MP467: Astrophysics and Cosmology

(1) Calculate the acceleration due to gravity on the surface of the Moon.

(2) Using the value of the acceleration due to gravity at the surface of the Earth, $g = 9.81\ m\ s^{-2}$, and the radius of the Earth, $R_\oplus = 6,380\ km$, calculate the mass of the Earth. How does your answer compare with the accepted value of $5.972 \times 10^{24}\ kg$?

(3) Using the fact that the Moon takes 27.3 days to orbit the Earth, at a distance of $384,000\ km$, use Newton's universal law of gravitation to calculate the mass of the Earth. Compare your answer with (2), what has gone wrong?

Hint: determine where the centre of gravity of the Earth-Moon system lies in relation to the centre of the Earth.

(4) Using the fact that the Earth takes 365 days to orbit the Sun, at a distance of $150,000,000\ km$, use Newton's universal law of gravitation to calculate the mass of the Sun.

(5) Given the radius of the Earth's orbit around the Sun, $150 \times 10^6\ km$ calculate the distance at which a star would exhibit a parallax of 1 arc second.

(6) Show the that gravitational potential energy stored in a sphere of uniform density with mass M and radius R is $E_{Grav} = -\frac{3}{5}\frac{GM^2}{R}$.

(7) The equation governing the internal dynamics of a spherical star in the lecture notes is

$$\rho(r)\ddot{r} = -\frac{GM(r)\rho(r)}{r^2} - \frac{dP(r)}{dr}.$$

Apply this equation to the surface of a spherical cloud of gas particles with radius R and constant total mass $M_0 = M(R)$, assuming uniform density, obeying the ideal gas law with constant temperature T, to show that the cloud will collapse if M_0 is greater than or of the order of

$$M_0 \gtrsim \frac{9}{2\sqrt{\pi}}\left(\frac{k_B T}{Gm}\right)^{3/2}\frac{1}{\sqrt{\rho}} := M_J$$

where m is the mass of the gas particles. The critical mass for collapse, M_J, is known as the Jean's Mass. (Hint: the critical mass is given by $\ddot{r} = 0$).

Calculate the Jean's mass for a dust cloud at a temperature of $100°K$ with 10^8 Hydrogen molecules per cubic metre.

(8) Calculate the ratio of the kinetic to the potential energy of a planet in stable circular orbit around the Sun. Compare you answer to the virial theorem.

(9) Using the equations for stellar equilibrium given in the lectures derive the second order differential equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho(r)$$

for the pressure P inside a star as a function of distance from the centre r.

(10) Solve the differential equation for $P(r)$ derived in question (9) for the case of a spherically symmetric star of total mass M and radius R with uniform density $\rho = \frac{3M}{4\pi R^3}$, assuming that the pressure at the surface, $P(R) = 0$ and the pressure at the centre is finite. Show that the pressure as a function of radius is

$$P(r) = P_c \left(1 - \frac{r^2}{R^2}\right)$$

where $P_c = G\left(\frac{\pi}{6}\right)^{1/3} M^{2/3}\rho^{4/3}$ is the central pressure. Using this model calculate the value of P_c for the Sun.

(11) In the lectures the ideal gas law was used to determine the temperature at the centre of a star. Given that the temperature is high enough to ionise the material in the core there will be a Coulomb repulsion between protons at the centre of the Sun which would be expected to give deviations from the ideal gas law. Calculate the average separation between two protons in the centre of the Sun and compare their Coulomb energy with their thermal energy. Do you expect a significant deviation from the ideal gas law due to Coulomb repulsion?

(12) Assuming a linear decrease in density inside a spherically symmetric star, $\rho(r) = \rho_c\left(1 - \frac{r}{R}\right)$ with ρ_c the central density and R the radius of the star, show that:

i) the mass contained inside a radius r is

$$m(r) = \frac{4\pi}{3} r^3 \rho_c - \frac{\pi r^4}{R}\rho_c$$

ii) the total mass of the star is

$$M = M(R) = \frac{\pi R^3}{3}\rho_c$$

iii) derive the equation

$$\frac{dP}{dr} = -\pi G \rho_c^2 \left(\frac{4}{3} r - \frac{7}{3}\frac{r^2}{R} + \frac{r^3}{R^2}\right)$$

for the pressure

iv) deduce that, as a function of r,

$$P(r) = P_c - \pi G \rho_c^2 \left(\frac{2r^2}{3} - \frac{7r^3}{9R} + \frac{r^4}{4R^2}\right)$$

where P_c is the central pressure. Show that $P_c = \frac{5\pi}{36} G\rho_c^2 R^2 = \frac{5}{4\pi}\frac{M^2 G}{R^4}$. Compare this central pressure and the central density in ii) with those for a star of mass M and radius R as given by the simple model in the lectures.

v) Assuming an ideal gas law equation of state, $P = \frac{2\rho}{m_p} k_B T$, derive the following expression for the temperature as a function of r

$$T = \frac{5\pi}{72}\frac{G m_p}{k_B}\rho_c R^2 \left(1 + \frac{r}{R} - \frac{19}{5}\frac{r^2}{R^2} + \frac{9}{5}\frac{r^3}{R^3}\right).$$

(13) Use the formulae for the pressure gradient and and the temperature gradient in a normal star given in the lecture notes,

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \qquad \text{and} \qquad \frac{d(T^4)}{dr} = -\frac{3\kappa(r)\rho(r)L(r)}{4\pi acr^2},$$

to show that, when radiation pressure, $P_r = \frac{a}{3}T^4$ ($a = 4\sigma_{SB}/c$, where σ_{SB} is the Stefan-Boltzmann constant), dominates over gas pressure, $P_g = \frac{\rho}{m}k_BT$, the luminosity is given by

$$L(r) = \frac{4\pi cGM(r)}{\kappa(r)}.$$

How does this answer compare, at the surface of the star, with the mass-luminosity relation derived in the lectures?

(14) The total flux of energy from the Sun at the Earth's orbit is $\mathcal{F} = 1400\ Wm^{-2}$. Use this to determine the luminosity of the Sun. Compare your answer with the number $4 \times 10^{26}\ W$ quoted in the lectures.

(15) Show that the equation for conservation of energy,

$$\frac{dL(r)}{dr} = 4\pi r^2 w(r)$$

where $w(r)$ is the rate of energy production per unit volume, and the equation for the temperature gradient when energy transport is radiative,

$$\frac{d(T^4)}{dr} = -\frac{3\kappa(r)\rho(r)L(r)}{4\pi acr^2},$$

can be combined into a single second order differential equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)\kappa(r)}\frac{d(T^4)}{dr}\right) = -\frac{3w(r)}{ac}.$$

(16) Using the equations for the pressure and temperature gradient in question (13) derive the relation,

$$\frac{dP}{dT} = \frac{16\pi acGMT^3}{3\kappa L}$$

where the pressure $P(T)$ is considered as function of T.

Assuming that the opacity is related to the temperature by $\kappa = \kappa_0\rho T^{-7/2}$ where κ_0 is a constant (this is known as Kramer's law) and that the medium is an ideal gas $P = \frac{\rho}{m}k_BT$ of particles with average mass m, derive the following expression for the density near the surface of a star, where the mass M and the luminosity L can be taken to be constant,

$$\rho = \left(\frac{64\pi acGMm}{51\kappa_0 k_B L}\right)^{1/2} T^{13/4}.$$

(17) In the calculation of a minimum stellar mass given in the lectures, prove that $T_c(\rho_c)$ is a *maximum* when $\frac{dT_c}{d\rho_c} = 0$.

(18) In the *pp*-chain process in the core of the Sun it was shown in the lectures that the reaction rate per unit volume for $p + p \rightarrow d + e^+ + \nu_e$ is $\approx 1.1 \times 10^{14}\ m^{-3}\, s^{-1}$. Given that each ^4He nucleus produced releases 26 MeV and the total luminosity of the Sun is $3.8 \times 10^{24}\ J\, s^{-1}$, estimate the volume of the solar core that is involved in energy production via nuclear reactions, assuming that the *pp*-chain is the dominant process.

The creation of each ^4He nucleus in the *pp*-chain results in the production of two neutrinos that immediately escape from the solar core. Estimate the total number of neutrinos that are produced every second. What is the flux of neutrinos (the number of neutrinos per second per square metre) at the distance of the Earth's orbit, assuming isotropic production?

(19) Calculate the pressure in a white dwarf (assuming an ideal gas equation of state) and compare it to the degeneracy pressure, using the parameters of a typical white dwarf: $\rho = 10^9 kg\ m^{-3}$, $T = 10^6 K$ and assume the star is made up mostly of ^4He nuclei. How would your answer change if the star were composed mostly of ^{12}C?

(20) Calculate the acceleration due to gravity on the surface of a white dwarf with the mass of the Sun and radius $10,000\ km$ and on the surface of a neutron star with the same mass and a radius of $10\ km$.

(21) A neutron star has a rocky crust consisting mostly of iron. If the nuclear binding energy of an iron nucleus in the crust is 10 MeV, estimate the height of a mountain on a neutron star.

(22) A neutron star with a radius of $10km$ and mass $1.5M_\odot$ is rotating 100 times a second. Calculate the ratio of the centrifugal acceleration at the equator to the acceleration due to gravity.

(23) Estimate the escape velocity at the surface of a spherical neutron star with mass $1.5M_\odot$ and radius $10km$.

(24) Calculate the total power radiated in the rotating magnetic dipole model of the Crab pulsar discussed in the lectures and compare it with the luminosity of the Sun.

(25) Suppose that our galaxy is a typical size, containing about 10^{11} stars all about the same mass as the Sun. Given that the galaxies are typically one megaparsec apart, estimate the density of the Universe.

(26) A cluster of about 300 galaxies in the constellation of Leo has an average redshift of 0.065. Using the redshift-distance relation estimate the distance of the cluster from the Earth.

(27) Show that the deceleration parameter in a radiation dominated universe, with $K = \Lambda = 0$, is $q = 1/2$.

(28) With the density of matter $\rho_{Mat} = \frac{3A}{8\pi G R^3}$, where A is a positive constant, the Friedmann equation is

$$\dot{R}^2 = \frac{A}{R} - c^2 K + \frac{\Lambda c^2 R^2}{3}.$$

Derive the following solutions using the initial condition $R(0) = 0$.

i) $A \neq 0,\ \Lambda = 0,\ K = 0.$

$$R(t) = \left(\frac{9A}{4}\right)^{1/3} t^{2/3}.$$

ii) $A \neq 0,\ \Lambda > 0,\ K = 0.$

$$R(t) = \left(\frac{3A}{\Lambda c^2}\right)^{1/3} \left\{\sinh\left(\sqrt{3\Lambda}\frac{ct}{2}\right)\right\}^{2/3}.$$

iii) $A \neq 0,\ \Lambda < 0,\ K = 0.$

$$R(t) = \left(\frac{3A}{|\Lambda| c^2}\right)^{1/3} \left\{\sin\left(\sqrt{3|\Lambda|}\frac{ct}{2}\right)\right\}^{2/3}.$$

Show that both (ii) and (iii) reduce to the matter dominated form (i) $R(t) = \left(\frac{9A}{4}\right)^{1/3} t^{2/3}$ for early times.

Prove that the following parametric forms of $R(t)$, with parameter ψ, satisfy the Friedmann equation with the stated conditions:

iv) $A \neq 0,\ \Lambda = 0,\ K = 1.$

$$R(\psi) = \frac{A}{2c^2}(1 - \cos\psi) \qquad t(\psi) = \frac{A}{2c^3}(\psi - \sin\psi).$$

iv) $A \neq 0,\ \Lambda = 0,\ K = -1.$

In parametric form

$$R(\psi) = \frac{A}{2c^2}(\cosh\psi - 1) \qquad t(\psi) = \frac{A}{2c^3}(\sinh\psi - \psi).$$

(29) The exact red-shift distance relation between the red-shift z of a galaxy and its distance d, derived in the lectures, is

$$d = \frac{c}{H_0}\int_0^z \frac{dz}{\sqrt{(1+z)^3\Omega_M + (1+z)^2\Omega_K + \Omega_\Lambda}},$$

where H_0 is Hubble's constant and Ω_M, Ω_K and Ω_Λ are constants with $\Omega_M + \Omega_K + \Omega_\Lambda = 1$.

Determine the functional form of $d(z)$ each the following three cases:

i) $\Omega_M = \Omega_\Lambda = 0$;

ii) $\Omega_K = \Omega_M = 0$;

iii) $\Omega_K = \Omega_\Lambda = 0$.

In each case solve the corresponding Friedmann equation, with $R(t_0) = R_0$, to find $R(t)$ and hence express the distance as a function of time, $d(t)$, with $t < t_0$.

Sketch d as a function of z in each of the three cases, on the same graph. Our Universe is observed to have $\Omega_K = 0$ and $\Omega_\Lambda \approx 2\Omega_M$, sketch this case on the same graph as the three above.

Sketch the case $\Omega_M = 0$ on the same graph (this integral cannot be evaluated analytically).

(30) Estimate the wavelength of a typical photon in a thermal radiation bath at $2.7\ K$.

(31) The temperature at the core of the Sun is around $10^7\ K$. How old was the Universe when it was at this temperature? Was it matter dominated or radiation dominated at this time?

The Large Hadron Collider at CERN in Geneva will collide lead nuclei at energies of order $3\ TeV$ (3,000 GeV) per nucleon, in the hope of producing a hot plasma of quarks and gluons. What temperature would this correspond to? How old was the Universe when photons in the thermal background were this temperature?