MP465 – Advanced Electromagnetism

Solutions to Problem Set 3

1. The point of this problem is just to get you used to the types of manipulations you need to do when using the Levi-Civita symbol, because it appears surprisingly frequently throughout maths and phsyics...

   (a) The curl of a vector field $\vec{b}$ is, component-wise,

$$\begin{aligned}\left(\vec{\nabla}\times\vec{b}\right)_m &= \sum_{k\ell}\epsilon_{mk\ell}(\vec{\nabla})_k b_\ell \\ &= \sum_{mn}\epsilon_{mk\ell}\frac{\partial b_\ell}{\partial x_k}\end{aligned}$$

   where all sums will be from 1 to 3 (or over x, y and z – same thing). Thus,

$$\begin{aligned}\left[\vec{a}\times\left(\vec{\nabla}\times\vec{b}\right)\right]_i &= \sum_{jm}\epsilon_{ijm}a_j\left(\vec{\nabla}\times\vec{b}\right)_m \\ &= \sum_{jm}\epsilon_{ijm}a_j\left(\sum_{k\ell}\epsilon_{mk\ell}\frac{\partial b_\ell}{\partial x_k}\right) \\ &= \sum_{jk\ell m}\epsilon_{ijm}\epsilon_{mk\ell}a_j\frac{\partial b_\ell}{\partial x_k}\end{aligned}$$

   Now, because the Levi-Civita symbol is totally antisymmetric, $\epsilon_{mk\ell} = -\epsilon_{km\ell} = \epsilon_{k\ell m}$ and so the above sum may be rewritten as

$$\begin{aligned}\left[\vec{a}\times\left(\vec{\nabla}\times\vec{b}\right)\right]_i &= \sum_{jk\ell}\left(\sum_m\epsilon_{ijm}\epsilon_{k\ell m}\right)a_j\frac{\partial b_\ell}{\partial x_k} \\ &= \sum_{jk\ell}\left(\delta_{ik}\delta_{j\ell}-\delta_{i\ell}\delta_{jk}\right)a_j\frac{\partial b_\ell}{\partial x_k}.\end{aligned}$$

Now, if we do the sums over j, k and ℓ successively, we get

$$\begin{aligned}\left[\vec{a}\times\left(\vec{\nabla}\times\vec{b}\right)\right]_i &= \sum_{k\ell}\left(\delta_{ik}a_\ell\frac{\partial b_\ell}{\partial x_k}-\delta_{i\ell}a_k\frac{\partial b_\ell}{\partial x_k}\right)\\ &= \sum_{\ell}\left(a_\ell\frac{\partial b_\ell}{\partial x_i}-\delta_{i\ell}\vec{a}\cdot\vec{\nabla}b_\ell\right)\\ &= \vec{a}\cdot\frac{\partial\vec{b}}{\partial x_i}-\vec{a}\cdot\vec{\nabla}b_i.\end{aligned}$$

If we swap the two vectors, we obviously get

$$\left[\vec{b}\times\left(\vec{\nabla}\times\vec{a}\right)\right]_i = \vec{b}\cdot\frac{\partial\vec{a}}{\partial x_i}-\vec{b}\cdot\vec{\nabla}a_i$$

so adding the two gives

$$\left[\vec{a}\times\left(\vec{\nabla}\times\vec{b}\right)+\vec{b}\times\left(\vec{\nabla}\times\vec{a}\right)\right]_i = \vec{a}\cdot\frac{\partial\vec{b}}{\partial x_i}-\vec{a}\cdot\vec{\nabla}b_i+\vec{b}\cdot\frac{\partial\vec{a}}{\partial x_i}-\vec{b}\cdot\vec{\nabla}a_i.$$

The second and last terms together are the i^{th}-component of $-(\vec{a}\cdot\vec{\nabla})\vec{b}-(\vec{b}\cdot\vec{\nabla})\vec{a}$, and the product rule tells us that $[\vec{\nabla}(\vec{a}\cdot\vec{b})]_i$ is the first plus the third terms, and so we've proved what we wanted:

$$\vec{a}\times\left(\vec{\nabla}\times\vec{b}\right)+\vec{b}\times\left(\vec{\nabla}\times\vec{a}\right) = \vec{\nabla}\left(\vec{a}\cdot\vec{b}\right)-\left(\vec{a}\cdot\vec{\nabla}\right)\vec{b}-\left(\vec{b}\cdot\vec{\nabla}\right)\vec{a}.$$

(b) For this problem, we need to be cogniscent of the fact that the derivative in the curl acts on a *product* of the components of $\vec{a}$ and $\vec{b}$ and to use the appropriate theorem of calculus. More specifically, since

$$\left(\vec{a}\times\vec{b}\right)_m = \sum_{k\ell}\epsilon_{mk\ell}a_kb_\ell,$$

then

$$\left[\vec{\nabla}\times\left(\vec{a}\times\vec{b}\right)\right]_i = \sum_{jm}\epsilon_{ijm}(\vec{\nabla})_j(\vec{a}\times\vec{b})_m$$
$$= \sum_{jm}\epsilon_{ijm}\frac{\partial}{\partial x_j}\left(\sum_{k\ell}\epsilon_{mk\ell}a_k b_\ell\right)$$
$$= \sum_{jk\ell m}\epsilon_{ijm}\epsilon_{mk\ell}\left[\frac{\partial}{\partial x_j}(a_k b_\ell)\right]$$
$$= \sum_{jk\ell m}\epsilon_{ijm}\epsilon_{mk\ell}\left(\frac{\partial a_k}{\partial x_j}b_\ell + a_k\frac{\partial b_\ell}{\partial x_j}\right).$$

Again using $\epsilon_{mk\ell} = \epsilon_{k\ell m}$ and summing over m gives

$$\left[\vec{\nabla}\times\left(\vec{a}\times\vec{b}\right)\right]_i = \sum_{jk\ell}\left(\delta_{ik}\delta_{j\ell}-\delta_{i\ell}\delta_{jk}\right)\left(\frac{\partial a_k}{\partial x_j}b_\ell + a_k\frac{\partial b_\ell}{\partial x_j}\right).$$

As we did in (a), we now do the sums over j, k and ℓ, in that order:

$$\left[\vec{\nabla}\times\left(\vec{a}\times\vec{b}\right)\right]_i = \sum_{k\ell}\left[\delta_{ik}\left(\frac{\partial a_k}{\partial x_\ell}b_\ell + a_k\frac{\partial b_\ell}{\partial x_\ell}\right) - \delta_{i\ell}\left(\frac{\partial a_k}{\partial x_k}b_\ell + a_k\frac{\partial b_\ell}{\partial x_k}\right)\right]$$
$$= \sum_{\ell}\left[\frac{\partial a_i}{\partial x_\ell}b_\ell + a_i\frac{\partial b_\ell}{\partial x_\ell} - \delta_{i\ell}\left((\vec{\nabla}\cdot\vec{a})b_\ell + (\vec{a}\cdot\vec{\nabla})b_\ell\right)\right]$$
$$= (\vec{b}\cdot\vec{\nabla})a_i + a_i(\vec{\nabla}\cdot\vec{b}) - (\vec{\nabla}\cdot\vec{a})b_i - (\vec{a}\cdot\vec{\nabla})b_i$$

which we recognise as the i^{th} component of

$$\vec{\nabla}\times\left(\vec{a}\times\vec{b}\right) = \vec{a}\left(\vec{\nabla}\cdot\vec{b}\right) + \left(\vec{b}\cdot\vec{\nabla}\right)\vec{a} - \vec{b}\left(\vec{\nabla}\cdot\vec{a}\right) - \left(\vec{a}\cdot\vec{\nabla}\right)\vec{b}$$

and that's exactly what we wanted to prove.

2. For this problem, we need to determine the polarisability and magnetisation induced by the given fields in the materials presented. Recall that the definitions of the electric and magnetic susceptibilities are

$$\vec{P} = \epsilon_0\chi_e\vec{E}, \qquad \vec{M} = \chi_m\vec{H}.$$

The first of these will give the polarisability directly from the electric field, but we have to manipulate the second a bit: recall that $\vec{H} = \vec{B}/\mu$, where $\mu = (1 + \chi_m)\mu_0$ is the material's permeability. This means that we get the magnetisation from the magnetic field via $\vec{M} = \chi_m \vec{B}/\mu$.

But this isn't all: we want the electric and magnetic dipole of each constituent atom/molecule in our material, which means we need a *number density* ρ_N for each material, i.e. the number of particles per unit volume. Now, if the total volume is V, then it has a total mass $\rho_M V$, where ρ_M is the mass density of the material. If the mass of each constituent is A (we're using m, M and μ for other quantities, so best to use a nonstandard variable for mass here), then $N = \rho_M V/A$, so $\rho_N = N/V = \rho_M/A$ is the number density. Thus, if $\vec{p}$ and $\vec{m}$ are the constituent dipoles, then $\vec{P} = \rho_N \vec{p}$ and $\vec{M} = \rho_N \vec{m}$. Thus,

$$\vec{p} = \frac{\epsilon_0 \chi_e A}{\rho_M} \vec{E}, \qquad \vec{m} = \frac{\chi_m A}{\mu \rho_M} \vec{B}.$$

So now we're in a position to put in some numbers. Me, whenever I'm confronted with the need for physical quantities, I go straight to `www.engineeringtoolbox.com`, and I get these:

material	mass density	relative permittivity	relative permeability
water	1000	80	0.999992
wood	500-800	2-6	1.00000043
air	1.204	1.000536	1.00000037

(All densities are in kilogrammes per cubic metre, and for air and water, I've taken the values at 20°C.) Note that it's the relative permittivity ϵ/ϵ_0 and relative permeability μ/μ_0 listed above, and we can see that $\mu \approx \mu_0$ holds for all three materials (as we mentioned before). However, $\mu = (1+\chi_m)\mu_0$ gives $\chi_m = \mu/\mu_0 - 1$, so we can't use that approximation to get that, because they'd all be zero. But they're easily computed from the exact values above: for water, wood and air, we get magnetic susceptibilities of -8.0×10^{-6}, 4.3×10^{-7} and 3.7×10^{-7} respectively. (So water is diamagnetic and wood and air are paramagnetic.) Similarly, $\chi_e = \epsilon/\epsilon_0 - 1$ and so we get electric susceptibilities of 79, 1-5 and 5.36×10^{-4}.

Now we need constituent masses. Water is easy: it's very close to 18 times the proton mass 1.67×10^{-27} kg, so we'll use that. Similarly, if we take air to be 20% O_2 and 80% N_2, we get 28.8 times the proton mass. Thus, we have everything we need for (a) and (c), and putting in the numbers gives the electric dipoles to be $3.15 \times 10^{-34}\,\mathrm{A \cdot m \cdot s}$ for water and $2.84 \times 10^{-36}\,\mathrm{A \cdot m \cdot s}$ for air, both in the same direction as the electric field. The magnetic dipoles are $9.57 \times 10^{-31}\,\mathrm{A \cdot m^2}$ for water and $5.88 \times 10^{-29}\,\mathrm{A \cdot m^2}$ for air, with the former in the opposite direction to $\vec{B}$ and the latter in the same direction.

Now, wood is the tricky one, because obviously not all wood is the same, which is why there's a range of values above. So let's take the midpoints, i.e. a mass density of $650\,\mathrm{kg \cdot m^{-3}}$ and an electric susceptibility of 3. Also, what we take as a constituent mass is not obvious, so we'll make a *very* crude approximation that it's made entirely of cellulose, which Wikipedia tells me has a molar mass of 162.1406 grammes per mole, so we'll take 162 proton masses as its constituent mass. With these assumptions, we get an electric dipole of $1.66 \times 10^{-36}\,\mathrm{A \cdot m \cdot s}$ and $7.12 \times 10^{-31}\,\mathrm{A \cdot m^2}$ for the magnetic dipole, both in the same direction as the fields inducing them.

3. We'll do this and the next problem by computing both sides of the equations given and showing that they're equal rather than deriving them from scratch.

   First off, for a linear medium, the energy density and Poynting vector are

$$u = \frac{\epsilon}{2}\vec{E} \cdot \vec{E} + \frac{1}{2\mu}\vec{B} \cdot \vec{B}, \qquad \vec{S} = \frac{1}{\mu}\vec{E} \times \vec{B}.$$

   So the time derivative of the energy density is

$$\frac{\partial u}{\partial t} = \epsilon\vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \frac{1}{\mu}\vec{B} \cdot \frac{\partial \vec{B}}{\partial t}$$

   and since Maxwell's equations tell us that

$$\begin{aligned} \mu\epsilon\frac{\partial \vec{E}}{\partial t} &= \vec{\nabla} \times \vec{B} - \mu\vec{J}, \\ \frac{\partial \vec{B}}{\partial t} &= -\vec{\nabla} \times \vec{E}, \end{aligned}$$

we get

$$\frac{\partial u}{\partial t} \;=\; \vec{E}\cdot\left(\frac{1}{\mu}\vec{\nabla}\times\vec{B}-\vec{J}\right)-\frac{1}{\mu}\vec{B}\cdot\vec{\nabla}\times\vec{E}$$

so

$$\frac{\partial u}{\partial t}+\vec{J}\cdot\vec{E} \;=\; -\frac{1}{\mu}\left(\vec{B}\cdot\vec{\nabla}\times\vec{E}-\vec{E}\cdot\vec{\nabla}\times\vec{B}\right).$$

Now let's compute the divergence of the Poynting vector using the Levi-Civita symbol:

$$\begin{aligned}
\vec{\nabla}\cdot\vec{S} &= \frac{1}{\mu}\vec{\nabla}\cdot(\vec{E}\times\vec{B})\\
&= \frac{1}{\mu}\sum_i \frac{\partial}{\partial x_i}(\vec{E}\times\vec{B})_i\\
&= \frac{1}{\mu}\sum_{ijk} \frac{\partial}{\partial x_i}(\epsilon_{ijk}E_jB_k)\\
&= \frac{1}{\mu}\sum_{ijk} \epsilon_{ijk}\left(\frac{\partial E_j}{\partial x_i}B_k+E_j\frac{\partial B_k}{\partial x_i}\right)\\
&= \frac{1}{\mu}\left(\sum_{kij} B_k\epsilon_{kij}\frac{\partial E_j}{\partial x_i}-\sum_{jik} E_j\epsilon_{jik}\frac{\partial B_k}{\partial x_i}\right)
\end{aligned}$$

since $\epsilon_{ijk}=\epsilon_{kij}=-\epsilon_{jik}$. In the first term, doing the sums over i and j first gives $\sum_k B_k(\vec{\nabla}\times\vec{E})_k=\vec{B}\cdot\vec{\nabla}\times\vec{E}$, and in the second, doing the (ik) sums gives $\sum_j E_j(\vec{\nabla}\times\vec{B})_j=\vec{E}\cdot\vec{\nabla}\times\vec{B}$, so

$$\vec{\nabla}\cdot\vec{S} \;=\; \frac{1}{\mu}\left(\vec{B}\cdot\vec{\nabla}\times\vec{E}-\vec{E}\cdot\vec{\nabla}\vec{B}\right)$$

and therefore, as desired,

$$\frac{\partial u}{\partial t}+\vec{J}\cdot\vec{E} \;=\; -\vec{\nabla}\cdot\vec{S}.$$

4. The first step for this problem is quite similar to what we did in Problem

3, namely,

$$\begin{aligned}
\frac{\partial}{\partial t}(\mu\epsilon S_i) &= \epsilon\sum_{jk}\epsilon_{ijk}\frac{\partial}{\partial t}(E_jB_k) \\
&= \epsilon\sum_{jk}\epsilon_{ijk}\left[\left(\frac{\partial\vec{E}}{\partial t}\right)_j B_k + E_j\left(\frac{\partial\vec{B}}{\partial t}\right)_k\right] \\
&= \sum_{jk}\epsilon_{ijk}\left(\frac{1}{\mu}\vec{\nabla}\times\vec{B}-\vec{J}\right)_j B_k - \epsilon\sum_{jk}\epsilon_{ijk}E_j\left(\vec{\nabla}\times\vec{E}\right)_k \\
&= \left[\frac{1}{\mu}(\vec{\nabla}\times\vec{B})\times\vec{B}-\vec{J}\times\vec{B}-\epsilon\vec{E}\times(\vec{\nabla}\times\vec{E})\right]_i .
\end{aligned}$$

Thus, using the definition of $\vec{f}$,

$$\begin{aligned}
\frac{\partial}{\partial t}(\mu\epsilon S_i)+f_i &= \left[\frac{1}{\mu}(\vec{\nabla}\times\vec{B})\times\vec{B}-\epsilon\vec{E}\times(\vec{\nabla}\times\vec{E})+\rho\vec{E}\right]_i \\
&= \left[-\frac{1}{\mu}\vec{B}\times(\vec{\nabla}\times\vec{B})-\epsilon\vec{E}\times(\vec{\nabla}\times\vec{E})+\epsilon\vec{E}(\vec{\nabla}\cdot\vec{E})\right]_i
\end{aligned}$$

since Gauss' law for electric fields gives $\rho=\epsilon\vec{\nabla}\cdot\vec{E}$. Now, if we invoke Problem 1(a) above, we see that

$$\begin{aligned}
\left[\vec{a}\times\left(\vec{\nabla}\times\vec{a}\right)\right]_i &= \vec{a}\cdot\frac{\partial\vec{a}}{\partial x_i}-\vec{a}\cdot\vec{\nabla}a_i \\
&= \frac{1}{2}\frac{\partial}{\partial x_i}(\vec{a}\cdot\vec{a})-\vec{a}\cdot\vec{\nabla}a_i.
\end{aligned}$$

If we use this, we see that

$$\begin{aligned}
\frac{\partial}{\partial t}(\mu\epsilon S_i)+f_i &= -\frac{\partial}{\partial x_i}\left(\frac{\epsilon}{2}\vec{E}\cdot\vec{E}+\frac{1}{2\mu}\vec{B}\cdot\vec{B}\right)+\frac{1}{\mu}(\vec{B}\cdot\vec{\nabla})B_i \\
&\quad +\epsilon(\vec{E}\cdot\vec{\nabla})E_i+\epsilon(\vec{\nabla}\cdot\vec{E})E_i.
\end{aligned}$$

Now to compute the right-hand side of the equation we want to verify: for a linear medium, the Maxwell stress tensor will be

$$T_{ij} = \left(\frac{\epsilon}{2}\vec{E}\cdot\vec{E}+\frac{1}{2\mu}\vec{B}\cdot\vec{B}\right)\delta_{ij}-\epsilon E_iE_j-\frac{1}{\mu}B_iB_j.$$

Therefore,

$$\begin{aligned}
\sum_j \frac{\partial T_{ij}}{\partial x_j} &= \sum_j \frac{\partial}{\partial x_j}\left[\left(\frac{\epsilon}{2}\vec{E}\cdot\vec{E}+\frac{1}{2\mu}\vec{B}\cdot\vec{B}\right)\delta_{ij}-\epsilon E_i E_j-\frac{1}{\mu}B_i B_j\right] \\
&= \sum_j \left[\delta_{ij}\frac{\partial}{\partial x_j}\left(\frac{\epsilon}{2}\vec{E}\cdot\vec{E}+\frac{1}{2\mu}\vec{B}\cdot\vec{B}\right)\right. \\
&\qquad \left.-\left(\epsilon\frac{\partial E_i}{\partial x_j}E_j+\epsilon E_i\frac{\partial E_j}{\partial x_j}+\frac{1}{\mu}\frac{\partial B_i}{\partial x_j}B_j+\frac{1}{\mu}B_i\frac{\partial B_j}{\partial x_j}\right)\right].
\end{aligned}$$

If we now do the sum, we get

$$\begin{aligned}
\sum_j \frac{\partial T_{ij}}{\partial x_j} &= \frac{\partial}{\partial x_i}\left(\frac{\epsilon}{2}\vec{E}\cdot\vec{E}+\frac{1}{2\mu}\vec{B}\cdot\vec{B}\right) \\
&\qquad -\left(\epsilon(\vec{E}\cdot\vec{\nabla})E_i+\epsilon E_i(\vec{\nabla}\cdot\vec{E})+\frac{1}{\mu}(\vec{B}\cdot\vec{\nabla})B_i+\frac{1}{\mu}B_i(\vec{\nabla}\cdot\vec{B})\right).
\end{aligned}$$

But $\vec{\nabla}\cdot\vec{B}=0$, so this is

$$\begin{aligned}
\sum_j \frac{\partial T_{ij}}{\partial x_j} &= \frac{\partial}{\partial x_i}\left(\frac{\epsilon}{2}\vec{E}\cdot\vec{E}+\frac{1}{2\mu}\vec{B}\cdot\vec{B}\right)-\frac{1}{\mu}(\vec{B}\cdot\vec{\nabla})B_i \\
&\qquad -\epsilon(\vec{E}\cdot\vec{\nabla})E_i-\epsilon(\vec{\nabla}\cdot\vec{E})E_i
\end{aligned}$$

and this proves that

$$\frac{\partial}{\partial t}(\mu\epsilon S_i)+f_i = -\sum_j \frac{\partial T_{ij}}{\partial x_j}$$

as we'd hoped.