MP465 – Advanced Electromagnetism

Solutions to Problem Set 1

1. (a) We see the potential given diverges at $r = 0$, so we expect some funny business there, but for $r > 0$, it's finite so we can do any calculations without worry.

   In particular, if we want $\nabla^2\Phi$, we can just use the spherical-coordinate form given; furthermore, since Φ depends only on the radial coordinate r and not the angular coordinates θ and ϕ, we just have

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)$$

   This is bog-standard differentiation, and the result is $qe^{-2r/a}/\pi a^3\epsilon_0$. Thus, for $r > 0$, we have

$$\begin{aligned}\rho &= -\epsilon_0\nabla^2\Phi \\ &= -\frac{q}{\pi a^3}\exp\left(-\frac{2r}{a}\right).\end{aligned}$$

   Since $\delta^{(3)}(\vec{r}) = 0$ for $r > 0$, this means that this is the function $B(r)$ that we're after.

   (b) Let $\mathcal{B}$ be the region $r < R$ (called a "ball of radius R" in maths-speak). The surface of this is the sphere of radius R centred at the origin, S^2. Thus, we see that

$$\begin{aligned}\int_{\mathcal{B}}\rho\,\mathrm{d}^3\vec{r} &= -\epsilon_0\int_{\mathcal{B}}\nabla^2\Phi\,\mathrm{d}^3\vec{r} \\ &= -\epsilon_0\int_{\mathcal{B}}\vec{\nabla}\cdot\left(\vec{\nabla}\Phi\right)\,\mathrm{d}^3\vec{r} \\ &= -\epsilon_0\int_{S^2}\vec{\nabla}\Phi\cdot\mathrm{d}\vec{\sigma} \\ &= -\epsilon_0\int_{S^2}\frac{\mathrm{d}\Phi}{\mathrm{d}r}\,\mathrm{d}\sigma\end{aligned}$$

since both $\vec{\nabla}\Phi$ and $\mathrm{d}\vec{\sigma}$ both point radially outward. Now, since this is evaluated on the spherical boundary, $r = R$ throughout and so

$$\begin{aligned}\int_{\mathcal{B}} \rho\, \mathrm{d}^3\vec{r} &= -\epsilon_0 \int_{S^2} \left[-\frac{q}{4\pi\epsilon_0} \left(\frac{1}{R^2} + \frac{2}{aR} + \frac{2}{a^2} \right) \exp\left(-\frac{2R}{a} \right) \right] \left(R^2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi \right) \\ &= q \left(1 + \frac{2R}{a} + \frac{2R^2}{a^2} \right) \exp\left(-\frac{2R}{a} \right).\end{aligned}$$

This is the total charge contained in the region $r < R$, and notice that in the $R \to 0$ limit, we get q, i.e. a nonzero charge.

We found $B(r)$ in (a), and so we know

$$\rho = A\delta^{(3)}(\vec{r}) - \frac{q}{\pi a^3} \exp\left(-\frac{2r}{a} \right)$$

and since the origin is contained in $\mathcal{B}$, we have

$$\int_{\mathcal{B}} \rho\, \mathrm{d}^3\vec{r} = A - \frac{q}{\pi a^3} \int_{\mathcal{B}} \exp\left(-\frac{2r}{a} \right) r^2 \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi.$$

But because the remaining integrand is continuous in r, it vanishes when integrated from $r = 0$ to $r = R \to 0$. Therefore, $A = q$ and the charge density is

$$\rho = q\delta^{(3)}(\vec{r}) - \frac{q}{\pi a^3} \exp\left(-\frac{2r}{a} \right)$$

(c) Now let $q = e$. The fact that there is a nonzero charge in any region, no matter how tiny, containing the origin, indicates that the $e\delta^{(3)}(\vec{r})$ term describes a point charge of magnitude $+e$ there. But notice that if $a = a_0$, the second term may be written as $-e|\psi_{100}(\vec{r})|^2$. The Born interpretation says that the modulus-squared of a QM wavefunction is the probability density, i.e. when integrated over a region, it gives the chance of finding the particle there. But since an electron has charge $-e$, then $-e|\psi_{100}(\vec{r})|^2$ gives the charge density due to the electron being "smeared out" into a probability cloud.

Thus, this ρ that we've found is the sum of the charge density of the hydrogen nucleus (a proton) plus the effective charge density of the quantum-mechanical electron, and so the potential we started with is the correct one to describe a ground-state hydrogen atom. Neat!

2. We want to look at the distribution given by the density $\rho(r, \phi, z) = \sigma\delta(z)$ for $r \leq a$, and zero otherwise, and compute the electric field along the z-axis using

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, \mathrm{d}^3\vec{r}'.$$

So if we're looking on the z-axis, $\vec{r} = z\hat{e}_z$, and since the position vector of each bit of charge is $\vec{r}' = r'\hat{e}_{r'} + z'\hat{e}_z$, we have

$$\begin{aligned} \vec{r} - \vec{r}' &= (z - z')\,\hat{e}_z - r'\hat{e}_{r'}, \\ |\vec{r} - \vec{r}'| &= \sqrt{(\vec{r} - \vec{r}') \cdot (\vec{r} - \vec{r}')} \\ &= \sqrt{(r')^2 + (z - z')^2} \end{aligned}$$

and so the electric field on the z-axis is

$$\begin{aligned} \vec{E}(0, \phi, z) &= \frac{1}{4\pi\epsilon_0} \int \frac{\sigma\delta(z')\left[(z - z')\,\hat{e}_z - r'\hat{e}_{r'}\right]}{\left[(r')^2 + (z - z')^2\right]^{3/2}} \, r'\mathrm{d}r'\mathrm{d}\phi'\mathrm{d}z' \\ &= \frac{\sigma}{4\pi\epsilon_0} \int \frac{r'(z\hat{e}_z - r'\hat{e}_{r'})}{\left[(r')^2 + z^2\right]^{3/2}} \mathrm{d}r'\mathrm{d}\phi' \end{aligned}$$

where we've done the z'-integral in the second step. Now, the only explicit ϕ'-dependence is in $\hat{e}_{r'} = \cos\phi'\hat{e}_x + \sin\phi'\hat{e}_y$, but since both the sine and cosine integrate to zero, only the z-component survives, giving

$$\begin{aligned} \vec{E}(0, \phi, z) &= \frac{\sigma z\hat{e}_z}{4\pi\epsilon_0} \int \frac{r'}{\left[(r')^2 + z^2\right]^{3/2}} \mathrm{d}r'\mathrm{d}\phi' \\ &= \frac{\sigma z\hat{e}_z}{4\pi\epsilon_0} \left[-\frac{1}{\sqrt{(r')^2 + z^2}} \right|_0^a \left[\phi'\right|_0^{2\pi} \\ &= \frac{\sigma z\hat{e}_z}{2\epsilon_0} \left(\frac{1}{\sqrt{z^2}} - \frac{1}{\sqrt{z^2 + a^2}} \right). \end{aligned}$$

Since $\sqrt{z^2} = |z|$ and $z/|z| = \mathrm{sgn}(z)$, we recover the same result we got by computing Φ and then using $\vec{E} = -\vec{\nabla}\Phi$, namely

$$\vec{E}(0, \phi, z) = \frac{\sigma}{2\epsilon_0} \left(\mathrm{sgn}(z) - \frac{z}{\sqrt{z^2 + a^2}} \right) \hat{e}_z.$$

3. (a) First, the obvious: since the rod is only on the z-axis, the density must be zero if $r \neq 0$, which suggests that there'll be a $\delta(r)$ in the density. Since it's length L centred on the origin, then obviously $\rho = 0$ if $z > L/2$ or $z < -L/2$, so for $|z| < L/2$ it must have a form like $A(r, \phi, z)\delta(r)$. However, since it's cylindrically symmetric, there can be no ϕ-dependence, and since the charge is uniformly-distributed, A can't depend on z either, so $\rho = A(r)\delta(r)$ for some $A(r)$.

Now, let's integrate this density over the cylindrical region $r < R$, $-L/2 < z < L/2$. This contains the entire rod, so the result must be q, and thus

$$\begin{aligned} q &= \int A(r)\delta(r)\, r\, \mathrm{d}r\, \mathrm{d}\phi\, \mathrm{d}z \\ &= \left[\lim_{r\to 0} rA(r)\right](2\pi)(L) \end{aligned}$$

since the delta-function evaluates the integrand at $r = 0$. For this to be finite, $rA(r)$ must approach $q/2\pi L$ as r goes to zero, and so $A(r) = q/2\pi Lr$ does the trick. Thus, the correct form for the density is

$$\rho(r, \phi, z) = \frac{\lambda\delta(r)}{2\pi r}$$

for $-L/2 < z < L/2$ and zero otherwise, where $\lambda = q/L$

(b) The potential is therefore

$$\begin{aligned} \Phi(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\vec{r}'\right)}{|\vec{r} - \vec{r}'|}\, \mathrm{d}^3\vec{r}' \\ &= \frac{\lambda}{8\pi^2\epsilon_0}\int \frac{\delta(r')/r'}{\sqrt{r^2 - 2rr'\cos(\phi' - \phi) + (r')^2 + (z - z')^2}}\, r'\mathrm{d}r'\mathrm{d}\phi'\mathrm{d}z' \\ &= \frac{\lambda}{8\pi^2\epsilon_0}\int \frac{1}{\sqrt{r^2 + (z' - z)^2}}\, \mathrm{d}\phi'\mathrm{d}z' \\ &= \frac{\lambda}{4\pi\epsilon_0}\int_{-L/2}^{L/2} \frac{\mathrm{d}z'}{\sqrt{r^2 + (z' - z)^2}} \end{aligned}$$

where the last two lines were obtained by first doing the r'-integral and then the ϕ' one.

This integral is of the form $\int(s^2+1)^{-1/2}\mathrm{d}s$, and is common enough that you can look it up if need be, but here's how to do it: make the substitution $s = \tanh u$, so that

$$\begin{aligned}\int \frac{\mathrm{d}s}{\sqrt{s^2+1}} &= \int \frac{\mathrm{sech}\,^2 u}{\sqrt{\tanh^2 u + 1}}\mathrm{d}u \\ &= \int \mathrm{sech}\, u\, \mathrm{d}u \\ &= \int \frac{\tanh u\,\mathrm{sech}\, u + \mathrm{sech}\,^2 u}{\mathrm{sech}\, u + \tanh u}\mathrm{d}u \\ &= \int \frac{\mathrm{d}(\mathrm{sech}\, u + \tanh u)}{\mathrm{sech}\, u + \tanh u} \\ &= \ln(\mathrm{sech}\, u + \tanh u) \\ &= \ln(\sqrt{s^2+1} + s).\end{aligned}$$

Thus, if we define $s = (z' - z)/r$ in the integral for Φ, we obtain

$$\begin{aligned}\Phi(r,\phi,z) &= \frac{\lambda}{4\pi\epsilon_0}\int_{-(L/2+z)/r}^{(L/2-z)/r} \frac{r\mathrm{d}s}{\sqrt{r^2+r^2s^2}} \\ &= \frac{\lambda}{4\pi\epsilon_0}\left[\ln(\sqrt{s^2+1}+s)\right|_{-(L/2+z)/r}^{(L/2-z)/r} \\ &= \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{\sqrt{(L/2-z)^2+r^2}+L/2-z}{\sqrt{(L/2+z)^2+r^2}-L/2-z}\right)\end{aligned}$$

as the scalar potential.

This is good *everywhere*; we didn't have to assume that we're only on the z-axis or the like. So the contrast between this problem and the previous one is that in some cases we can get an explicit expression for the general potential (and thus electric field), whereas in others we can only get such an expression under certain assumptions. (Note, however, if we're willing to deal with special functions like elliptic integrals, then we can sometimes get full expressions even in the latter cases.)

4. (a) We know that the scalar potential due to N point charges $q_1, \ldots, q_N$ located at $\vec{r}_1, \ldots, \vec{r}_N$ is simply

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\vec{r}-\vec{r}_i|}$$

so the potential for the single real charge and three image charges in the order given in the problem is

$$\Phi(x,y,z) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{(x-a)^2+(y-b)^2+z^2}} - \frac{q}{\sqrt{(x-a)^2+(y+b)^2+z^2}} - \frac{q}{\sqrt{(x+a)^2+(y-b)^2+z^2}} + \frac{q}{\sqrt{(x+a)^2+(y+b)^2+z^2}}\right).$$

If we evaluate this in the plane $x = 0$, we see the first and third terms above cancel each other out, as do the second and fourth terms. Similarly, if $y = 0$, the first and second cancel out and the third and fourth do as well. Thus, the above potential vanishes if either $x = 0$ or $y = 0$. Since the surface of the conductor lies entirely in these two planes, then $\Phi = 0$ on the conductor's surface, as desired. Thus, this gives the scalar potential outside the conductor.

(b) Since $\vec{E} = -\vec{\nabla}\Phi$, this is just a matter of calculus and should be pretty standard (if tedious) work for all of you, so I'll just state the result component by component (since it's easier that way):

$$E_x(x,y,z) = \frac{q}{4\pi\epsilon_0}\left\{\frac{x-a}{[(x-a)^2+(y-b)^2+z^2]^{3/2}} - \frac{x-a}{[(x-a)^2+(y+b)^2+z^2]^{3/2}} - \frac{x+a}{[(x+a)^2+(y-b)^2+z^2]^{3/2}} + \frac{x+a}{[(x+a)^2+(y+b)^2+z^2]^{3/2}}\right\}$$

$$E_y(x,y,z) = \frac{q}{4\pi\epsilon_0}\left\{\frac{y-b}{[(x-a)^2+(y-b)^2+z^2]^{3/2}} - \frac{y+b}{[(x-a)^2+(y+b)^2+z^2]^{3/2}} - \frac{y-b}{[(x+a)^2+(y-b)^2+z^2]^{3/2}} + \frac{y+b}{[(x+a)^2+(y+b)^2+z^2]^{3/2}}\right\}$$

$$E_z(x,y,z) = \frac{q}{4\pi\epsilon_0}\left\{\frac{z}{[(x-a)^2+(y-b)^2+z^2]^{3/2}} - \frac{z}{[(x-a)^2+(y+b)^2+z^2]^{3/2}} - \frac{z}{[(x+a)^2+(y-b)^2+z^2]^{3/2}} + \frac{z}{[(x+a)^2+(y+b)^2+z^2]^{3/2}}\right\}.$$

(And with the above, you can easily see that $E_y = E_z = 0$ on $x = 0$ and $E_x = E_z = 0$ on $y = 0$; therefore, on the conductor's surface, the electric field is normal to it, as it must be.)

(c) The surface charge density on a conductor is given by the formula $\sigma = \epsilon_0 \hat{n}\cdot\vec{E}$, where $\hat{n}$ is a unit normal pointing out of the conductor at the point on the surface we're considering. For the case at hand, the surface is piecewise defined (one part is $x = 0$, $y > 0$ and the other is $x > 0$, $y = 0$), so the surface charge density will also be piecewise.

First, on the surface $x = 0$, $y > 0$ (which we'll call surface 1): everywhere on this part of the surface, the unit normal is $\hat{e}_x$, so $\sigma = \epsilon_0 E_x$. All points on this surface have coordinates $(0, y, z)$ with $y > 0$, so if we use our results from (b), we see

$$\begin{aligned}\sigma_1(y,z) &= \epsilon_0 E_x(0,y,z)\\ &= \frac{q}{4\pi}\left\{\frac{-a}{[a^2+(y-b)^2+z^2]^{3/2}} - \frac{-a}{[a^2+(y+b)^2+z^2]^{3/2}}\right.\\ &\qquad \left. -\frac{a}{[a^2+(y-b)^2+z^2]^{3/2}} + \frac{a}{[a^2+(y+b)^2+z^2]^{3/2}}\right\}\\ &= \frac{qa}{2\pi}\left\{\frac{1}{[a^2+(y+b)^2+z^2]^{3/2}} - \frac{1}{[a^2+(y-b)^2+z^2]^{3/2}}\right\}.\end{aligned}$$

On the surface $x > 0$, $y = 0$ (surface 2): $\hat{e}_y$ is the unit normal everwhere, so $\sigma = \epsilon_0 E_x$. All points have coordinates $(x, 0, z)$ with $y > 0$, so if we denote the

$$\begin{aligned}\sigma_2(x,z) &= \epsilon_0 E_y(x,0,z)\\ &= \frac{q}{4\pi}\left\{\frac{-b}{[(x-a)^2+b^2+z^2]^{3/2}} - \frac{b}{[(x-a)^2+b^2+z^2]^{3/2}}\right.\\ &\qquad \left. -\frac{-b}{[(x+a)^2+b^2+z^2]^{3/2}} + \frac{b}{[(x+a)^2+b^2+z^2]^{3/2}}\right\}\\ &= \frac{qb}{2\pi}\left\{\frac{1}{[(x+a)^2+b^2+z^2]^{3/2}} - \frac{1}{[(x-a)^2+b^2+z^2]^{3/2}}\right\}\end{aligned}$$

so we see that the full expression is

$$\sigma(x,y,z) = \begin{cases}\sigma_1(y,z) & \text{for } x=0, y>0\\ \sigma_2(x,z) & \text{for } x>0, y=0\end{cases}$$

Now to compute the induced charge on the surface: the total charge on surface 1 is

$$
\begin{aligned}
q_1 &= \int_{\text{surface 1}} \sigma_1(y,z)\,\mathrm{d}y\,\mathrm{d}z \\
&= \frac{qa}{2\pi}\int_0^\infty \left(\int_{-\infty}^\infty \left\{ \frac{1}{[a^2+(y+b)^2+z^2]^{3/2}} - \frac{1}{[a^2+(y-b)^2+z^2]^{3/2}} \right\} \mathrm{d}z\right) \mathrm{d}y
\end{aligned}
$$

where we've written in a way to show that we'll do the z-integration first. Using

$$
\int \frac{\mathrm{d}s}{(s^2+1)^{3/2}} = \frac{s}{\sqrt{s^2+1}}
$$

(which you should be able to verify; it's easier to show that the similar one I did above), we see that

$$
\begin{aligned}
\int_{-\infty}^\infty \frac{1}{[a^2+(y+b)^2+z^2]^{3/2}}\,\mathrm{d}z &= 2\int_0^\infty \frac{1}{[a^2+(y+b)^2+z^2]^{3/2}}\,\mathrm{d}z \\
&= 2\left[\frac{1}{(y+b)^2+a^2}\,\frac{z}{\sqrt{z^2+(y+b)^2+a^2}}\right|_0^\infty \\
&= \frac{2}{(y+b)^2+a^2}.
\end{aligned}
$$

Now for the y-integral, which just gives us an arctangent:

$$
\begin{aligned}
\int_0^\infty \frac{2}{(y+b)^2+a^2}\,\mathrm{d}y &= 2\left[\frac{1}{a}\arctan\left(\frac{y+b}{a}\right)\right|_0^\infty \\
&= \frac{2}{a}\left[\frac{\pi}{2} - \arctan\left(\frac{b}{a}\right)\right].
\end{aligned}
$$

The other part of the integral over surface 1 is exactly the same except with a $-b$ instead of a b, so doing both the z- and y-integrals just gives the above with $\arctan(-b/a) = -\arctan(b/a)$. The overall charge q_1 that we're after is the difference between these two integrals and thus

$$
\begin{aligned}
q_1 &= \frac{qa}{2\pi}\left\{\frac{2}{a}\left[\frac{\pi}{2} - \arctan\left(\frac{b}{a}\right)\right] - \frac{2}{a}\left[\frac{\pi}{2} + \arctan\left(\frac{b}{a}\right)\right]\right\} \\
&= -\frac{2q}{\pi}\arctan\left(\frac{b}{a}\right).
\end{aligned}
$$

(A fair bit of work for a pretty simple answer, eh? Welcome to Theoretical Physics!)

On surface 2, we need to use the other part of σ, giving a total charge

$$\begin{aligned} q_2 &= \int_{\text{surface 2}} \sigma_2(x,z)\,\mathrm{d}x\,\mathrm{d}z \\ &= \frac{qb}{2\pi}\int_0^\infty \left(\int_{-\infty}^\infty \left\{\frac{1}{[(x+a)^2+b^2+z^2]^{3/2}} - \frac{1}{[(x-a)^2+b^2+z^2]^{3/2}}\right\}\mathrm{d}z\right)\mathrm{d}x \end{aligned}$$

But notice that this is *exactly* the same integral we just did for surface 1, except with a and b swapped! Sure, we're calling one of the integration variables x rather than y, but the integrand has the same functional dependence on x as it did on y and this variable is integrated over the same range, 0 to ∞. Because of this, we barely have to do any work other than switch a and b to get

$$q_2 = -\frac{2q}{\pi}\arctan\left(\frac{a}{b}\right).$$

Thus, the total induced charge on the conductor's surface is

$$\begin{aligned} \tilde{q} &= q_1 + q_2 \\ &= -\frac{2q}{\pi}\left[\arctan\left(\frac{b}{a}\right) + \arctan\left(\frac{a}{b}\right)\right] \end{aligned}$$

which, if we use the arctangent identity given in the problem, gives $\tilde{q} = -q$. Which is precisely the sum of the three image charges $-q$, $-q$ and q. QED, as they say.