = 1, \dot{R} = \sqrt{\frac{A}{R} - c^2}$; solution in parametric form:

$$R(\psi) = \frac{A}{2c^2}(1 - \cos \psi)$$

$$t(\psi) = \frac{A}{2c^3}(\psi - \sin \psi) \quad (\text{Cycloid})$$

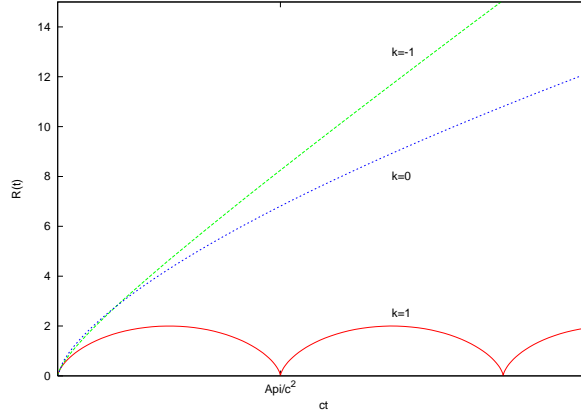
Check:

$$\begin{aligned} \dot{R} &= \frac{dR}{d\psi} \frac{d\psi}{dt} = \frac{\frac{A}{2c^2} \sin \psi}{\frac{A}{2c^3}(1 - \cos \psi)} = \frac{c \sin \psi}{1 - \cos \psi} \\ &= \frac{c \sqrt{1 - \left(\frac{2c^2}{A} R - 1\right)^2}}{\frac{2c^2 R}{A}} = \frac{A}{2cR} \sqrt{\frac{4c^2 R}{A} - \frac{4c^4}{A^2} R^2} \\ &= \sqrt{\frac{A}{R} - c^2} \end{aligned}$$

c) $K = -1, \dot{R} = \sqrt{\frac{A}{R} + c^2}$, a parametric solution is

$$R(\psi) = \frac{A}{2c^2}(\cosh \psi - 1)$$

$$t(\psi) = \frac{A}{2c^3}(\sinh \psi - \psi).$$



Models with $\Lambda = 0$ and $\rho_{Mat} \neq 0$. $R(t)$ is plotted against ct .

iii) $K = 0$ (favoured observationally).

In this case the Friedmann equation reduces to

$$\dot{R}^2 = \frac{A}{R} + \frac{\Lambda c^2 R^2}{3}$$

$$\begin{aligned} \Rightarrow \quad \dot{R}^2 R &= A + \frac{\Lambda c^2 R^3}{3} & \Rightarrow \quad 2\dot{R}\ddot{R}R + \dot{R}^3 &= \Lambda c^2 R^2 \dot{R} \\ \Rightarrow \quad \frac{\ddot{R}}{R} + \frac{\dot{R}^2}{2R^2} &= \frac{\Lambda c^2}{2} & \Rightarrow \quad \frac{1}{R^{3/2}} \frac{d}{dt} \left(\dot{R} R^{1/2} \right) &= \frac{\Lambda c^2}{2} \\ \Rightarrow \quad \frac{1}{R^{3/2}} \frac{d}{dt} \left(\frac{2}{3} \frac{d}{dt} \left(R^{3/2} \right) \right) &= \frac{\Lambda c^2}{2} & \Rightarrow \quad \frac{d^2}{dt^2} \left(R^{3/2} \right) &= \frac{3\Lambda c^2}{4} R^{3/2} \end{aligned}$$

a) $\Lambda = 0$

$$\frac{d^2}{dt^2} \left(R^{3/2} \right) = 0 \Rightarrow R^{3/2} = a + bt \Rightarrow \frac{3}{2} \dot{R} R^{1/2} = b$$

$\frac{\dot{R}}{R} = \frac{2}{3} \frac{b}{R^{3/2}} = \frac{2}{3} \frac{b}{a+bt}$ and $\left(\frac{\dot{R}}{R}\right)^2 = \frac{A}{R^3}$. Therefore

$$\frac{4}{9} \frac{b^2}{(a+bt)^2} = \frac{A}{(a+bt)^2}$$

$$b = \frac{3}{2} \sqrt{A}$$

Take $R(0) = 0 \Rightarrow a = 0$,

$$R^{3/2}(t) = \frac{3}{2} \sqrt{At}$$

as before.

b) $\Lambda > 0, \Rightarrow R^{3/2} = a'e^{\sqrt{\frac{3\Lambda}{4}}ct} + b'e^{-\sqrt{\frac{3\Lambda}{4}}ct}$ or:

$$R^{3/2} = a \cosh\left(\sqrt{\frac{3\Lambda}{4}}ct\right) + b \sinh\left(\sqrt{\frac{3\Lambda}{4}}ct\right)$$

Taking the initial condition $R(0) = 0 \Rightarrow a = 0$

$$\Rightarrow R = b^{2/3} \left\{ \sinh\left(\sqrt{\frac{3\Lambda}{4}}ct\right) \right\}^{2/3}$$

so

$$\frac{\dot{R}}{R} = \frac{2}{3} \sqrt{\frac{3\Lambda c^2}{4}} \frac{\cosh\left(\sqrt{\frac{3\Lambda}{4}}ct\right)}{\sinh\left(\sqrt{\frac{3\Lambda}{4}}ct\right)} \Rightarrow \left(\frac{\dot{R}}{R}\right)^2 = \frac{4}{9} \frac{3\Lambda c^2}{4} \frac{\cosh^2\left(\sqrt{\frac{3\Lambda}{4}}ct\right)}{\sinh^2\left(\sqrt{\frac{3\Lambda}{4}}ct\right)}$$

$$\text{and } \left(\frac{\dot{R}}{R}\right)^2 = \frac{A}{R^3} + \frac{\Lambda c^2}{3}$$

$$\Rightarrow \frac{\Lambda c^2}{3} \frac{\cosh^2\left(\sqrt{\frac{3\Lambda}{4}}ct\right)}{\sinh^2\left(\sqrt{\frac{3\Lambda}{4}}ct\right)} = \frac{A}{b^2 \left\{ \sinh\left(\sqrt{\frac{3\Lambda}{4}}ct\right) \right\}^2} + \frac{\Lambda c^2}{3}$$

$$\Rightarrow \frac{\Lambda c^2}{3} \frac{1 + \sinh^2 X - \sinh^2 X}{\sinh^2 X} = \frac{A}{b^2 \sinh^2 X}, \quad X = \sqrt{\frac{3\Lambda}{4}}ct$$

$$\Rightarrow \frac{\Lambda c^2}{3} = \frac{A}{b^2} \Rightarrow b^2 = \frac{3A}{\Lambda c^2} \Rightarrow b = \sqrt{\frac{3A}{\Lambda}} \frac{1}{c}.$$

Giving the solution

$$R(t) = \left(\frac{3A}{\Lambda c^2}\right)^{1/3} \left\{ \sinh\left(\sqrt{\frac{3\Lambda}{4}}ct\right) \right\}^{2/3}$$

For small t and very large t we get the limits:

$$R(t) = \begin{cases} \left(\frac{3A}{\Lambda c^2}\right)^{1/3} \left(\sqrt{\frac{3\Lambda}{4}} ct\right)^{2/3} = \left(\frac{9A}{4}\right)^{1/3} t^{2/3}, & t \rightarrow 0 \\ \left(\frac{3A}{\Lambda c^2}\right)^{1/3} \frac{1}{2^{2/3}} \exp\left(\frac{2}{3}\sqrt{\frac{3\Lambda}{4}} ct\right) = \left(\frac{3A}{4\Lambda c^2}\right)^{1/3} \exp\left(\sqrt{\frac{\Lambda}{3}} ct\right), & \text{late } t. \end{cases}$$

The real Universe is believed to be described by a solution of this form at the present day, intermediate between these two limits.

- c) $\Lambda < 0$, $R(t) = \left(\frac{3A}{c^2|\Lambda|}\right)^{1/3} \sin^{2/3}\left(\sqrt{\frac{3|\Lambda|}{4}} ct\right)$ with the same boundary conditions as a) and b) above.

3.5 Microwave background

The universe is "glowing" at $T = 2.725 \pm 0.001K$ ($\Rightarrow \lambda = 2mm$)

- i) Ordinary matter: In sphere of radius R are N galaxies, each mass m_G . The sphere contains Mass $M = Nm_G = \frac{4\pi}{3} R^3 \rho_{Mat}$, $\Rightarrow \rho_{Mat} = \frac{3Nm_G}{4\pi R^3} = \frac{3A}{8\pi G R^3}$

$$\rho_{Mat} \propto \frac{1}{R^3}.$$

- ii) Thermal Radiation, energy density ϵ . A sphere of radius R contains energy $E = \frac{4\pi}{3} R^3 \epsilon$, but $E \propto \nu$ (for thermal radiation $k_B T \approx h\nu$). As R increases, wavelength λ stretches, $\lambda = \frac{c}{\nu} \Rightarrow \nu$ decreases $\Rightarrow \nu \propto \frac{1}{R}$, $\Rightarrow E \propto \frac{1}{R}$, $\Rightarrow R^4 \epsilon = const$, $\epsilon \propto \frac{1}{R^4}$. Let $\rho_{Rad} = \frac{\epsilon}{c^2} = \frac{3B}{8\pi G R^4}$, with B a constant, be the mass density equivalent to ϵ . Then

$$\rho_{Rad} \propto \frac{1}{R^4}.$$

Radiation mass density contributes to the Friedmann equation,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{A}{R^3} + \frac{B}{R^4} + \frac{\Lambda c^2}{3} - \frac{c^2 K}{R^2} \quad (3.11)$$

At the present day, assuming $K = 0$, $\rho_{Mat} = 2.5 \times 10^{-27} kg m^{-3}$ For thermal radiation $\epsilon = \frac{4\sigma_{SB} T^4}{c}$ so $\rho_{Rad} = \frac{\epsilon}{c^2} = 4.642 \times 10^{-31} kg m^{-3}$, with $T = 2.725^\circ K$. So, at the present day,

$$\frac{\rho_{Rad}}{\rho_{Mat}} = 1.9 \times 10^{-4}.$$

At earlier times

$$\frac{\rho_{Rad}(t)}{\rho_{Mat}(t)} = \frac{\rho_{Rad}(t_0)}{\rho_{Mat}(t_0)} \frac{R_0}{R(t)} = (1.9 \times 10^{-4}) \frac{R_0}{R(t)}.$$

For matter dominated expansion (ignoring Λ),

$$R(t) = \left(\frac{9A}{4}\right)^{1/3} t^{2/3} \propto t^{2/3},$$

where t is in seconds. The energy density in radiation equals that in matter when

$$\left(\frac{t_0}{t}\right)^{2/3} = \frac{1}{1.9 \times 10^{-4}} \quad \Rightarrow \quad t = (1.9 \times 10^{-4})^{3/2} t_0 = 2.6 \times 10^{-6} t_0,$$

a few millionths of its present age. With $t_0 = 13.7 \times 10^9 yr$ we get that $\rho_{Rad} = \rho_{Mat}$ when $t = 36,000 yr$. For $t < 36,000 yr$ the energy density in radiation dominates over that in matter, so it is safe ignore A and the Friedmann equation becomes

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{B}{R^4} + \frac{\Lambda c^2}{3}.$$

But we also have $\frac{B}{R^4} \gg \frac{\Lambda c^2}{3}$ at the same time (though not today), so in the early universe ($t < 10^4 yr$) a very good approximation is

$$\dot{R}^2 = \frac{B}{R^2}$$

$\Rightarrow \dot{R}R = \sqrt{B} \Rightarrow \frac{1}{2} \frac{d}{dt}(R^2) = \sqrt{B} \Rightarrow R^2 = 2\sqrt{B}t + a$. With initial condition $R(0) = 0$ we get

$$R(t) = (2\sqrt{B})^{1/2} t^{1/2}.$$

Summary

$t > t_0 = 10^{10} yr$	$\left(\frac{\dot{R}}{R}\right)^2 = \frac{\Lambda c^2}{3}$	$R(t) \propto \exp\left(\sqrt{\frac{\Lambda}{3}} ct\right)$
$36,000 yr < t < t_0$	$\left(\frac{\dot{R}}{R}\right)^2 = \frac{A}{R^3}$	$R(t) \propto t^{2/3}$ matter dominated
$t < 36,000 yr$	$\left(\frac{\dot{R}}{R}\right)^2 = \frac{B}{R^4}$	$R(t) \propto t^{1/2}$ radiation dominated

$\frac{K}{R^2}$ never was and never will be significant.

When the universe was younger, it was hotter

$$T \propto \nu \propto \frac{1}{R}.$$

At the present day: $R_0 \approx 10^{26} m$, $T = 3K$, at $t = 36,000 yr$, $\frac{R_0}{R} = 10^4 \Rightarrow T = 3 \times 10^4 K$. All the matter (mostly Hydrogen and Helium) was ionised, when $T > 4000K$. This occurred when

$$\frac{R_0}{R} \approx 10^3 \Rightarrow \frac{t_0}{t} = 10^{9/2} = 3 \times 10^4 \Rightarrow t = \frac{1.4 \times 10^{10} yr}{3 \times 10^4} = 500,000 yr$$

(a more accurate figure is $t_s = 370,000 \text{ yrs}$). Before this time the matter in the Universe was an ionised plasma, afterwards it is mostly neutral Hydrogen unless it gets re-ionised due to the heat from stars.

As $R \propto t^{1/2}$ for small t we get $\dot{R} \xrightarrow{t \rightarrow 0} \infty$. We cannot trust the Friedmann equation back to $t = 0$. We can only go back to some time t_i (a fraction of a second) where the temperature and energies are still understood and replace our ignorance of $t < t_i$ with initial conditions $R(t_i)$.

3.6 The Horizon Problem and Inflation

Microwave photons from two points 180° apart in the sky come from the "surface of last scattering" when neutral hydrogen was formed at $t_s \approx 370,000 \text{ yr}$ when $T \approx 4000 \text{ K}$. So two diametrically opposite points in the sky appear to be the same temperature (to within a few parts in 10^6), so presumably they were in thermal contact at some point in the past. But this is inconsistent with our model.

The radius R_s of the surface of last scattering can be calculated. Let our own Galaxy sit at $r = 0$ and we observe photons coming in from the surface of last scattering at a co-ordinate distance r_s . In a short time interval dt at an intermediate time t , a photon travels a physical distance $-R(t)dr = cdt \Rightarrow dr = -c \frac{dt}{R(t)}$ ($dr < 0$ because the photon is travelling inwards, towards us)

$$\int_{r_s}^0 dr = -c \int_{t_s}^{t_0} \frac{dt}{R(t)}.$$

The physical radius at t_s is

$$R_s = R(t_s)r_s = cR(t_s) \int_{t_s}^{t_0} \frac{dt}{R(t)}.$$

For $t_s < t < t_0$ we have $R(t) = bt^{2/3}$. Therefore

$$\begin{aligned} R_s &= R(t_s) \frac{1}{b} c \int_{t_s}^{t_0} \frac{dt}{t^{2/3}} = bt_s^{2/3} \frac{c}{b} \left[3t^{1/3} \right]_{t_s}^{t_0} = \\ &= 3ct_s^{2/3} \left(t_0^{1/3} - t_s^{1/3} \right) = 3ct_s \left[\left(\frac{t_0}{t_s} \right)^{1/3} - 1 \right]. \end{aligned}$$

With $t_0 \approx 1.4 \times 10^{10} \text{ yr}$, $t_s \approx 3.7 \times 10^5 \text{ yr}$ we get

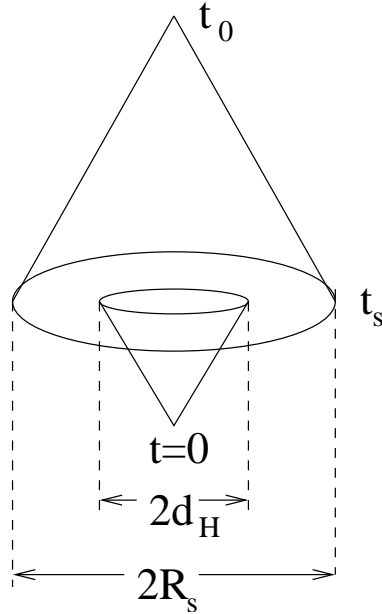
$$R_s \approx c \times 10^5 \times 30 \text{ yr} = (3 \times 10^8) \frac{m}{s} \times (3 \times 10^6) \times (3 \times 10^7 s) \approx 3 \times 10^{22} m.$$

Let the distance a photon could have travelled at time t_s , since $t = 0$, be d_H . For $t < t_s$ let $R(t) = b't^n$ ($n = \frac{2}{3}$ for matter dominated and $n = \frac{1}{2}$ for radiation dominated). Now

$$d_H = cb't_s^n \int_0^{t_s} \frac{dt}{b't^n} = ct_s^n \left[\frac{t^{-n+1}}{-n+1} \right]_0^{t_s} = \frac{c}{1-n} t_s$$

e.g. for $n = \frac{1}{2} \Rightarrow d_H = 2ct_s$ but then

$$\frac{R_s}{d_H} = \frac{3}{2} \left[\left(\frac{t_0}{t_s} \right)^{1/3} - 1 \right] \approx 50.$$



How can two points on the surface of last scattering, 180° apart in the sky and therefore a physical distance $2R_s$ apart be in thermal equilibrium with each other when $2R_s \approx 50(2d_H)$ is 50 times the distance a photon could have travelled since the beginning of the Universe? This is known as “the Horizon Problem”.

3.6.1 Inflationary Universe

Possible solutions

1. Friedmann equation is wrong - it breaks down for some $t < t_s$ (must happen at some early time anyway, since cannot allow $\dot{R} \rightarrow \infty$) (e.g quantum gravity).
2. Friedmann equation is correct, but change the R.H.S. In the "inflationary universe" picture it is assumed that a very large positive cosmological constant "switched on" for a very brief period at a very early time ($t \sim 10^{-35} s$) and was large enough to dominate the dynamics. If Λ dominates $\dot{R}^2 = \frac{\Lambda c^2 R^2}{3}$ and $R = R(0)e^{\sqrt{\frac{\Lambda}{3}}ct}$ exponential expansion for a period of time between t_1 and t_2 . This solves the horizon problem if $R(t_2)/R(t_1) \approx 10^{25}$.

As a by-product of inflation we get a natural explanation of why $K \approx 0$. Non-zero K is associated with curvature of 3-dimensional space, for example positive curvature,

$K > 0$, results in parallel lines converging, like great circles intersecting on the surface of a 2-dimensional sphere. The greater the radius of the sphere the smaller the curvature, for example it is difficult to detect the curvature of the Earth's surface on length-scales of a few metres. If $K \neq 0$ and is significant before inflation, when $t < 10^{-35}$ s, then its significance is decreased by a factor of 10^{25} if $R(t)$ increases by 10^{25} and the relevance of K in the Friedmann becomes negligible for all times after the period of inflation. This could explain why attempts to measure K today give a null result.

3.7 The first 3 minutes (Weinberg)

Looking back at $t < 36,000yr \approx 10^{12}s$ when $T = 10^4 \text{ }^\circ K$, $R \propto \frac{1}{T} \propto t^{1/2}$. $k_B T$ is an energy, in fact $k_B T \approx \frac{1}{\sqrt{t}}$ if the energy is given in *MeV* ($1 \text{ MeV} \approx 10^{10} \text{ K}$) and t in seconds.

$$T (10^{10} \text{ K}) \approx T (\text{MeV}) \approx \frac{1}{\sqrt{t (\text{secs})}}$$

It is remarkable how much we can deduce about the early Universe from this equation.

Time	Temperature/ Energy	
		$R \propto e^{\sqrt{\frac{\Lambda}{3}} ct} \uparrow$
$1.37 \times 10^{10} yr$	$2.7K$	t_0 , present day
$4.56 \times 10^9 yr$	$4K$	$R \propto t^{2/3}$ Matter dominated \downarrow Solar System formed
$10^9 yr$	$5K$	Galaxies formed. Era of Quasars and active galactic nuclei Heavy elements are created in stars and supernovae
$370,000 yr$	$4000K$	Hydrogen ionises, surface of last scattering $R \propto t^{2/3}$ Matter dominated \uparrow
$10^4 yr$	$25,000K$	$R \propto t^{1/2}$ Radiation dominated \downarrow No heavy elements; Plasma, p , ${}^4\text{He}$, e^- , ${}^7\text{Li}$, D , γ , ν , $\bar{\nu}$ 75% p and 25% ${}^4\text{He}$ by mass, $\frac{1}{8}$ neutrons, $\frac{7}{8}$ protons
100s	$0.1 MeV$ ($10^9 K$)	α -particles disintegrate as do D and Li , $N_n/N_p = 1/7$ Era of nucleo-synthesis
4s	$0.5 MeV$ ($5 \times 10^9 K$)	photons in thermal background can produce e^+ , e^- pairs $p, n, e^+, e^-, \nu, \bar{\nu}, \gamma$
1s	$1 MeV$	β -decay, $n \rightarrow p + e^- + \bar{\nu}$, starts to deplete neutrons
$10^{-2} s$	$10 MeV$ ($10^{11} K$)	$p + e^- \leftrightarrow n + \nu$ works both ways protons and neutrons in thermal equilibrium, $N_p/N_n = 1$
$5 \times 10^{-5} s$	$150 MeV$ ($1.5 \times 10^{11} K$)	p and n evaporate into quarks and gluons Plasma: $q, \bar{q}, e^-, e^+, \mu^-, \mu^+, \nu, \bar{\nu}, \gamma, g$
$10^{-10} s$	$100 GeV$	Electromagnetism and weak nuclear force unify into the electro-weak force
$10^{-33} s$		Era of inflation
$10^{-38} s$	$10^{16} GeV$	Grand Unification (electromagnetism, strong and weak); Grand Unified Theory, Supersymmetry, Superstrings?
$10^{-43} s$	$10^{19} GeV$	Era of Quantum Gravity; Friedmann equation cannot be correct

Between 2 s and 100 s free neutrons can decay to protons, $n \rightarrow p + e^- + \bar{\nu}_e$. After 100 s neutrons are bound into Helium nuclei and are stable because the only energy levels available to a proton arising from neutron decay are blocked, due to the Pauli exclusion principle, by the other two protons already present in the Helium nucleus. We can estimate how many neutrons are left at 100s using the lifetime for neutron decay in free space, $n \rightarrow p + e^- + \bar{\nu}$, which is $\tau = 900 s$. We should allow for the fact that the neutron to proton ratio N_n/N_p is not quite 1 at 2 s, due to the neutron-proton mass difference $\Delta E := (m_n - m_p)c^2 = 1.3 MeV$. The Maxwell-Boltzmann distribution for particles of energy E in a gas at temperature T is $n(E) \sim e^{-E/k_B T}$. Using $T = 1/\sqrt{t}$, with T measured in MeV and t in seconds, gives

$$\frac{N_n}{N_p} = e^{-\Delta E \sqrt{t}} \quad \Rightarrow \quad N_n = N_p e^{-\Delta E \sqrt{t}}.$$

If neutrons drop out of equilibrium at a time t_e and subsequently decay to protons until

they are bound into Helium nuclei at time t_{He} , then we expect the neutron-proton ratio at t_{He} to be

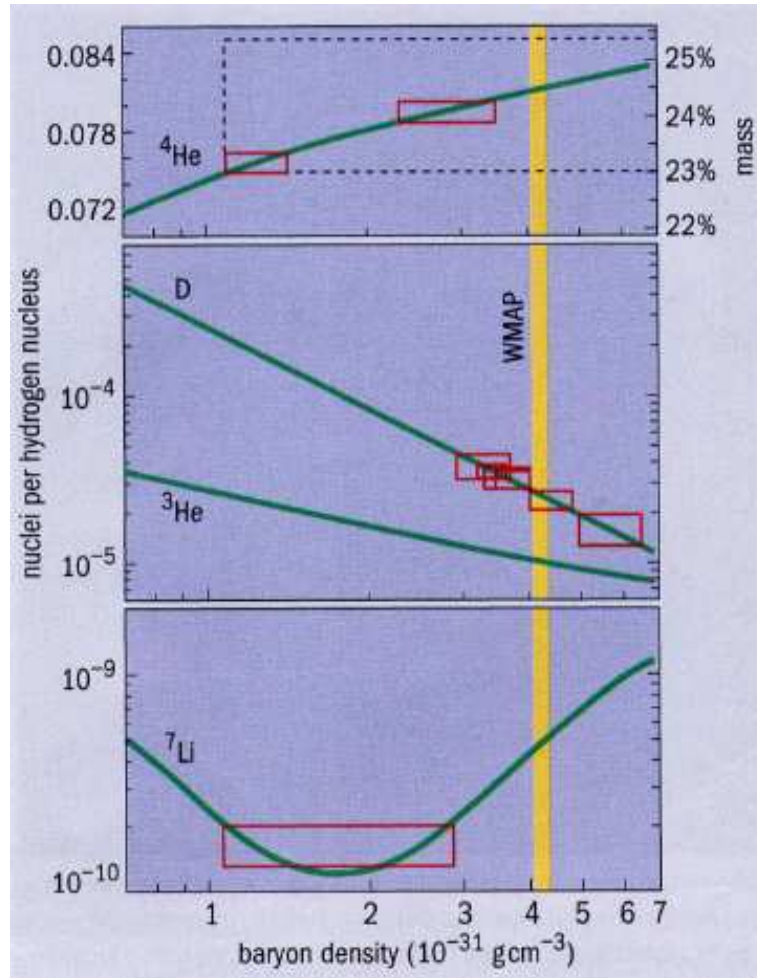
$$\frac{N_n}{N_p} = e^{-\frac{(t_{He}-t_e)}{\tau}} e^{-\Delta E \sqrt{t_e}}.$$

Using the values $\Delta E = 1.3 \text{ MeV}$, $\tau = 900 \text{ s}$, $t_e = 2 \text{ s}$ and $t_{He} = 100 \text{ s}$ gives

$$\frac{N_n}{N_p} \approx \frac{1}{7},$$

as observed.

If the rate of Helium production in the early Universe were larger, more Helium would be produced earlier and the neutrons would have less time to decay, resulting in a larger N_n/N_p ratio and hence a larger He/H ratio. If the rate of Helium production were smaller more of the Helium would be produced later and the neutrons would have more time to decay, resulting in a smaller N_n/N_p ratio and hence smaller He/H ratio. The rate for Helium production increases if the density of neutrons and protons increases and the observed ratio of primordial Hydrogen to Helium in the Universe puts a limit on the maximum allowed density of neutrons and protons consistent with observations. At the present day it cannot be more than 15% of the Dark Matter. There are similar considerations for other light elements that were produced in the Big Bang — deuterium, ^3He and ^7Li .



Abundances of primordial elements: the vertical axis is the abundance of each isotope relative to hydrogen. The red rectangles reflect the observational data — their vertical extent are the measured primordial abundances and their horizontal extent is obtained by comparing their vertical extent with the theoretical predictions as represented by the various green curves. The top curve shows the mass fraction of primordial ${}^4\text{He}$ relative to Hydrogen: since 1990 the abundances as measured by independent observations do not agree, probably indicating that uncertainties have been underestimated and the dotted black box may be a more accurate reflection of the observational uncertainties, giving a ${}^4\text{He}$ abundance lying between 0.23 and 0.25. Notice that the theoretical prediction for ${}^4\text{He}$ increases as the baryon density (the density of protons and neutrons) increases, as described above. The tightest constraints come from observations of inhomogeneities in the cosmic microwave background from a microwave detector called the Wilkinson Microwave Anisotropy Probe (WMAP - the vertical yellow band) indicating a value for the baryon density of $(4.1 \pm 0.1) \times 10^{-28} \text{kgm}^{-3}$, putting the density of protons plus neutrons at about 15% of the total mass density (or 4% of the critical density), implying the existence of another, unknown, type of matter. Direct observations of ${}^7\text{Li}$ are also somewhat lower than the WMAP value, perhaps due to the re-processing of ${}^7\text{Li}$ in stars not being fully understood. (Taken from Physics World, Vol. 28, No. 8, August 2007.)

At extremely high temperatures and energies the energy density in thermal radiation is so large that every photon behaves like a black-hole. Suppose a thermal photon has wavelength λ and frequency ν , so $\lambda = c/\nu$, and energy $E = h\nu$, with mass equivalent $M = E/c^2 = h\nu/c^2$. The wavelength is the same as the Schwarzschild radius when

$$\lambda \approx \frac{GM}{c^2} \quad \Rightarrow \quad E = h\nu = \frac{hc^3}{GM} = \frac{hc^5}{GE} \quad \Rightarrow \quad E = \sqrt{\frac{\hbar c^5}{G}}.$$

It is conventional to use \hbar rather than h (this is only an order of magnitude estimate) and define the *Planck Energy* as

$$E_{Planck} = \sqrt{\frac{\hbar c^5}{G}} = 2 \times 10^9 \text{ J} = 1.2 \times 10^{19} \text{ GeV}$$

and the mass equivalent is the *Planck Mass*,

$$M_{Planck} = E_{Planck}/c^2 = \sqrt{\frac{\hbar c}{G}}$$

which is about $10^{19} m_{proton}$ or 10^{-5} gm . Dividing the Planck energy by Planck's constant gives one over the *Planck Time*,

$$t_{Planck} = \sqrt{\frac{\hbar G}{c^5}} \approx 10^{-43} \text{ s},$$

and multiplying this by the speed of light gives the *Planck Length*,

$$l_{Planck} = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-35} \text{ m}.$$

The Friedmann equation is unlikely to be valid when the energy reaches the Planck energy over the Planck volume (the Planck length cubed). At these fantastic energy densities quantum effects probably require some, as yet unknown, quantum theory of gravity.

In fact the Planck energy appears in the Friedman equation naturally, at much more modest energies. Using the explicit expression for the Stefan-Boltzmann constant,

$$\sigma_{S-B} = \frac{\pi^2 k_B^4}{60 \hbar^3 c^2},$$

the energy density in thermal photons is

$$\epsilon_{Rad} = \frac{4}{c} \sigma_{S-B} T^4 = \frac{\pi^2}{15 c^3 \hbar^3} (k_B T)^4,$$

equivalent to a mass density

$$\rho_{Rad} = \frac{\pi^2}{15 c^5 \hbar^3} (k_B T)^4,$$

At early times, when the universe was less than about 10,000 years old, ρ_{Rad} dominates the Friedmann equation,

$$H^2 = \frac{8\pi G}{3} \rho_{Rad} = \frac{8\pi^3}{45} \frac{G}{c^5 \hbar^3} (k_B T)^4 = \frac{8\pi^3}{45 \hbar^2} \frac{(k_B T)^4}{(E_{Planck})^2}.$$

Then the cosmic scale factor $a(t) \propto t^{1/2}$, so $H = \frac{1}{2t}$, and¹

$$(k_B T)^4 = \frac{45}{32\pi^3} \left(\frac{\hbar E_{Planck}}{t} \right)^2.$$

This formula is valid for times as late as 10,000 years, the appearance of the Planck energy here is not a signal of quantum gravity effects, it is merely due to the fact that a classical gravitational field is being sourced by quantum matter (thermal photons).

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¹Actually, as written, this equation is only valid for temperatures for which $k_B T \ll m_e c^2$, that is times later than about 4s. For temperatures of order $2m_e c^2/k_B$ and greater electron-positron pairs can be created out of thermal energy and they contribute to ρ_{Rad} . This modifies the prefactor $\frac{8\pi^3}{45\hbar^2}$ but the general conclusion is unchanged.

Quantity	Symbol	Value
Speed of light (in vacuum)	c	$299\,792\,458\text{ m s}^{-1}$ (exact)
Newton's constant	G	$6.673 \times 10^{-11}\text{ kg}^{-1}\text{ m}^3\text{ s}^{-2}$
Planck's constant	h	$6.626 \times 10^{-34}\text{ J s}$
Electron charge (magnitude)	e	$1.602 \times 10^{-19}\text{ C}$
Electric permittivity (vacuum)	$\epsilon_0 = \frac{1}{\mu_0 c^2}$	$8.854 \times 10^{-12}\text{ C}^2\text{ N}^{-1}\text{ m}^{-2}$
Magnetic permeability (vacuum)	μ_0	$4\pi \times 10^{-7}\text{ N s}^2\text{ C}^{-2}$
Fine structure constant	$\alpha = \frac{e^2}{2\epsilon_0 h c}$	7.297×10^{-3}
Thompson cross-section	σ_T	$6.652 \times 10^{-29}\text{ m}^2$
Electron mass	m_e	$9.109 \times 10^{-31}\text{ kg}$
Proton mass	m_p	$1.673 \times 10^{-27}\text{ kg}$
Neutron mass	m_n	$1.675 \times 10^{-27}\text{ kg}$
Atomic mass unit (mass of ^{12}C atom /12)	$a.m.u.$	$1.661 \times 10^{-27}\text{ kg}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\text{ J K}^{-1}$
Stefan-Boltzmann constant	σ_{SB}	$5.670 \times 10^{-8}\text{ J s}^{-1}\text{ m}^{-2}\text{ K}^{-4}$
Avagadro's number	N_A	$6.022 \times 10^{23}\text{ mol}^{-1}$
Earth mass	M_{\oplus}	$5.97 \times 10^{24}\text{ kg}$
Earth radius (equatorial)	R_{\oplus}	$6.38 \times 10^3\text{ km}$
Lunar mass	M_{C}	$7.35 \times 10^{22}\text{ kg}$
Lunar radius	R_{C}	$1.74 \times 10^3\text{ km}$
Earth-Moon distance (mean)	$d_{\oplus-\text{C}}$	$3.84 \times 10^5\text{ km}$
Earth-Sun distance (mean)	$d_{\oplus-\odot}$	$1.50 \times 10^8\text{ km}$
Solar mass	M_{\odot}	$1.99 \times 10^{30}\text{ kg}$
Solar radius (equatorial)	R_{\odot}	$6.961 \times 10^5\text{ km}$
Solar luminosity	L_{\odot}	$3.85 \times 10^{26}\text{ J s}^{-1}$
Temperature of microwave background	T_0	$2.725 \pm 0.002^\circ\text{ K}$
Hubble constant ($H_0 = 100h\text{ km s}^{-1}\text{ Mpc}^{-1}$)	H_0 h	$70.4 \pm 1.4\text{ km s}^{-1}\text{ Mpc}^{-1}$ 0.704 ± 0.014
Critical density	$\rho_c = \frac{3H_0^2}{8\pi G}$	$1.88 \times 10^{-26}\text{ h}^2\text{ kg m}^{-3}$
Dark energy density (Cosmological constant)	Ω_{Λ}	0.728 ± 0.016
Baryon density	$\Omega_B = \rho_B/\rho_{crit}$	0.0456 ± 0.0016
Dark matter density	$\Omega_M = \rho_M/\rho_{crit}$	0.227 ± 0.014
Total density	Ω_{tot}	1.001 ± 0.014
Age of the Universe	t_0	$13.7 \pm 0.2 \times 10^9\text{ yr}$
Electron Volt	eV	$1.602 \times 10^{-19}\text{ J}$
year	yr	$3.156 \times 10^7\text{ s}$
light year	lyr	$9.461 \times 10^{15}\text{ m}$
parsec ($1pc=3.26\text{ lyr}$)	pc	$3.086 \times 10^{16}\text{ m}$