

Angular Momentum in quantum Mechanics

In classical physics angular momentum is a vector \mathbf{J} with a magnitude J and a direction specified by three components (J_x, J_y, J_z) . Of course these are not independent, since

$$J^2 = \mathbf{J} \cdot \mathbf{J} = J_x^2 + J_y^2 + J_z^2,$$

and angular momentum is specified by three real numbers. Vectors are added using the usual triangle law from which

$$|J_1 - J_2| \leq |\mathbf{J}_1 + \mathbf{J}_2| \leq J_1 + J_2. \quad (1)$$

Quantum mechanically things are different in a number of ways:

- Angular momentum is *quantised*. J is not a continuous variable but can take only discrete values which are multiples of the fundamental quantum unit \hbar . In fact

$$J^2 = j(j+1)\hbar^2$$

where j is either a non-negative integer, $j = 0, 1, 2, \dots$ or a positive half-integer $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$

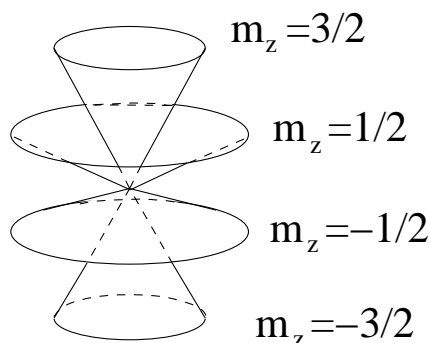
- The uncertainty principle says that J_x , J_y and J_z cannot be simultaneously specified — there are only two degrees of freedom in quantum angular momenta. If J is specified then we can measure only one linear combination of J_x , J_y or J_z , not all three independently. For example if J and J_z are known for a given quantum state then J_x and J_y are completely undetermined and have no physical value. Furthermore J_z is also quantised and has only $2j + 1$ possible values

$$J_z = m_z \hbar$$

where

$$m_z = -j, -j + 1, \dots, j - 1, j. \quad (2)$$

For $j = \frac{3}{2}$, for example, there are 4 possible values of J_z , given by $m_z = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$, but J_x and J_y are completely undetermined. Pictorially the situation looks something like this:



where \mathbf{J} is represented by a cone whose height is determined by m_z .

- Because of the above two restrictions, addition of angular momenta in quantum mechanics is also different to the classical case. If two angular momenta \mathbf{J}_1 and \mathbf{J}_2 , are added to give a third \mathbf{J} then the triangle inequality (1) still holds, but J can only take the discrete values

$$J = |J_1 - J_2|, |J_1 - J_2| + \hbar, \dots, J_1 + J_2 - \hbar, J_1 + J_2,$$

or equivalently

$$j = |j_1 - j_2|, |j_1 - j_2| + 1, \dots, j_1 + j_2 - 1, j_1 + j_2.$$

If the z -components are known there are further constraints on the sum: if \mathbf{J}_1 has third component $J_{1,z}$ and \mathbf{J}_2 has third component $J_{2,z}$ then these do just add like the classical case and \mathbf{J} will have third component

$$J_z = J_{1,z} + J_{2,z}$$

or $m = m_1 + m_2$. This requires $j \geq |m|$ because of (2).

In general a particle need not be in a definite state of angular momentum, it may be in a linear superposition of different j and m . A general state is a vector in a Hilbert space and we can use definite states of j and m as basis vectors, denoted by $|j; m\rangle$. This basis can be chosen so that it is orthonormal with

$$\langle j; m | j'; m' \rangle = \delta_{jj'} \delta_{mm'}.$$

A general state is then a linear sum of basis vectors

$$\Psi = \sum_j \sum_{m=-j}^j \psi_{jm} |j; m\rangle$$

where ψ_{jm} are complex numbers. For example adding a state with definite angular momentum $|j_1; m_1\rangle$ to a second state with definite angular momentum $|j_2; m_2\rangle$ produces a linear superposition of states with angular momenta j between $|j_2 - j_1|$ and $j_2 + j_1$. A standard notation for this is

$$|j_1; m_1\rangle \otimes |j_2; m_2\rangle = \sum_{j=|j_2-j_1|}^{j_2+j_1} C_{j_1 j_2; m_1 m_2}^{j; m} |j; m\rangle \quad (3)$$

where the numbers $C_{j_1 j_2; m_1 m_2}^{j; m}$ are called *Clebsch-Gordon coefficients*. Conservation of angular momentum requires that only states with $m = m_1 + m_2$ appear in the sum on the right-hand side, so $C_{j_1 j_2; m_1 m_2}^{j; m} = 0$ unless $m = m_1 + m_2$.

The reason for the \otimes symbol is that, for fixed j_1 and j_2 , there are $2j_1 + 1$ possible values of m_1 and $2j_2 + 1$ possible values of m_2 . Thus the set

$$\{|j_1; m_1\rangle; m_1 = -j_1, -j_1 + 1, \dots, j_1 - 1, j_1\}$$

is a basis for a $2j_1 + 1$ dimensional vector space and

$$\{|j_2; m_2 \rangle; m_2 = -j_2, -j_2 + 1, \dots, j_2 - 1, j_2\}$$

is a basis for a $2j_2 + 1$ dimensional vector space. There are therefore $(2j_1 + 1)(2j_2 + 1)$ possible states on the left-hand side of equation (3) so vector addition in quantum mechanics is like multiplication in terms of Hilbert space dimensions.¹ A slightly more compact notation, which we shall often use, is

$$|j_1, j_2; m_1, m_2 \rangle = |j_1; m_1 \rangle \otimes |j_2; m_2 \rangle .$$

For each value of j on the right-hand side of (3) there are $2j + 1$ possible values of m and j runs from $|j_2 - j_1|$ to $j_2 + j_1$ giving, assuming $j_2 \geq j_1$ for example,

$$\sum_{j=j_2-j_1}^{j_2+j_1} (2j + 1) = \{(j_2 + j_1) + (j_2 - j_1)\}(2j_1 + 1) + (2j_1 + 1) = (2j_1 + 1)(2j_2 + 1)$$

different $|j; m \rangle$.

The quantum Hilbert space associated with these angular momentum states is a finite dimensional complex vector space with dimension $(2j_1 + 1)(2j_2 + 1)$ and either

$$\{|j; m \rangle; j = |j_1 - j_2|, |j_1 - j_2| + 1, \dots, j_1 + j_2 - 1, j_1 + j_2, m = -j, -j + 1, \dots, j - 1, j\}$$

or

$$\{|j_1, j_2; m_1, m_2 \rangle; m_1 = -j_1, -j_1 + 1, \dots, j_1 - 1, j_1, m_2 = -j_2, -j_2 + 1, \dots, j_2 - 1, j_2\}$$

can be used as an orthonormal basis for the space. The Clebsch-Gordon coefficients just give a linear transformation between two different orthonormal bases.

Consider for example combining two spin-1/2 electrons, so $j_1 = j_2 = 1/2$. Each electron has a spin which is a vector in a two-dimensional Hilbert space of possible spins, its spin is some linear combination of two possible spin states, often called spin up ($|\uparrow \rangle$) and spin down ($|\downarrow \rangle$) relative to some fiducial direction such as the z -axis of a Cartesian co-ordinate system. In our notation

$$\left| \frac{1}{2}; \frac{1}{2} \right\rangle = |\uparrow \rangle, \quad \text{and} \quad \left| \frac{1}{2}; -\frac{1}{2} \right\rangle = |\downarrow \rangle .$$

There are only two possible results for the total angular momentum when the electron spins are combined, $j = \frac{1}{2} - \frac{1}{2} = 0$ or $j = \frac{1}{2} + \frac{1}{2} = 1$. If both electrons are spin up, $m_1 = m_2 = 1/2$, the combined spin must have $m = 1$ and only $j = 1$ is allowed in the combined state:

$$\left| \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2} \right\rangle = |1; 1 \rangle \quad \Leftrightarrow \quad |\uparrow \rangle \otimes |\uparrow \rangle = |\uparrow\uparrow \rangle .$$

¹ For historical reasons the \otimes operation is called a *tensor product* — a term that comes originally from the theory of elasticity in solids.

Similarly if both electrons are spin down, $m_1 = m_2 = -1/2$, the combined spin must have $m = -1$ and again only $j = 1$ is allowed in the combined state:

$$\left| \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}, -\frac{1}{2} \right\rangle = |1; -1\rangle \quad \Leftrightarrow \quad |\downarrow\rangle \otimes |\downarrow\rangle = |\downarrow\downarrow\rangle.$$

Obviously these two states are symmetric under interchange of the two electrons.²

If however one electron has $m_1 = 1/2$ and the other $m_2 = -1/2$, or *vice versa*, then the combination has $m = 0$ and this could be either $j = 1$ or $j = 0$: $j = 1$ is the symmetric combination

$$|1; 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

and $j = 0$ the orthogonal combination

$$|0; 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle),$$

which is necessarily anti-symmetric.

These can be inverted to give

$$\begin{aligned} |\uparrow\downarrow\rangle &= \left| \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, -\frac{1}{2} \right\rangle = \frac{1}{\sqrt{2}} (|1; 0\rangle + |0; 0\rangle), \\ |\downarrow\uparrow\rangle &= \left| \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}, \frac{1}{2} \right\rangle = \frac{1}{\sqrt{2}} (|1; 0\rangle - |0; 0\rangle), \end{aligned}$$

from which the Clebsch-Gordon coefficients can be read off

$$C_{\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, -\frac{1}{2}}^{1; 0} = \frac{1}{\sqrt{2}}, \quad C_{\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, -\frac{1}{2}}^{0; 0} = \frac{1}{\sqrt{2}}, \quad C_{\frac{1}{2}, \frac{1}{2}; -\frac{1}{2}, \frac{1}{2}}^{1; 0} = \frac{1}{\sqrt{2}}, \quad C_{\frac{1}{2}, \frac{1}{2}; -\frac{1}{2}, \frac{1}{2}}^{0; 0} = -\frac{1}{\sqrt{2}}.$$

The total Hilbert space for the electrons' spin is 4-dimensional and one can use either

$$\left\{ \left(\left| \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2} \right\rangle \right), \left(\left| \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, -\frac{1}{2} \right\rangle \right), \left(\left| \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}, \frac{1}{2} \right\rangle \right), \left(\left| \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}, -\frac{1}{2} \right\rangle \right) \right\}$$

or

$$\{|1; 1\rangle, |1; 0\rangle, |1; -1\rangle, |0; 0\rangle\}$$

as a set of basis vectors. The decomposition into $j = 1$ (triplet) and $j = 0$ (singlet) sectors is

$$\begin{aligned} |1; 1\rangle &= |\uparrow\uparrow\rangle \\ |1; 0\rangle &= \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |1; -1\rangle &= |\downarrow\downarrow\rangle \end{aligned} \tag{4}$$

² The total quantum state corresponding to two Fermions should of course be anti-symmetric under interchange of the two particles. Spin is only part of the story, a complete quantum description should also include position. For example electrons with $j=1$ could have a relative orbital angular momentum of $l=1$ making the total quantum state antisymmetric under interchange of the two electrons.

and

$$|0; 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle). \quad (5)$$

It is often useful to exhibit the linear transformation that the Clebsch-Gordon coefficients represent in the form of a table.

		$j = 1$			$j = 0$
m_1	m_2	$m = 1$	$m = 0$	$m = -1$	$m = 0$
1/2	1/2	1	0	0	0
1/2	-1/2	0	$1/\sqrt{2}$	0	$1/\sqrt{2}$
-1/2	1/2	0	$1/\sqrt{2}$	0	$-1/\sqrt{2}$
-1/2	-1/2	0	0	1	0

A table for $j = \frac{1}{2}$ combined with $j = \frac{3}{2}$ will be derived later.