MP465 – Advanced Electromagnetism

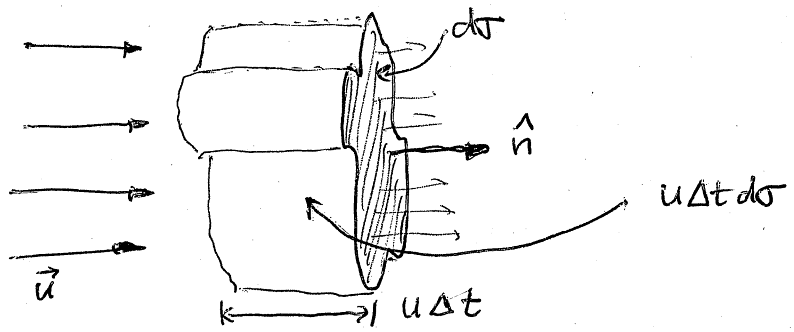
Tutorial 6 (24 March 2020)

Moving Charge Distributions

Another factoid about magnetostatic systems: we know (hopefully) at this point that currents are due to moving charges. How exactly does that work? More specifically, if we have a charge distribution characterised by a density $\rho(\vec{r})$ and the velocity field of this distribution is $\vec{u}(\vec{r})$, what current density $\vec{J}(\vec{r})$ does this give?

Those of you who took Fluid Mechanics may already know the answer, but let's figure it out here anyway: we pick a point $\vec{r}$ and look at a very small region around it so that all quantities may be approximated by their values at that point. Now, let $\hat{n}$ be a unit vector in the same direction as $\vec{u}$ at this point. We now pick some tiny surface element $\mathrm{d}\vec{\sigma}$ whose normal is $\hat{n}$ and ask how much charge Δq will flow through this area in a small tims Δt.

If $u = |\vec{u}|$, the charge distribution will travel a distance $u\Delta t$ in this time. The total volume of charge moving through the area $\mathrm{d}\sigma = |\mathrm{d}\vec{\sigma}|$ in this time will therefore be $u\Delta t\,\mathrm{d}\sigma$, as shown below:



But since $\vec{u}$ and $\mathrm{d}\vec{\sigma}$ are parallel, $u\,\mathrm{d}\sigma = \vec{u}\cdot\mathrm{d}\vec{\sigma}$ so the total volume crossing the area element in Δt is $\vec{u}\cdot\mathrm{d}\vec{\sigma}\Delta t$, which means that the total *charge* flowing through this element is $\Delta q = \rho\vec{u}\cdot\mathrm{d}\vec{\sigma}\Delta t$. Since $\Delta q/\Delta t$ is the current, we thus find the little bit of current $\mathrm{d}I$ flowing through the surface element $\mathrm{d}\vec{\sigma}$ is

$$\mathrm{d}I = \rho\vec{u}\cdot\mathrm{d}\vec{\sigma}.$$

But recall the definition of the current density: if $\mathrm{d}\vec{\sigma}$ is a surface element located at $\vec{r}$, then the current flowing through this element is $\vec{J}(\vec{r})\cdot\mathrm{d}\vec{\sigma}$.

Therefore, we have what we're after: if a charge configuration with density $\rho(\vec{r})$ moves with a velocity field $\vec{u}(\vec{r})$, then it has an associated current density

$$\vec{J}(\vec{r}) = \rho(\vec{r})\,\vec{u}(\vec{r}).$$

(Important: we assumed a *static* system here, i.e. no time dependence. This result only holds for such systems.)

Let's use this fact to find the magnetic field and magnetic dipole moment of a spinning disc of charge. Take an infinitely-thin disc of radius a with a charge q distributed uniformly, giving a surface charge density of $\sigma = q/\pi a^2$. If we define a cylindrical coordinate system such the disc lies in the xy-plane with its centre at the origin, then the correct charge density for this is $\rho(r,\phi,z) = \sigma\delta(z)$ for $r < a$ and zero everywhere else.

Now we spin the disc around the z-axis with a constant angular velocity $\vec{\omega} = \omega\hat{e}_z$. We know from basic rotational mechanics that the velocity at a position $\vec{r} = r\hat{e}_r + z\hat{e}_z$ is $\vec{u} = \vec{\omega}\times\vec{r}$, so we get the velocity field $\vec{u}(r,\phi,z) = \omega r\hat{e}_\phi$. From what we just derived, this gives us a current density

$$\vec{J}(r,\phi,z) = \sigma\omega r\delta(z)\hat{e}_\phi$$

for $r < a$ and zero elsewhere.

To get the magnetic field, we'll of course use the formula

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,\mathrm{d}^3\vec{r}\,',$$

but once again we'll find that we need elliptic integrals to get this for arbitrary $\vec{r}$. So let's limit ourselves to finding this field only on the z-axis, namely, $\vec{r} = z\hat{e}_z$. It's easy to show that

$$\begin{aligned}
\vec{J}(\vec{r}\,')\times(\vec{r}-\vec{r}\,') &= \sigma\omega r'\delta(z')\left[(z-z')\,\hat{e}_{r'} + r'\hat{e}_z\right],\\
|\vec{r}-\vec{r}\,'| &= \sqrt{(r')^2 + (z-z')^2}
\end{aligned}$$

so

$$\begin{aligned}
\vec{B}(0,\phi,z) &= \frac{\mu_0}{4\pi}\int \frac{\sigma\omega r'\delta(z')\left[(z-z')\,\hat{e}_{r'} + r'\hat{e}_z\right]}{\left[(r')^2 + (z-z')^2\right]^{3/2}}\,r'\,\mathrm{d}r'\,\mathrm{d}\phi'\,\mathrm{d}z'\\
&= \frac{\mu_0\sigma\omega}{4\pi}\int \frac{(r')^2\left(z\hat{e}_{r'} + r'\hat{e}_z\right)}{\left[(r')^2 + z^2\right]^{3/2}}\,\mathrm{d}r'\,\mathrm{d}\phi'
\end{aligned}$$

after doing the z' integral.

Now, notice that the only dependence on ϕ' in the integrand is in $\hat{e}_{r'} = \cos\phi'\,\hat{e}_x + \sin\phi'\,\hat{e}_y$, but since both the sine and cosine integrate to zero over the interval $[0, 2\pi]$, the first term in the numerator vanishes, leaving

$$\begin{aligned} \vec{B}\,(0,\phi,z) &= \frac{\mu_0\sigma\omega}{4\pi}\int\frac{(r')^3\,\hat{e}_z}{\left[(r')^2+z^2\right]^{3/2}}\,\mathrm{d}r'\,\mathrm{d}\phi' \\ &= \frac{\mu_0\sigma\vec{\omega}}{2}\int_0^a\frac{(r')^3}{\left[(r')^2+z^2\right]^{3/2}}\,\mathrm{d}r' \end{aligned}$$

since $\omega\hat{e}_z = \vec{\omega}$. Thus, as expected, the magnetic field is purely in the z-direction. This integral is easily done by noticing that

$$\begin{aligned} \frac{(r')^3}{\left[(r')^2+z^2\right]^{3/2}} &= \frac{r'\left[(r')^2+z^2-z^2\right]}{\left[(r')^2+z^2\right]^{3/2}} \\ &= \frac{r'}{\sqrt{(r')^2+z^2}} - \frac{r'z^2}{\left[(r')^2+z^2\right]^{3/2}} \end{aligned}$$

giving

$$\begin{aligned} \vec{B}\,(0,\phi,z) &= \frac{\mu_0\sigma\vec{\omega}}{2}\left[\sqrt{(r')^2+z^2}+\frac{z^2}{\sqrt{(r')^2+z^2}}\right|_0^a \\ &= \frac{\mu_0\sigma\vec{\omega}}{2}\left[\sqrt{z^2+a^2}+\frac{z^2}{\sqrt{z^2+a^2}}-2|z|\right] \end{aligned}$$

where we used $\sqrt{z^2} = z^2/\sqrt{z^2} = |z|$.

Now, if $|z| \gg a$, then expanding the first two terms gives

$$\begin{aligned} \sqrt{z^2+a^2} &= |z|\sqrt{1+\frac{a^2}{|z|^2}} \approx |z| + \frac{a^2}{2|z|} - \frac{a^4}{8|z|^3}, \\ \frac{z^2}{\sqrt{z^2+a^2}} &= \frac{|z|}{\sqrt{1+\frac{a^2}{|z|^2}}} \approx |z| - \frac{a^2}{2|z|} + \frac{3a^4}{8|z|^3} \end{aligned}$$

which gives

$$\vec{B}(0,\phi,z) \approx \frac{\mu_0\sigma a^4\vec{\omega}}{8|z|^3}.$$

Why do we care about this? Well, we know the leading term in the multipole expansion for a magnetic field is the dipole contribution, so we can use the above approximation to check this: the magnetic dipole moment is

$$\begin{aligned}
\vec{m} &= \frac{1}{2}\int \vec{r}' \times \vec{J}(\vec{r}')\, \mathrm{d}^3\vec{r}' \\
&= \frac{\sigma\omega}{2}\int (r'\hat{e}_{r'} + z'\hat{e}_z) \times r'\delta(z')\,\hat{e}_{\phi'}\, r'\,\mathrm{d}r'\,\mathrm{d}\phi'\,\mathrm{d}z' \\
&= \frac{\sigma\omega}{2}\int (r')^3\,\hat{e}_z\,\mathrm{d}r'\,\mathrm{d}\phi' \\
&= \frac{\pi\sigma a^4\omega}{4}\hat{e}_z.
\end{aligned}$$

The dipole field given in *spherical* coordinates (i.e. where $r = |\vec{r}|$) is

$$\vec{B}_1 = \frac{\mu_0}{4\pi}\frac{3(\vec{m}\cdot\vec{r})\vec{r} - r^2\vec{m}}{r^5},$$

which, for the dipole we have here, is

$$\vec{B}_1 = \frac{\mu_0\sigma a^4\omega}{16}\frac{3z\vec{r} - r^2\hat{e}_z}{r^5}$$

so if we're on the z-axis, $\vec{r} = z\hat{e}_z$ and $r = |z|$, and you can show quickly that

$$\vec{B}_1 = \frac{\mu_0\sigma a^4\vec{\omega}}{8|z|^3}$$

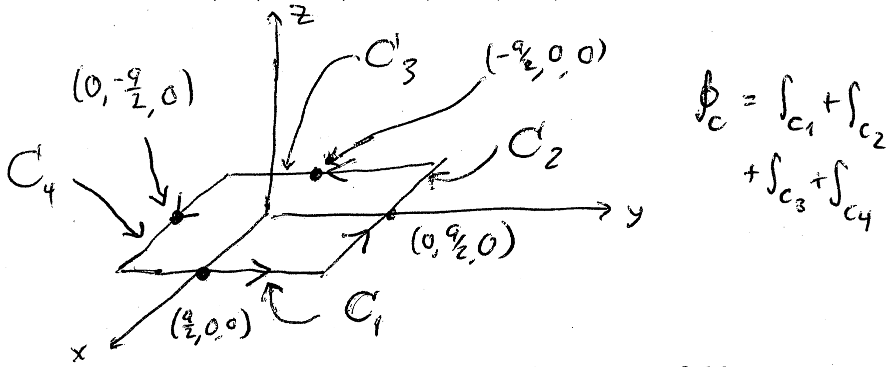
which is exactly what we got from our explicit computation in the $|z| \gg a$ case. Yay!

Potential Energy of a Magnetic Dipole

In lecture, I said that the potential energy of a magnetic dipole moment $\vec{m}$ in a magnetic field $\vec{B}$ is $V = -\vec{m}\cdot\vec{B}$. As promised, we'll prove this now...

We start with a point dipole $\vec{m}$. We're free to choose any coordinate system we like for this discussion, so pick one such the dipole is at the origin and it points in the positive z-direction: $\vec{m} = m\hat{e}_z$. We know from lecture that one way to model such a dipole is to take a tiny loop of constant current in the xy-plane, flowing anticlockwise, and taking the loop size to zero while keeping the quantity $m =$ (current) $\times$ (area inside loop) constant.

So let's take a square loop of side length a with a constant current I flowing around it, so $m = Ia^2$ is the dipole magnitude. If the square is centred at the origin with its sides parallel to the x- and y-axes, then the midpoints of each side are $(\pm a/2, 0, 0)$ and $(0, \pm a/2, 0)$, as shown below:



To find the force on this loop due to an external magnetic field, we use the Lorentz force law in the form $\vec{F} = \int \vec{J} \times \vec{B}\, \mathrm{d}^3\vec{r}$. However, as we've fone several times for a 1-dimensional system, we replace $\vec{J}\,\mathrm{d}^3\vec{r}$ with $I\mathrm{d}\vec{r}$, and so the total force on a closed loop $\mathcal{C}$ of current is

$$\vec{F} = \oint_{\mathcal{C}} I(\vec{r})\, \mathrm{d}\vec{r} \times \vec{B}(\vec{r})\,.$$

In this case, $\mathcal{C}$ is a square so the integral can be broken up into four line integrals, one for each side, indicated in the picture above. Now the key point: because the loop is so small, the magnetic field does not change much over each side and so we approximate it by its value at the midpoint of each side. So on segment $\mathcal{C}_1$, the field is approximately $\vec{B}(a/2.0.0)$. On this segment, $\mathrm{d}\vec{r} = \hat{e}_y \lceil y$ with y going from $-a/2$ to $a/2$, so the contribution to the force is

$$\begin{aligned}
\vec{F}_1 &= \oint_{\mathcal{C}_1} I(\vec{r})\, \mathrm{d}\vec{r} \times \vec{B}(\vec{r}) \\
&\approx \int_{-a/2}^{a/2} I\hat{e}_y \times \vec{B}(a/2, 0, 0)\, \mathrm{d}y \\
&= Ia\left[B_z(a/2, 0, 0)\hat{e}_x - B_x(a/2, 0, 0)\hat{e}_z\right].
\end{aligned}$$

Very similar arguments give the other three contributions as

$$\begin{aligned}
\vec{F}_2 &\approx Ia\left[B_z(0,a/2,0)\hat{e}_y - B_y(0,a/2,0)\hat{e}_z\right]. \\
\vec{F}_3 &\approx Ia\left[B_x(-a/2,0,0)\hat{e}_z - B_z(-a/2,0,0)\hat{e}_x\right]. \\
\vec{F}_4 &\approx Ia\left[B_y(0,-a/2,0)\hat{e}_z - B_z(0,-a/2,0)\hat{e}_y\right]
\end{aligned}$$

so the total force is approximately

$$\begin{aligned}
\vec{F} \approx\; & Ia\left\{\left[B_z(a/2,0,0) - B_z(-a/2,0,0)\right]\hat{e}_x + \left[B_z(0,a/2,0) - B_z(0,-a/2,0)\right]\hat{e}_y\right. \\
& \left. + \left[B_x(-a/2,0,0) - B_x(a/2,0,0) + B_y(0,-a/2,0) - B_y(0,a/2,0)\right]\hat{e}_z\right\}.
\end{aligned}$$

It's easy to show that, for any function $f(s)$, if h is very small, $f(s+h/2) - f(s-h/2) = hf(s) + O(h^2)$, so we see the above is, to $O(a^3)$,

$$\vec{F} \approx Ia^2\left\{\frac{\partial B_z}{\partial x}(0,0,0)\hat{e}_x + \frac{\partial B_z}{\partial y}(0,0,0)\hat{e}_y - \left[\frac{\partial B_x}{\partial x}(0,0,0) + \frac{\partial B_y}{\partial y}(0,0,0)\right]\hat{e}_z\right\}.$$

Notice that the z component is $\partial B_z/\partial z - \vec{\nabla}\cdot\vec{B}$ evaluated at the origin, but since $\vec{\nabla}\cdot\vec{B} = 0$ everywhere, we get

$$\begin{aligned}
\vec{F} &\approx Ia^2\left\{\frac{\partial B_z}{\partial x}(0,0,0)\hat{e}_x + \frac{\partial B_z}{\partial y}(0,0,0)\hat{e}_y + \frac{\partial B_z}{\partial z}(0,0,0)\hat{e}_z\right\} \\
&= Ia^2\vec{\nabla}B_z(0,0,0).
\end{aligned}$$

But $Ia^2 = m$, the dipole magnitude, and $B_z = \hat{e}_z\cdot\vec{B}$, so the force on the dipole at the origin (i.e. where the dipole is located) is

$$\begin{aligned}
\vec{F} &\approx \vec{\nabla}\left(m\hat{e}_z\cdot\vec{B}\right) \\
&= \vec{\nabla}\left(\vec{m}\cdot\vec{B}\right)
\end{aligned}$$

with the approximation becoming better as $a\to 0$. Thus, the above is the force on a dipole in a magnetic field.

However, we notice that the force is *conservative*, and thus we propose a potential energy function V such that $\vec{F} = -\vec{\nabla}V$. From the above, we see exactly what we need, and so

$$V(\vec{r}) = -\vec{m}\cdot\vec{B}(\vec{r})$$

gives the potential energy of a point magnetic dipole placed at a position $\vec{r}$ in an external magnetic field.