MP465 – Advanced Electromagnetism

Tutorial 10 (30 April 2020)

Another Radiation Example

In lecture, we've looked at the far-zone approximation for localised oscillating sources and found expressions for all sorts of quantities. All of them are based on having an expression for the electric dipole moment of the charge/current configuration, which, for a single-frequency oscillation, has the form $\vec{p}(t) = \mathrm{Re}[\tilde{\vec{p}}_0 e^{-i\omega t}]$ for some constant complex vector (which we call the dipole amplitude) $\tilde{\vec{p}}_0$. In most cases, what we'll be given is a current density of the form $\vec{J}(t, \vec{r}) = \mathrm{Re}[\tilde{\vec{J}}(\vec{r}) e^{-i\omega t}]$ and we showed that the dipole amplitude can be immediately computed from

$$\tilde{\vec{p}}_0 \;\; = \;\; \frac{i}{\omega} \int \tilde{\vec{J}}(\vec{r}) \, \mathrm{d}^3\vec{r}.$$

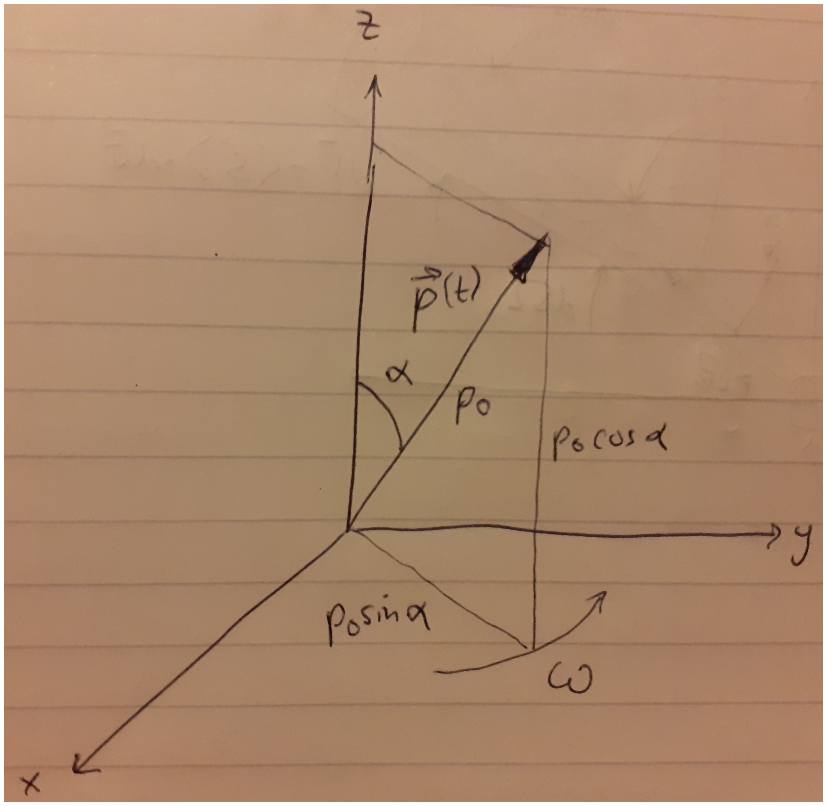
However, the example we'll do here will start by assuming we already have a dipole, specifically, one of constant magnitude p_0 spinning around a fixed axis with an angular speed ω, as shown on the next page.

Note that we're allowing the angle between the dipole and the axis of spin to be arbitrary, and we denote it by α, What would $\vec{p}(t)$ be in this case? First, pick a coordinate system so that the dipole's "tail" is at the origin and the axis of spin is the z-axis. This makes the z-component of the dipole constant in time, and if the dipole's magnitude is p_0, simple trigonometry says that it's $p_0 \cos\alpha$. If we project the tip of the dipole onto the xy-plane, we see that it traces a circle of radius $p_0 \sin\alpha$, and if we assume the motion is anticlockwise, the unit vector giving its direction in the xy-plane is $\cos\omega t \, \hat{e}_x + \sin\omega t \, \hat{e}_y$. All of this together gives the time-dependent dipole moment as

$$\vec{p}(t) \;\; = \;\; p_0 \sin\alpha \cos\omega t \, \hat{e}_x + p_0 \sin\alpha \sin\omega t \, \hat{e}_y + p_0 \cos\alpha \, \hat{e}_z.$$

However, as we discovered, it's only *moving* dipoles that radiate EM waves, so the constant z-component will not contribute to this and can be ignored in finding $\tilde{\vec{p}}_0$. Thus, what we want is a dipole amplitude such that

$$\mathrm{Re}\left[\tilde{\vec{p}}_0 e^{-i\omega t}\right] \;\; = \;\; p_0 \sin\alpha \cos\omega t \, \hat{e}_x + p_0 \sin\alpha \sin\omega t \, \hat{e}_y.$$



We can always write any complex vector in terms of two real vectors, so let's do that here: there are two real vectors $\vec{p}_R$ and $\vec{p}_I$ such that $\tilde{\vec{p}}_0 = \vec{p}_R + i\vec{p}_R$, and it's easy to see that this gives

$$\text{Re}\left[\tilde{\vec{p}}_0 e^{-i\omega t}\right] \;=\; \vec{p}_R \cos\omega t + \vec{p}_I \sin\omega t$$

and thus we have $\vec{p}_R = p_0 \sin\alpha\, \hat{e}_x$ and $\vec{p}_I = p_0 \sin\alpha\, \hat{e}_y$. We now have our dipole amplitude: $\tilde{\vec{p}}_0 = p_0 \sin\alpha\, (\hat{e}_x + i\hat{e}_y)$.

We're now in a position to pull out the expression for the far-zone magnetic field, because everything else comes from that. It's $\vec{B}(t, \vec{r}) = \text{Re}[\tilde{\vec{B}}(\vec{r}) e^{-i\omega t}]$ where

$$\tilde{\vec{B}}(\vec{r}) \;\approx\; \frac{\mu_0 \omega^2}{4\pi c} \left(\frac{\vec{r} \times \tilde{\vec{p}}_0}{r^2} \right) e^{ikr}.$$

This will give us the electric field using $\vec{E}(t, \vec{r}) = \text{Re}[\tilde{\vec{E}}(\vec{r}) e^{-i\omega t}]$ with $\tilde{\vec{E}} \approx$

$c\tilde{\vec{B}} \times \hat{e}_r$. These can both be computed using the dipole we've derived, but what we're most interested in is the time-averaged power distribution $\mathrm{d}\bar{P}/\mathrm{d}\Omega$, because this will give us an idea as to how much EM radiation we get in any given direction.

We derived this quantity in lecture, but let's remind ourselves quickly of how we get it: the time-averaged Poynting vector is

$$\begin{aligned}
\langle\vec{S}\rangle &= \left\langle \frac{1}{\mu_0}\vec{E}\times\vec{B}\right\rangle \\
&= \frac{1}{2\mu_0}\tilde{\vec{E}}(\vec{r})\times\tilde{\vec{B}}^*(\vec{r}) \\
&\approx \frac{c}{2\mu_0}|\tilde{\vec{B}}(\vec{r})|^2\hat{e}_r \\
&\approx \frac{\mu_0\omega^4}{32\pi^2 c}\frac{|\hat{e}_r\times\tilde{\vec{p}}_0|^2}{r^2}\,\hat{e}_r
\end{aligned}$$

and this gives us the energy current density associated with the far-zone EM radiation. Now we compute the cross-product: using Cartesian unit vectors but writing their coefficients in spherical coordinates gives

$$\begin{aligned}
\hat{e}_r\times\tilde{\vec{p}}_0 &= (\sin\theta\cos\phi\,\hat{e}_x+\sin\theta\sin\phi\,\hat{e}_y+\cos\theta\,\hat{e}_z)\times(p_0\sin\alpha\,\hat{e}_x+ip_0\sin\alpha\,\hat{e}_y) \\
&= p_0\sin\alpha\left[-i\cos\theta\,\hat{e}_x+\cos\theta\,\hat{e}_y+\sin\theta(-\sin\phi+i\cos\phi)\hat{e}_z\right].
\end{aligned}$$

Recall that $|\vec{a}|^2=\vec{a}^*\cdot\vec{a}$, so

$$\begin{aligned}
|\hat{e}_r\times\tilde{\vec{p}}_0|^2 &= \left(\hat{e}_r\times\tilde{\vec{p}}_0\right)^*\cdot\left(\hat{e}_r\times\tilde{\vec{p}}_0\right) \\
&= \{p_0\sin\alpha\left[i\cos\theta\,\hat{e}_x+\cos\theta\,\hat{e}_y+\sin\theta(-\sin\phi-i\cos\phi)\hat{e}_z\right]\} \\
&\quad\cdot\{p_0\sin\alpha\left[-i\cos\theta\,\hat{e}_x+\cos\theta\,\hat{e}_y+\sin\theta(-\sin\phi+i\cos\phi)\hat{e}_z\right]\} \\
&= p_0^2\sin^2\alpha\left[\cos^2\theta+\cos^2\theta+\sin^2\theta(\sin^2\phi+\cos^2\phi)\right] \\
&= p_0^2\sin^2\alpha\,(1+\cos^2\theta)
\end{aligned}$$

where we've used $\sin^2+\cos^2=1$ a couple of times. So our time-averaged Poynting vector is

$$\langle\vec{S}\rangle \approx \frac{\mu_0 p_0^2\omega^4\sin^2\alpha}{32\pi^2 c r^2}\left(1+\cos^2\theta\right)\hat{e}_r.$$

We now surround our spinning dipole with a sphere of (very large) radius r centred on the origin, so the above is the energy current density on the surface of this sphere. If we pick a small area $\mathrm{d}\sigma$ on this surface, then the solid angle $\mathrm{d}\Omega$ which subtends it is given by $\mathrm{d}\sigma = r^2\mathrm{d}\Omega$. The unit normal to the sphere's surface is $\hat{e}_r$, so the surface area element ia $\mathrm{d}\vec{\sigma} = r^2\mathrm{d}\Omega\,\hat{e}_r$. By definition, the time-averaged power – the energy flow rate – through this bit of surface is thus

$$\begin{aligned}\mathrm{d}\bar{P} &= \langle\vec{S}\rangle\cdot\mathrm{d}\vec{\sigma}\\ &\approx \frac{\mu_0 p_0^2\omega^4\sin^2\alpha}{32\pi^2 c}\left(1+\cos^2\theta\right)\mathrm{d}\Omega\end{aligned}$$

and thus the power per solid angle – the power distribution – is

$$\begin{aligned}\frac{\mathrm{d}\bar{P}}{\mathrm{d}\Omega} &= \langle\vec{S}\rangle\cdot\mathrm{d}\vec{\sigma}\\ &\approx \frac{\mu_0 p_0^2\omega^4\sin^2\alpha}{32\pi^2 c}\left(1+\cos^2\theta\right).\end{aligned}$$

Note that this is maximised when $\theta = 0$ or $\theta = \pi$ and minimised at $\theta = \pi/2$, so we see most of the radiated energy is along the axis of the dipole's spin, with very little radiated in directions perpendicular to this axis.

To find the total radiated power, we integrate the above power distribution over all 4π steradians of the sphere. This is equivalent to taking $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ and integrating θ from 0 to π and ϕ from 0 to 2π, giving

$$\begin{aligned}\bar{P} &= \int_{4\pi}\frac{\mathrm{d}\bar{P}}{\mathrm{d}\Omega}\,\mathrm{d}\Omega\\ &\approx \int \frac{\mu_0 p_0^2\omega^4\sin^2\alpha}{32\pi^2 c}\left(1+\cos^2\theta\right)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\\ &= \frac{\mu_0 p_0^2\omega^4\sin^2\alpha}{32\pi^2 c}\left(\int_0^\pi(\sin\theta+\cos^2\theta\sin\theta)\mathrm{d}\theta\right)\left(\int_0^{2\pi}\mathrm{d}\phi\right)\\ &= \frac{\mu_0 p_0^2\omega^4\sin^2\alpha}{32\pi^2 c}\left[-\cos\theta-\frac{1}{3}\cos^3\theta\right|_0^\pi\left[\phi\right|_0^{2\pi}\\ &= \frac{p_0^2\omega^4\sin^2\alpha}{6\pi\epsilon_0 c^3}.\end{aligned}$$

In the last step, we've replaced μ_0 by $1/\epsilon_0 c^2$. We didn't have to do this, but it's fairly common to have expressions in terms of μ_0 when the sources are magnetic in nature (currents and magnetic dipoles) and in terms of ϵ_0 when the sources are electric in nature (charges and electric dipoles).

The Instability of the Classical Hydrogen Atom

We can now use this formula to do a famous compution that lead, in part, to the formulation of quantum mechanics. By 1910 or so, the structure of the atom had been extensively studied and physicist and chemists were trying to come up with a consistent way to model an atom. J. J. Thompson had discovered the electron and realised that there were tiny, discrete bits of negative charge in every atom. But an (nonionised) atom is electrically neutral, so there had to be an equal amount of positive change. Thompson proposed his "plum pudding" model, in which the electrons were embedded like bits of fruit in a positively-charged lump of pudding. Heating the atom could then release the electrons as cathode rays.

But about a decade later, Ernest Rutherford's experiments showed that, rather than being spread throughout the atom, the positive charge was actually all concentrated in an extremely small region at the atom's centre, the nucleus. So then the prevailing model was something like a solar system, with most of the atom's mass as the positive nucleus and the electrons orbiting the nucleus like planets, with the attractive Coulomb force taking the place of gravity. Seems reasonable, and this model still persists in popular culture to this day.

The problem is, it can't work. Why? Because, as we've seen, an accelerating charge *radiates* energy away. And an electron (or anything else, for that matter) moving in a circular orbit does indeed accelerate. Thus, the planetary model of an atom would imply that all atoms radiate energy.

The formula we just derived allows us to compute how much energy is radiated, at least for a simple atom. So, take hydrogen. We assume the proton is fixed at the centre with the electron moving in a circle of radius R around it. But two charges separated by a distance defines an electric dipole! So a charge of $+e$ at the origin and $-e$ located at a position $\vec{R}$ gives a dipole of magnitude $p_0 = eR$ (pointing from the electron toward the proton). But the movement of the electron around the nucleus means this dipole spins, and so fits the example we've done. Since the orbit lies in a plane, $\alpha = \pi/2$, and so this tells us the rate at which the atom emits energy should be

$$\bar{P} \approx \frac{e^2 R^2 \omega^4}{6\pi\epsilon_0 c^3}.$$

But what's ω? The Coulomb force between the electron and the proton is $e^2/4\pi\epsilon_0 R^2$, and this must equal the centripetal force $m_e\omega^2 R$, so $\omega^2 =$

$e^2/4\pi\epsilon_0 R^3 m_e$, giving

$$\bar{P} \approx \frac{e^6}{96\pi^3\epsilon_0^3 c^3 m_e^2}\frac{1}{R^4}.$$

If the atom's total energy is E, this power is $-\dot{E}$ because it's the rate at which the atom *loses* energy due to radiation. But we know from Newtonian classical mechanics that a central potential of the form $V(r) = -k/r$ for a positive k leads to bound circular orbits of radius R with total energy $E = -k/2R$. For the Coulomb force, $k = e^2/4\pi\epsilon_0$, so $E = -e^2/8\pi\epsilon_0 R$ and thus the rate at which atom's energy changes is

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{e^2}{8\pi\epsilon_0 R}\right) \\ &= \frac{e^2}{8\pi\epsilon_0 R^2}\frac{\mathrm{d}R}{\mathrm{d}t} \\ &= -\bar{P} \\ &\approx -\frac{e^6}{96\pi^3\epsilon_0^3 c^3 m_e^2}\frac{1}{R^4}. \end{aligned}$$

Solving this for $\dot{R}$ gives

$$\begin{aligned} \frac{\mathrm{d}R}{\mathrm{d}t} &\approx -\frac{4}{3m_e^2 c^3}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{1}{R^2} \\ &= -\frac{4\alpha^2\lambda_e^2 c}{3}\frac{1}{R^2} \end{aligned}$$

where $\alpha = e^2/4\pi\epsilon_0\hbar c \approx 1/137$ is the fine-structure constant and $\lambda_e = \hbar/m_e c \approx 3.86 \times 10^{-13}$ m is the Compton wavelength of the electron. Now, suppose the initial radius of the orbit at $t = 0$ is R_0; then since the above differential equation is separable, we see that the radius R_1 at a time t_1 is given by the equation

$$\int_0^{t_1}\mathrm{d}t = -\frac{3}{4\alpha^2\lambda_e^2 c}\int_{R_0}^{R_1} R^2\mathrm{d}R$$

which gives

$$t_1 = \frac{R_0^3 - R_1^3}{4\alpha^2\lambda_e^2 c}.$$

So now we ask the following question: if the orbit's initial radius was the Bohr radius $a_0 = 5.29 \times 10^{-11}$ m, how long would it take for the electron to spiral into the nucleus? This uses $R_0 = a_0$ and $R_1 = 0$, and putting all the numbers in gives an answer of about 1.6×10^{-11} seconds. So a classical hydrogen atom has a lifetime of around *sixteen trillionths of a second*, and if this was the correct model of an atom, all hydrogen (not to mention every other element) in the universe would be long gone by now.

Now, some approximations were made here, obviously, but the basic maths is correct and we're correctly applied the laws of electromagnetism. The problem is that we've assumed that Newtonian classical mechanics holds at the subatomic level, and this result is an indication that maybe that's just not true. And that seems to be the case: we need quantum mechanics to describe the behaviour of fundamental particles. This calculation we've just done is virtually identical to the one that made Niels Bohr start thinking about his own model of the atom, and the rest, as they say, is history.

Types of EM Radiation

In lecture, I used a Fourier transform-based argument to derive the far-zone approximation for the magnetic field for *any* time-dependence of the electric dipole moment, and came up with

$$\vec{B}(t,\vec{r}) \approx \frac{\mu_0}{4\pi c}\left(\frac{\ddot{\vec{p}}(t-r/c)\times\vec{r}}{r^2}\right)$$

and $\vec{E}$ still approximately given by $c\vec{B}\times\hat{e}_r$. Now, since we don't assume the sources are rapidly oscillating, there's no need to time-average anything, so we get a Poynting vector

$$\begin{aligned}\vec{S} &\approx \frac{1}{\mu_0}\left(c\vec{B}\times\hat{e}_r\right)\times\vec{B}\\ &= \frac{c}{\mu_0}|\vec{B}|^2\hat{e}_r\\ &\approx \frac{\mu_0}{16\pi^2 c r^2}|\ddot{\vec{p}}(t-r/c)\times\hat{e}_r|^2.\end{aligned}$$

Thus, we see that the sources radiate only if the second derivative of the dipole moment is nonzero. So it's *acceleration* that causes radiation, not just movement. For the case of a single charge, this means that there are two very common cases where we expect electromagnetic radiation.

The first is the type we just saw in the classical hydrogen atom, where the charge is going around in a circle. But this isn't just confined to a charge in orbit around another, it's also true when we use other means – like, say magnetic fields – to force a charge to follow a circular path. Because this is exactly what's done in many particle accelerators (like the LHC at CERN), this radiation is named after the first circular accelerator originally developed in Berkeley: *cyclotron radiation*. This is a very real issue at big accelerators; as an exercise for yourself, look up some of the experiments done with the LHC and use the calculations we've done to find the power radiated. In many cases, it's enough to easily fry a human instantly, and so all big circular accelerators must be heavily shielded in order to prevent incinerating all those poor little physicists and technicians working on them.

The other common type of radiation shows up when a charge is travelling on a more-or-less straight path, but its speed changes. Since this generally consists of a charged particle being created and then slowing down as it moves through a detector (or some other medium), it's called *Brehmmstrahlung*, German for "braking radiation". If the rate of deceleration of the particle is small, this radiation isn't too big, but if a charged particle hits something and slows down very rapidly, it can be extreme and thus in such situations, shielding will also be required. Problem 2 on Problem Set 5 is a simplified version of this. If you're curious, you can put in some numbers that might come up in a real lab setting into the result of that problem to get an idea of the power that would be radiated as Brehmmstrahlung.

(These obviously aren't the only two types of time-dependence that will give a nonzero dipole second derivative, but they're the two most common, which is why they have their own names.)