MP465 – Advanced Electromagnetism

Tutorial 11 (7 May 2020)

The Electromagnetic Energy-Momentum Tensor

In Problem Set 3, we talked about the correct equations which encapsulate energy and momentum conservation, and we found that, in vacuum (i.e. $\epsilon = \epsilon_0$ and $\mu = \mu_0$), they take the form

$$\begin{aligned} \frac{\partial u}{\partial t} + \vec{J} \cdot \vec{E} &= -\vec{\nabla} \cdot \vec{S}, \\ \frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 S_i\right) + f_i &= -\sum_{j=1}^{3} \frac{\partial T_{ij}}{\partial x_j} \end{aligned}$$

where

$$\begin{aligned} \vec{S} &= \frac{1}{\mu_0}\, \vec{E} \times \vec{B}, \\ u &= \frac{\epsilon_0}{2}\, |\vec{E}|^2 + \frac{1}{2\mu_0}\, |\vec{B}|^2, \\ T_{ij} &= \frac{1}{2}\left(\epsilon_0 |\vec{E}|^2 + \frac{1}{\mu_0}\, |\vec{B}|^2\right) \delta_{ij} - \epsilon_0 E_i E_j - \frac{1}{\mu_0}\, B_i B_j, \\ \vec{f} &= \rho \vec{E} + \vec{J} \times \vec{B}. \end{aligned}$$

Given all of our discussion about the relativistic formulation of EM, we'd now like to see of all of these quantities and equations may be recast in a Lorentz-covariant form. We expect this to be doable, because all of the above come from Maxwell's equations and we were successful in twisting those into shape. Plus, there's one scalar and one 3-vector equation, which, as we've seen, suggests that a 4-vector equation is hidden in them.

We'll follow the same basic method as we did in lecture, namely, rewrite all the fields and sources in terms of the appropriate components of the field strength tensor $F^{\mu\nu}$ and the 4-current J^μ. We start with the Poynting vector, or at least the x-component:

$$\begin{aligned} S_x &= \frac{1}{\mu_0}\left[E_y B_z - E_z B_y\right] \\ &= \frac{1}{\mu_0}\left[(cF^{02})(F^{12}) - (cF^{03})(-F^{13})\right] \\ &= \frac{c}{\mu_0}\left[F^{02}F^{12} + F^{03}F^{13}\right] \\ &= \frac{c}{\mu_0}\left[F^{02}F^{1}{}_{2} + F^{03}F^{1}{}_{3}\right] \\ &= \frac{c}{\mu_0}\left[F^{00}F^{1}{}_{0} + F^{01}F^{1}{}_{1} + F^{02}F^{1}{}_{2} + F^{03}F^{1}{}_{3}\right] \end{aligned}$$

where in the last step we've simply added zero twice, since both F^{00} and $F^{1}{}_{1}$ are zero. In doing so, we see that the second index of each field strength is summed over and thus we can write $S_x = cF^{0\rho}F^{1}{}_{\rho}/\mu_0$. S_y is very similar:

$$\begin{aligned} S_y &= \frac{1}{\mu_0}\left[E_z B_x - E_x B_z\right] \\ &= \frac{1}{\mu_0}\left[(cF^{03})(F^{23}) - (cF^{01})(-F^{21})\right] \\ &= \frac{c}{\mu_0}\left[F^{01}F^{21} + F^{03}F^{23}\right] \\ &= \frac{c}{\mu_0}\left[F^{01}F^{2}{}_{1} + F^{03}F^{2}{}_{3}\right] \\ &= \frac{c}{\mu_0}\left[F^{00}F^{2}{}_{0} + F^{01}F^{2}{}_{1} + F^{02}F^{2}{}_{2} + F^{03}F^{2}{}_{3}\right] \\ &= \frac{c}{\mu_0}F^{0\rho}F^{2}{}_{\rho}. \end{aligned}$$

And, predictably enough, looking at the z-component leads to $S_z = cF^{0\rho}F^{3}{}_{\rho}/\mu_0$, meaning that the i^{th} component is $S_i = cF^{0\rho}F^{i}{}_{\rho}/\mu_0$. This suggests a possibility: if we define the quantity $M^{\mu\nu} = F^{\mu\rho}F^{\nu}{}_{\rho}/\mu_0$ – which transforms like any tensor with two upper indices – we see that $M^{0i} = S_i/c$ for $i = 1, 2, 3$, i.e. the components of the Poynting vector are accounted for inside this new object.

One other thing we can see is that this object is symmetric under interchange of its indices:

$$\begin{aligned}
M^{\nu\mu} &= \frac{1}{\mu_0} F^{\nu\rho} F^{\mu}{}_{\rho} \\
&= \frac{1}{\mu_0} F^{\mu}{}_{\rho} F^{\nu\rho} \\
&= \frac{1}{\mu_0} F^{\mu\rho} F^{\nu}{}_{\rho} \\
&= M^{\mu\nu}
\end{aligned}$$

and thus $M^{i0} = S_i/c$ as well.

But what are some of the other components of this object? Let's look at the 00 component; by definition,

$$\begin{aligned}
M^{00} &= \frac{1}{\mu_0} F^{0\rho} F^{0}{}_{\rho} \\
&= \frac{1}{\mu_0} \left[F^{00} F^{0}{}_{0} + F^{01} F^{0}{}_{1} + F^{02} F^{0}{}_{2} + F^{03} F^{0}{}_{3} \right] \\
&= \frac{1}{\mu_0} \left[\left(\frac{E_x}{c}\right)\left(\frac{E_x}{c}\right) + \left(\frac{E_y}{c}\right)\left(\frac{E_y}{c}\right) + \left(\frac{E_z}{c}\right)\left(\frac{E_z}{c}\right) \right] \\
&= \epsilon_0 |\vec{E}|^2
\end{aligned}$$

which is at least part of some of the quantities we're looking at, u and T_{ij}, so maybe we're onto something here. Let's see what the 12 component gives us:

$$\begin{aligned}
M^{12} &= \frac{1}{\mu_0} F^{1\rho} F^{2}{}_{\rho} \\
&= \frac{1}{\mu_0} \left[F^{10} F^{2}{}_{0} + F^{11} F^{2}{}_{1} + F^{12} F^{2}{}_{2} + F^{13} F^{2}{}_{3} \right] \\
&= \frac{1}{\mu_0} \left[\left(-\frac{E_x}{c}\right)\left(\frac{E_y}{c}\right) + (-B_y)(B_x) \right] \\
&= -\epsilon_0 E_x E_y - \frac{1}{\mu_0} B_x B_y
\end{aligned}$$

which also happens to be T_{xy}! It's easy to show that $M^{13} = T_{xz}$ and $M^{23} = T_{yz}$ as well. So we've got the Poynting vector and the off-diagonal components of the Maxwell stress tensor, but we don't quite have u, T_{xx}, T_{yy} and T_{zz} yet.

However, we did find $M^{00} = \epsilon_0|\vec{E}|^2$, and notice that we can rewrite this as

$$\begin{aligned} M^{00} &= \epsilon_0|\vec{E}|^2 \\ &= \frac{\epsilon_0}{2}|\vec{E}|^2 + \frac{1}{2\mu_0}|\vec{B}|^2 + \frac{\epsilon_0}{2}|\vec{E}|^2 - \frac{1}{2\mu_0}|\vec{B}|^2 \\ &= u + \left(\frac{\epsilon_0}{2}|\vec{E}|^2 - \frac{1}{2\mu_0}|\vec{B}|^2\right). \end{aligned}$$

True, but so what? If we look at M^{11} now, we can see why we write it this way:

$$\begin{aligned} M^{11} &= \frac{1}{\mu_0} F^{1\rho} F^{1}{}_{\rho} \\ &= \frac{1}{\mu_0}\left[F^{10}F^{1}{}_{0} + F^{11}F^{1}{}_{1} + F^{12}F^{1}{}_{2} + F^{13}F^{1}{}_{3}\right] \\ &= \frac{1}{\mu_0}\left[\left(-\frac{E_x}{c}\right)\left(\frac{E_x}{c}\right) + (B_z)(B_z) + (-B_y)(-B_y)\right] \\ &= -\epsilon_0 E_x^2 + \frac{1}{\mu_0}(B_y^2 + B_z^2) \\ &= \frac{1}{\mu_0}|\vec{B}|^2 - \epsilon_0 E_x^2 - \frac{1}{\mu_0} B_x^2. \end{aligned}$$

Note that the xx component of the stress tensor is

$$\begin{aligned} T_{xx} &= \frac{1}{2}\left(\epsilon_0|\vec{E}|^2 + \frac{1}{2\mu_0}|\vec{B}|^2\right)\delta_{xx} - \epsilon_0 E_x^2 - \frac{1}{\mu_0} B_x^2, \\ &= \frac{\epsilon_0}{2}|\vec{E}|^2 + \frac{1}{2\mu_0}|\vec{B}|^2 - \epsilon_0 E_x^2 - \frac{1}{\mu_0} B_x^2 \\ &= \frac{\epsilon_0}{2}|\vec{E}|^2 + \frac{1}{2\mu_0}|\vec{B}|^2 + \left(T_{xx} - \frac{1}{\mu_0}|\vec{B}|^2\right) \\ &= M^{11} + \frac{\epsilon_0}{2}|\vec{E}|^2 - \frac{1}{2\mu_0}|\vec{B}|^2 \end{aligned}$$

so

$$M^{11} = T_{xx} - \left(\frac{\epsilon_0}{2}|\vec{E}|^2 - \frac{1}{2\mu_0}|\vec{B}|^2\right).$$

The quantity in the brackets also showed up when we wrote M^{00} in terms of u, and it also shows up when we compute the remaining two components of $M^{\mu\nu}$:

$$\begin{aligned} M^{22} &= T_{yy} - \left(\frac{\epsilon_0}{2} |\vec{E}|^2 - \frac{1}{2\mu_0} |\vec{B}|^2 \right), \\ M^{33} &= T_{zz} - \left(\frac{\epsilon_0}{2} |\vec{E}|^2 - \frac{1}{2\mu_0} |\vec{B}|^2 \right). \end{aligned}$$

Therefore, if we write $M^{\mu\nu}$ as a 4×4 matrix as we've done before, we get

$$M^{\mu\nu} = \begin{pmatrix} u+K & S_x/c & S_y/c & S_z/c \\ S_x/c & T_{xx}-K & T_{xy} & T_{xz} \\ S_y/c & T_{yx} & T_{yy}-K & T_{yz} \\ S_z/c & T_{zx} & T_{zy} & T_{zz}-K \end{pmatrix}$$

where we've defined

$$K = \frac{\epsilon_0}{2} |\vec{E}|^2 - \frac{1}{2\mu_0} |\vec{B}|^2$$

for convenience. Notice that K only appears in the diagonal elements, so we can rewrite this as

$$\begin{aligned} M^{\mu\nu} &= \begin{pmatrix} u & S_x/c & S_y/c & S_z/c \\ S_x/c & T_{xx} & T_{xy} & T_{xz} \\ S_y/c & T_{yx} & T_{yy} & T_{yz} \\ S_z/c & T_{zx} & T_{zy} & T_{zz} \end{pmatrix} - K \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= T^{\mu\nu} - K\eta^{\mu\nu} \end{aligned}$$

where we define the *electromagnetic energy-momentum tensor* as

$$T^{\mu\nu} = \begin{pmatrix} u & S_x/c & S_y/c & S_z/c \\ S_x/c & T_{xx} & T_{xy} & T_{xz} \\ S_y/c & T_{yx} & T_{yy} & T_{yz} \\ S_z/c & T_{zx} & T_{zy} & T_{zz} \end{pmatrix}.$$

I've been a bit hasty here; there's no question $M^{\mu\nu}$ deserves two Lorentz indices, because it's constructed explictly from the field strength, and the metric tensor $\eta^{\mu\nu}$ is one of the fundamental building blocks of special relativity. But that doesn't mean the object we've just defined, $T^{\mu\nu}$, transforms as

$T^{\mu\nu} \mapsto T'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}$, as an object with two-upper indices (a "4-tensor") should.

The potential spanner in the works is K; we see that since $T^{\mu\nu} = M^{\mu\nu} + K\eta^{\mu\nu}$, if K changes under a Lorentz transformation, then $T^{\mu\nu}$ will *not* transform as a 4-tensor. If, however, K is a Lorentz-invariant quantity, then this energy-momentum thingie will indeed be a 4-tensor. Luckily, it is, as we can eventually see if we write K in terms of the field strength tensor:

$$\begin{aligned}
K &= \frac{\epsilon_0}{2}\left(E_x^2 + E_y^2 + E_z^2\right) - \frac{1}{2\mu_0}\left(B_x^2 + B_y^2 + B_z^2\right) \\
&= \frac{1}{2\mu_0 c^2}\left[(cF^{01})^2 + (cF^{02})^2 + (cF^{03})^2\right] - \frac{1}{2\mu_0}\left[(F^{23})^2 + (F^{31})^2 + (F^{12})^2\right] \\
&= \frac{1}{2\mu_0}\left[(F^{01})(-F_{01}) + (F^{02})(-F_{02}) + (F^{03})(-F_{03}) + (F^{12})(F_{12}) + (F^{23})(F_{23}) + (F^{31})(F_{31})\right] \\
&= -\frac{1}{2\mu_0}\left[F^{01}F_{01} + F^{02}F_{02} + F^{03}F_{03} + F^{12}F_{12} + F^{23}F_{23} + F^{31}F_{31}\right].
\end{aligned}$$

Because any terms like $F^{00}F_{00}$ are zero, you might be tempted to identify this with $-F^{\lambda\rho}F_{\lambda\rho}/2\mu_0$, but you'd be off by a factor of two. That's because that sum contains, for example, both $F^{12}F_{12}$ and $F^{21}F_{21}$ (we must sum over *all* pairs of indices). However, because $F^{21} = -F^{12}$, these two terms are the same, so adding them gives $2F^{12}F_{12}$. Because of this, we find

$$K = -\frac{1}{4\mu_0} F^{\lambda\rho}F_{\lambda\rho}.$$

Since both pairs of indices are contracted, this is indeed a Lorentz-invariant quantity, and therefore the electromagnetic energy-momentum tensor

$$\begin{aligned}
T^{\mu\nu} &= \begin{pmatrix} u & S_x/c & S_y/c & S_z/c \\ S_x/c & T_{xx} & T_{xy} & T_{xz} \\ S_y/c & T_{yx} & T_{yy} & T_{yz} \\ S_z/c & T_{zx} & T_{zy} & T_{zz} \end{pmatrix} \\
&= \frac{1}{\mu_0}\left(F^{\mu\rho}F^{\nu}{}_{\rho} - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\rho}F_{\lambda\rho}\right)
\end{aligned}$$

is indeed a tensor in the sense it Lorentz-transforms as we expect a 2-upper-index object must. Furthermore, it contains all the information about the energy, momentum and stress of the EM field.

Now, to the energy- and momentum-conservation equations. Using what we've just found, we can rewrite

$$\frac{\partial u}{\partial t} + \vec{J} \cdot \vec{E} \quad = \quad -\vec{\nabla} \cdot \vec{S}$$

as

$$(c\partial_0)(T^{00}) + (J_1)(-cF^{10}) + (J_2)(-cF^{20}) + (J_3)(-cF^{30}) \quad = \quad -(\partial_1)(cT^{10}) - (\partial_2)(cT^{20}) - (\partial_3)(cT^{30})$$

or, after dividing through by c and rearranging,

$$\begin{aligned} \partial_0 T^{00} + \partial_1 T^{10} + \partial_2 T^{20} + \partial_3 T^{30} &= \partial_\mu T^{\mu 0} \\ &= J_1 F^{10} + J_2 F^{20} + J_3 F^{30} \\ &= J_\mu F^{\mu 0} \end{aligned}$$

(where again we've used the fact that $F^{00} = 0$). Now look at the x-component of the energy conservation equation, namely,

$$\begin{aligned} \frac{\partial}{\partial t}(\mu_0 \epsilon_0 S_x) + f_x &= \frac{\partial}{\partial t}(\mu_0 \epsilon_0 S_x) + \rho E_x + J_y B_z - J_z B_y \\ &= -\sum_j \frac{\partial T_{xj}}{\partial x_j} \\ &= -\frac{\partial T_{xx}}{\partial x} - \frac{\partial T_{xy}}{\partial y} - \frac{\partial T_{xz}}{\partial z}. \end{aligned}$$

This becomes

$$(c\partial_0)\left(\frac{T^{01}}{c}\right) + \left(-\frac{J_0}{c}\right)(cT^{01}) + (J_2)(-F^{21}) - (J_3)(F^{31}) \quad = \quad -(\partial_1)(T^{11}) - (\partial_2)(T^{21}) - (\partial_3)(T^{31})$$

or

$$\begin{aligned} \partial_0 T^{01} + \partial_1 T^{11} + \partial_2 T^{21} + \partial_3 T^{31} &= \partial_\mu T^{\mu 1} \\ &= J_0 T^{01} + J_2 F^{21} + J_3 F^{31} \\ &= J_\mu F^{\mu 1} \end{aligned}$$

and at this point, you can probably guess what the y- and z-components are, with the upshot being that the four energy/momentum-conservation equations become

$$\partial_\mu T^{\mu\nu} \quad = \quad J_\mu T^{\mu\nu}.$$

Now, the importance of the energy-momentum tensor is that it contains all information about the energy and momentum of the EM field, and thus we can succunctly give the conservation law as we've just done. However, it's arguably the *most* important quantity to have in general relativity (GR). It's energy (usually in the form of mass) that causes spacetime to curve, and so we need to have a relativistic quantity that tells us what energy is around, and that's exactly what $T^{\mu\nu}$ is. In fact, the most fundamental equation in GR is Einstein's equation

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \Lambda g^{\mu\nu} \quad = \quad \frac{8\pi G}{c^4}T^{\mu\nu}$$

and lo and behold, there's the energy-momentum tensor appearing explicitly as a source term. (I won't explain fully what all the terms are, just that everything on the left-hand side depends on the spacetime metric g and its derivatives, so the above is a set of nonlinear partial differential equations which, when solved, tell us how spacetime is curved.)

But *all* systems with any sort of matter or energy – not just as electromagnetic fields – have an associated energy-momentum tensor, and it's that which goes into the above equation. For example, putting the appropriate tensor for a point mass into the above yields the Schwarzschild metric which describes an electrically-neutral nonrotaing black hole. A *charged* black hole would use the tensor we've just derived, with the field strength tensor needed to construct $T^{\mu\nu}$ being the one for a stationary point charge. So much like electromagnetism leads to special relatvity, it also has hints of what needs to be done in general relativity as well...