MP465 – Advanced Electromagnetism

Tutorial 8 (7 April 2020)

Plane Waves and Conductors

We've talked already about conductors, those media in which charges can move freely. In the case of a perfect conductor, like a superconductor, charges can move without any resistance if a force is applied to them, but most materials aren't perfect conductors. In fact, the word I just used – "resistance" – is precisely the term we use to say how freely (or not) these charges can move. At the more practical engineering level, this quantity appears in the most famous formulation of Ohm's law: $I = \mathcal{E}/R$. In other words, if we have a voltage difference $\mathcal{E}$ (the notation comes from the alternate term "electromotive force" or "EMF" for voltage) between two points, then the current I between those points is proportional to this voltage, with that constant defined to be the inverse of the resistance R. Thus, a material with a very low resistance will have a very high current, whereas a very good insulator with a large R will not have much of a current at all.

Grand, but how do we formulate this in terms of the quantities appearing in Maxwell's equations, which are supposed to be all we need to describe electromagnetism? What we need is the microscopic version of Ohm's law, which is actually quite simple: we say that every medium has a property called conductance, denoted σ (that letter gets used an awful lot, doesn't it?), such that

$$\vec{J} = \sigma\vec{E}.$$

In other words, if the electric field in a conductor is $\vec{E}$, then the free current density it induces is proportional to it.

σ is something like the inverse of resistance: a very conductive material will have a large σ and thus even a tiny electric field might get the charges moving easily, but an insulator with a very low conductance might need an extremely large $\vec{E}$ to get even a small current density going.

The key is that, inside a conductor, $\vec{J}$ is no longer an independent quantity and thus, if the conductor is also a linear medium, Ampère's law is now

$$\vec{\nabla} \times \vec{B} = \mu\sigma\vec{E} + \mu\epsilon\frac{\partial\vec{E}}{\partial t}.$$

Also notice that if σ is constant (it doesn't have to be, but let's assume it is) then the continuity equation becomes

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= -\sigma\vec{\nabla}\cdot\vec{E} \\ &= -\frac{\sigma}{\epsilon}\rho\end{aligned}$$

which shows that the charge density must decay exponentially with time: $\rho(t,\vec{r}) = \rho_0(\vec{r})e^{-\sigma t/\epsilon}$ for some function $\rho_0(\vec{r})$. This is intuitively consistent: if the charges can move freely, we do not expect any region to sustain a free charge density very long. The charges will redistribute themselves so as to tend toward overall electric neutrality.

So now let's look at how this influences the existence of plane wave solutions. First, we assume either the conductor was electrically neutral to begin with, or we've waited loing enough so that ρ is effectively zero. Then the equations we need to solve are

$$\begin{gathered}\vec{\nabla}\cdot\vec{E} = 0, \qquad \vec{\nabla}\cdot\vec{B} = 0, \\ \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \qquad \vec{\nabla}\times\vec{B} = \mu\sigma\vec{E} + \mu\epsilon\frac{\partial\vec{E}}{\partial t}.\end{gathered}$$

So are there still monochromatic plane wave solutions to these equations? If there are, then we have

$$\begin{aligned}\vec{E} &= \mathrm{Re}\left[\tilde{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right], \\ \vec{B} &= \mathrm{Re}\left[\tilde{\vec{B}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]\end{aligned}$$

for some vectors $\tilde{\vec{E}}_0$, $\tilde{\vec{B}}_0$ and $\vec{k}$. The first three Maxwell equations are unchanged from the case with no conductor, so we get the same results as before: $\vec{k}\cdot\tilde{\vec{E}}_0 = \vec{k}\cdot\tilde{\vec{B}}_0 = 0$ and $\tilde{\vec{B}}_0 = \vec{k}\times\tilde{\vec{E}}_0/\omega$. Ampère's law, however, gives

$$i\vec{k}\times\tilde{\vec{B}}_0 = \mu\sigma\tilde{\vec{E}}_0 - i\mu\epsilon\omega\tilde{\vec{E}}_0.$$

Using $\tilde{\vec{B}}_0 = \vec{k}\times\tilde{\vec{E}}_0/\omega$ and doing the usual calculations then yields

$$-\frac{ik^2}{\omega}\tilde{\vec{E}}_0 = \left(\mu\sigma - i\mu\epsilon\omega\right)\tilde{\vec{E}}_0$$

which means the relation between the wave vector and frequency is now

$$k^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega.$$

But since we've assumed that ω is real, this is only possible if k is a *complex* number. What complex number is it? Well, it must have the form $k = \kappa + i\gamma$, so squaring this, putting it into the above and equating the real and imaginary parts gives

$$\kappa^2 - \gamma^2 = \mu\epsilon\omega^2, \qquad 2\kappa\gamma = \mu\sigma\omega.$$

The second gives $\gamma = \mu\sigma\omega/2\kappa$, and putting this into the first and doing a bit of algebra gives the quartic equation

$$\kappa^4 - \mu\epsilon\omega^2\kappa^2 - \frac{\mu^2\sigma^2\omega^2}{4} = 0.$$

This is quadratic in κ^2, and solving gives the two solutions

$$\kappa^2 = \frac{\mu\epsilon\omega^2 \pm \sqrt{\mu^2\epsilon^2\omega^4 + \mu^2\sigma^2\omega^2}}{2}.$$

κ is real so the above must be positive. Since the square root is larger than $\mu\epsilon\omega^2$, we therefore have to choose the plus sign of the two solutions, giving

$$\begin{aligned}\kappa^2 &= \mu\epsilon\omega^2\left(\frac{\sqrt{1+\frac{\sigma^2}{\epsilon^2\omega^2}}+1}{2}\right)\\ &= \frac{n^2\omega^2}{c^2}\left(\frac{\sqrt{1+\frac{\sigma^2}{\epsilon^2\omega^2}}+1}{2}\right)\end{aligned}$$

since $\mu\epsilon = n^2/c^2$. Just to make things easy, let's assume the wave propagates in the positive z-direction, i.e. $\vec{k} = (\kappa + i\gamma)\hat{e}_z$ with $\kappa > 0$. This, together with $\gamma = \mu\sigma\omega/2\kappa$, gives

$$\begin{aligned}\kappa &= \frac{n\omega}{c}\sqrt{\frac{\sqrt{1+\frac{\sigma^2}{\epsilon^2\omega^2}}+1}{2}},\\ \gamma &= \frac{n\omega}{c}\sqrt{\frac{\sqrt{1+\frac{\sigma^2}{\epsilon^2\omega^2}}-1}{2}}.\end{aligned}$$

Yoicks. What's the interpretation of all this? Well, let's put this wave vector into the expression for the electric field:

$$\begin{aligned}\vec{E} &= \text{Re}\left[\tilde{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right], \\ &= \text{Re}\left[\tilde{\vec{E}}_0 e^{i((\kappa+i\gamma)z-\omega t)}\right] \\ &= e^{-\gamma z}\text{Re}\left[\tilde{\vec{E}}_0 e^{i(\kappa z-\omega t)}\right].\end{aligned}$$

The part in the Re describes a plane wave with wave vector $\kappa\hat{e}_z$, with the relation between κ and ω given above, but the factor in front says that its amplitude is decreasing exponentially with increasing z. In other words, the further the wave goes into the conductor, the weaker it becomes. In fact, we see that for each $1/\gamma$ further it goes into the conductor, its amplitude is attenuated by a factor of $1/e$. For this reason,

$$d = \frac{1}{\gamma} = \frac{1}{\sigma}\sqrt{\frac{2\epsilon}{\mu}\left(\sqrt{1+\frac{\sigma^2}{\epsilon^2\omega^2}}+1\right)}$$

is called the *penetration depth* of the wave.

Now, let's look at the two extreme cases: as $\sigma \to 0$, the medium tends toward a perfect insulator, which is equivalent to the absence of a free current since Ohm's law says $\vec{J} \to 0$ in this limit. And we do indeed recover the no-free-charges-or-currents case we discussed in lecture: $\kappa \to n\omega/c$ and $\gamma \to 0$. We have a unattentuated wave which merrily propagates as far as it wants (infinite penetration depth). For a perfect conductor, $\sigma \to \infty$; we see κ and γ both blow up, which means d drops to zero. The wave cannot penetrate the conductor at all, and so the electric and magnetic fields must *vanish* inside a perfect conductor. This is a result we invoked on intuitive grounds at the beginning of the module, and now we have a more rigourous justification for this assumption. Ain't physics great?

Solar Energy Flux

Now let's do a bit of actual number-crunching with one of the results from the last lecture, namely, the form of the average Poynting vector for an EM plane wave:

$$\langle\vec{S}\rangle = \frac{\vec{k}}{2\mu\omega}|\tilde{\vec{E}}_0|^2.$$

Note that the magnitude of this vector is

$$\begin{aligned}\left|\langle\vec{S}\rangle\right| &= \frac{k}{2\mu\omega}|\tilde{\vec{E}}_0|^2\\ &= \frac{n}{2\mu c}|\tilde{\vec{E}}_0|^2\end{aligned}$$

since $k/\omega = 1/v = n/c$. Thus, if we have a medium with a known n and μ and can somehow measure $|\langle\vec{S}\rangle|$, then we can get an idea of the magnitude of the electric field in the EM wave from this expression.

So let's actually do this for the light from the Sun! $|\langle\vec{S}\rangle|$ has been measured at the Earth's surface (or rather, just outside the Earth's atmosphere) and is called the "solar constant", having the value $1.36\,\mathrm{kW}\cdot\mathrm{m}^{-2}$. (Remember that it's an energy current and thus has units of (energy per time) per area, or watts per square metre in SI.) Can we determine the strength of the electric field in the Sun's EM radiation?

Only roughly, because our result was based on monochromatic waves, and the Sun's light is anything but monochromatic. But let's go with it anyway: if we assume that we can take just outside the Earth's atmosphere to be a true vacuum, then $n = 1$ and $\mu = \mu_0$, so we find

$$\begin{aligned}|\tilde{\vec{E}}_0| &= \sqrt{2\mu_0 c\left|\langle\vec{S}\rangle\right|}\\ &= 1.01\times 10^3\,\mathrm{V}\cdot\mathrm{m}^{-1}\end{aligned}$$

or about one kilovolt per metre.

Now that we have this, we can get some other quantities of interest. For example, the strength of the magnetic field part of the Sun's light is

$$\begin{aligned}|\tilde{\vec{B}}_0| &= \frac{|\vec{k}\times\tilde{\vec{E}}_0|}{\omega}\\ &= \frac{|\tilde{\vec{E}}_0|}{c}\\ &= 3.38\times 10^{-6}\,\mathrm{T}\end{aligned}$$

or a few microtesla. The energy density is $u = \epsilon_0|\tilde{\vec{E}}_0|^2/2 = 4.51\times 10^{-6}\,\mathrm{J}\cdot\mathrm{m}^{-3}$ and the magnitude of the momentum density is $|\langle\mu\epsilon\vec{S}\rangle| = |\langle\vec{S}\rangle|/c^2 = 1.51\times 10^{-14}\mathrm{N}\cdot\mathrm{s}\cdot\mathrm{m}^{-3}$.

Now, let's go back to this idea of seasonal temperature. I gave the general idea of why Dublin is warmer than Sydney in June, but let's see if we can estimate how much warmer. Or rather, how much more energy Dublin gets from the Sun than Sydney does at high noon on the Summer Solstice. Recall that the average power that a given area element $\mathrm{d}\vec{\sigma}$ (*not* the conductivity – I told you that σ is an overused letter) receives is $\mathrm{d}\bar{P} = |\langle\vec{S}\rangle|\,\mathrm{d}\sigma\,\cos\theta$, where θ is the angle between the wave's direction and the unit normal to the area element. At high summer in the northern hemisphere, this angle will be the latitude of the place in question minus the tilt of the Earth's axis, about $23.4°$, as shown in the below picture:

Thus, the relative power a given bit of area receives between two latitudes λ_1 and λ_2 will be

$$\frac{\mathrm{d}\bar{P}_1}{\mathrm{d}\bar{P}_2} = \frac{\cos\left(\lambda_1 - 23.4°\right)}{\cos\left(\lambda_2 - 23.4°\right)}.$$

Dublin's latitude is about $\lambda_1 = 53°$ and Sydney's is $\lambda_2 = -36°$, so putting these in gives a ratio of 1.7; at 12pm on June 21$^{\mathrm{st}}$, a given patch of ground in Dublin gets 1.7 times as much energy per second as the same patch would in Sydney.

This is, of course, a very rough estimate and doesn't even begin to take into account any atmospheric effects, but it gives the correct general idea of what's going on. Similar calculations could be made for relative power ratios for different places and at different times of the year, but the key lies in the fact that the Poynting vector is responsible for the energy current of an EM wave, and the magnitude of that vector can be empirically measured.