MP465 – Advanced Electromagnetism

Tutorial 7 (1 April 2020)

Fun with the Levi-Civita Symbol

In this tutorial, we'll look at an object that I suspect most of you have encountered before may not have suspected how useful it is in doing many of the types of calculations we come across in this (and other modules). It's called the Levi-Civita symbol (after Tullio Levi-Civita) ϵ_{ijk}, and is a three-index object, where each index can take on the value 1, 2 or 3 (or x, y, z if we think of the indices aas labelling the Cartesian components of a position vector) and is either 1, -1 or 0 depending on the indices. Here's the definition:

$$\epsilon_{ijk} = \begin{cases} +1 & \text{when } (ijk) \text{ is an even permutation of } (123), \\ -1 & \text{when } (ijk) \text{ is an odd permutation of } (123), \\ 0 & \text{when any two indices are the same,} \end{cases}$$

By "even" and "odd" permutation, we mean the number of pair-swaps you need to get from the given values of (ijk) to (123). For example, (312) is an even permutation because you can get to (123) in two pairwise-swaps: (312) $\rightarrow$ (132) $\rightarrow$ (123). So using this rule, the three even permutations are (123), (231) and (312), and the three odd ones are (321), (213) and (132).

This definition makes ϵ_{ijk} totally antisymmetric, i.e. any swap of two indices changes the sign. For example, for any choice of index values, $\epsilon_{kji} = -\epsilon_{ijk}$ because you can get (kji) from (ijk) by swapping i and k. In fact, *any* totally antisymmetric object A_{ijk} must be proportional to the Levi-Civita symbol (provided the indices can only take on the values 1, 2 and 3). And if M_{ij} is antisymmetric in i and j, then there must exist three numbers m_1, m_2 and m_3 such that

$$M_{ij} = \sum_{k=1}^{3} \epsilon_{ijk} m_k.$$

(Henceforth, all sums in this tutorial will be assumed to run from 1 to 3.)

We could have used this in lecture; recall that we showed that $\int x'_j J_i(\vec{r}')\,\mathrm{d}^3\vec{r}'$ was antisymmetric in its indices. Therefore, we would have known that there exists a vector $\vec{m}$ whose components are defined by

$$\int x'_j J_i(\vec{r}')\,\mathrm{d}^3\vec{r}' = \sum_k \epsilon_{jik} m_k$$

and this vector is precisely the magnetic dipole moment.

As I said, you've likely seen this symbol before. For example, if you've studied angular momentum (AM) in quantum mechanics, you've probably seen the commutation relations for the three AM operators J_x, J_y and J_z written as

$$[J_i, J_j] \;=\; i\hbar \sum_k \epsilon_{ijk} J_k.$$

However, I'm guessing that the main place you've seen it is in the definition of the cross product of two three-dimensional vectors: if $\vec{a}$ and $\vec{b}$ are both 3-vectors, then $\vec{c} = \vec{a} \times \vec{b}$ is also a 3-vector with components

$$\left(\vec{a} \times \vec{b}\right)_i \;=\; \sum_{j,k} \epsilon_{ijk} a_j b_k.$$

Thus,

$$\begin{aligned} c_x &= \sum_{jk} \epsilon_{1jk} a_j b_k \\ &= \epsilon_{123} a_2 b_3 + \epsilon_{132} a_3 b_2 \\ &= a_y b_z - a_z b_y, \end{aligned}$$

where in the second step we used the fact that the only nonzero values of ϵ in the sum are the ones with three different indices.

Similarly, the curl of any vector may be written as

$$\left(\vec{\nabla} \times \vec{a}\right)_i \;=\; \sum_{j,k=1}^{3} \epsilon_{ijk} \partial_j a_k$$

where ∂_j is a shorthand for $\partial/\partial x_j$. However, it's worth noting that this formula hold only for the Cartesian components of the curl. In a curvilinear coordinate system, it's more complicated.

Now, one of the most useful identities that the Levi-Civita symbol satisfies is the one involving the product of two of them summed over one of the indices, namely,

$$\sum_k \epsilon_{ijk} \epsilon_{mnk} \quad \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm},$$

where δ is our old friend, the Kronecker delta symbol. It's this which gives us one of the most famous vector identities of all, that for the vector triple product $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$ (or "back minus cab", as I learned it way back in the day). Here's the proof:

$$\begin{aligned}
\left[\vec{a} \times (\vec{b} \times \vec{c})\right]_i &= \sum_{k,j} \epsilon_{ijk} a_j (\vec{b} \times \vec{c})_k \\
&= \sum_{k,j} \epsilon_{ijk} a_j \left(\sum_{mn} \epsilon_{kmn} b_m c_n \right) \\
&= \sum_{j,m,n} a_j b_m c_n \left(\sum_k \epsilon_{ijk} \epsilon_{mnk} \right) \\
&= \sum_{j,m,n} a_j b_m c_n \left(\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm} \right) \\
&= \left(\sum_m b_m \delta_{im} \right) \left(\sum_{j,n} a_j c_n \delta_{jn} \right) - \left(\sum_n c_n \delta_{in} \right) \left(\sum_{j,m} a_j b_m \delta_{jm} \right) \\
&= b_i \left(\vec{a} \cdot \vec{c} \right) - c_i \left(\vec{a} \cdot \vec{b} \right)
\end{aligned}$$

and we're done. In Problem Set 3, I've given a couple of similar identities to you to prove.

Another place the Levi-Civita symbol appears is as a way to compute the determinant of a 3×3 matrix. If A is any such matrix with A_{ij} its entries, then

$$\sum_{\ell,m,n} \epsilon_{\ell mn} A_{i\ell} A_{jm} A_{kn} = \epsilon_{ijk} \det A$$

or, since $\epsilon_{123} = 1$,

$$\det A = \sum_{\ell,m,n} \epsilon_{\ell mn} A_{1\ell} A_{2m} A_{3n}$$

ant this is actually the formula you're using when you figure out minors and cofactors and the like when computing $\det A$, believe it or not.

And this also gives us the most common way to compute the cross product: the above formula for $(\vec{a} \times \vec{b})_i$ gives the vector form $\vec{a} \times \vec{b} = \sum_{ijk} \hat{e}_i a_j b_k$. But suppose we define a "matrix" M whose entries are given by $M_{1i} = \hat{e}_i$,

$M_2 j = a_j$ and $M_{3k} = bk$. This matrix has the three Cartesian unit vectors in its top row, its middle row are the components of $\vec{a}$ and its bottom row are the components of $vecb$. Using the determinant formula above, we see

$$\begin{aligned}
\vec{a} \times \vec{b} &= \sum_{ijk} \hat{e}_i a_j b_k \\
&= \sum_{ijk} M_{1i} M_{2j} M_{3k} \\
&= \det M \\
&= \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}
\end{aligned}$$

which is probably how a lot of you compute cross products. Now you know why it works!

To finish this tutorial, let's prove a vector calculus identity that I mentioned in the last lecture, namely,

$$\int_{\mathcal{V}} \vec{\nabla} \times \vec{a} \, \mathrm{d}^3 \vec{r} = \oint_{\Sigma} \mathrm{d}\vec{\sigma} \times \vec{a}.$$

The i^{th} component of this equation is

$$\int_{\mathcal{V}} \left(\vec{\nabla} \times \vec{a} \right)_i \mathrm{d}^3 \vec{r} = \oint_{\Sigma} \left(\mathrm{d}\vec{\sigma} \times \vec{a} \right)_i .$$

and this is what we'll prove.

We gave the formula for the i^{th} component of the curl of a vector above, and hopefully you all see it can be rewritten as

$$\begin{aligned}
\left(\vec{\nabla} \times \vec{a} \right)_i &= \sum_j \partial_j \left(\sum_k \epsilon_{ijk} a_k \right) \\
&= \vec{\nabla} \cdot \vec{b}_i
\end{aligned}$$

where $\vec{b}_i$ is the vector with components $(\vec{b}_i)_j = \sum_k \epsilon_{ijk} a_k$ (e.g. $\vec{b}_1 = a_z \hat{e}_y - a_y \hat{e}_z$). Thus, if we integrate this curl over a region of space $\mathcal{V}$ with boundary

Σ, we see

$$\begin{aligned}
\int_{\mathcal{V}} \left(\vec{\nabla} \times \vec{a}\right)_i \mathrm{d}^3\vec{r} &= \int_{\mathcal{V}} \vec{\nabla} \cdot \vec{b}_i \, \mathrm{d}^3\vec{r} \\
&= \oint_{\Sigma} \vec{b}_i \cdot \mathrm{d}\vec{\sigma} \\
&= \oint_{\Sigma} \sum_j \left(\vec{b}_i\right)_j \mathrm{d}\sigma_j \\
&= \oint_{\Sigma} \sum_{j,k} \epsilon_{ijk} \mathrm{d}\sigma_j a_k \\
&= \oint_{\Sigma} \left(\mathrm{d}\vec{\sigma} \times \vec{a}\right)_i
\end{aligned}$$

and so we have the vector calculus identity

$$\int_{\mathcal{V}} \vec{\nabla} \times \vec{a} \, \mathrm{d}^3\vec{r} = \oint_{\Sigma} \mathrm{d}\vec{\sigma} \times \vec{a}.$$