MP465 – Advanced Electromagnetism

Problem Set 5

Due by 5pm on Friday, 8 May 2020

1. A centre-fed linear antenna consists of two conducting wires, each of length L, arranged horizontally such that an alternating current is fed into the (insulated) gap between them. If we take the antenna to be along the y-axis, then the current density is given by

$$\vec{J}(t,x,y,z) \;=\; \begin{cases} I_0\left(1-\frac{|y|}{L}\right)\delta(x)\delta(z)\hat{e}_y\sin\omega t & \text{for } -L\le y\le L, \\ \vec{0} & \text{otherwise,} \end{cases}$$

where I_0 is a real constant.

   (a) Find the complex electric dipole amplitude associated with this current, namely, the vector $\tilde{\vec{p}}_0$ such that $\vec{p}(t) = \mathrm{Re}[\tilde{\vec{p}}_0 e^{-i\omega t}]$.

   (b) Find the time-averaged power distribution $\mathrm{d}\bar{P}/\mathrm{d}\Omega$ as a function of the spherical angles θ and ϕ, and use it to compute the total time-averaged radiated power $\bar{P}$.

2. If we do *not* assume that the time-dependence of our sources is periodic, then the far-zone approximation for the magnetic field is

$$\vec{B}(t,\vec{r}) \;\approx\; \frac{\mu_0}{4\pi c}\,\frac{\ddot{\vec{p}}(t-r/c)\times\hat{e}_r}{r},$$

where

$$\vec{p}(t) \;=\; \int \rho(t,\vec{r})\vec{r}\,\mathrm{d}^3\vec{r}.$$

is the time-dependent electric dipole moment of the sources. (The electric field is still given by $\vec{E}\approx c\vec{B}\times\hat{e}_r$.)

   (a) Without assuming anything about the way the charge density depends on time, show from the continuity equation that the derivative of $\vec{p}(t)$ is

$$\dot{\vec{p}}(t) \;=\; \int \vec{J}(t,\vec{r})\,\mathrm{d}^3\vec{r}.$$

(b) A point charge q undergoes a constant acceleration $\vec{a} = a\hat{e}_z$. Find the total power radiated by this charge.

3. We know from lecture that when we have an inertial frame $\mathcal{S}'$ moving at a constant velocity $\vec{v}$ relative to an inertial frame $\mathcal{S}$, then the electric and magnetic fields measured in $\mathcal{S}'$ are related to the ones in $\mathcal{S}$ by

$$E'_\parallel = E_\parallel, \qquad \vec{E}'_\perp = \gamma(v)\left(\vec{E}_\perp + \vec{v}\times\vec{B}\right),$$
$$B'_\parallel = B_\parallel, \qquad \vec{B}'_\perp = \gamma(v)\left(\vec{B}_\perp - \frac{\vec{v}}{c^2}\times\vec{E}\right),$$

where $\parallel$ and $\perp$ denote, respectively, the components parallel and perpendicular to $\vec{v}$.

In a particular frame $\mathcal{S}$, we have an electromagnetic field given by

$$\vec{E} = E_0\hat{e}_y, \qquad \vec{B} = \frac{\sqrt{2}E_0}{c}\left(\hat{e}_x - \hat{e}_y\right),$$

where E_0 is a constant. Consider the frame $\mathcal{S}'$ in which the electric and magnetic fields are antiparallel to each other; if $v_x = 0$, find v_y and v_z. (Hint: if $\vec{E}'$ and $\vec{B}'$ are antiparallel, this means that there is some positive constant λ such that $\vec{B}' = -\lambda\vec{E}'/c$.)

4. The field strength and dual field strength of an electromagnetic field described by the 4-potential $A^\mu = (\Phi/c, \vec{A})$ are, respectively,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \star F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}.$$

Show that

$$\star F^{\mu\nu}F_{\mu\nu} = -\frac{4}{c}\vec{E}\cdot\vec{B},$$

and explain why this implies that $\vec{E}\cdot\vec{B}$ is Lorentz-invariant.

VECTOR CALCULUS FORMULAE

1. Cartesian coordinates (x, y, z) with constant unit direction vectors $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$

   - position vector: $\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$
   - line element: $\mathrm{d}\vec{r} = \mathrm{d}x\,\hat{e}_x + \mathrm{d}y\,\hat{e}_y + \mathrm{d}z\,\hat{e}_z$
     surface element: $\mathrm{d}\vec{\sigma} = \mathrm{d}y\,\mathrm{d}z\,\hat{e}_x + \mathrm{d}x\,\mathrm{d}z\,\hat{e}_y + \mathrm{d}x\,\mathrm{d}y\,\hat{e}_z$
     volume element: $\mathrm{d}^3\vec{r} = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$
   - gradient of a function $f(x, y, z)$:

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{e}_x + \frac{\partial f}{\partial y}\hat{e}_y + \frac{\partial f}{\partial z}\hat{e}_z$$

   - divergence of a vector $\vec{A}(x, y, z) = A_x(x, y, z)\hat{e}_x + A_y(x, y, z)\hat{e}_y + A_z(x, y, z)\hat{e}_z$:

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

   - curl of a vector $\vec{A}(x, y, z) = A_x(x, y, z)\hat{e}_x + A_y(x, y, z)\hat{e}_y + A_z(x, y, z)\hat{e}_z$:

$$\vec{\nabla} \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{e}_z$$

   - Laplacian of a function $f(x, y, z)$:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

2. Cylindrical coordinates (r, ϕ, z) with unit direction vectors $\hat{e}_r$, $\hat{e}_\phi$, $\hat{e}_z$

   - relation to Cartesian coordinates: $x = r\cos\phi$, $y = r\sin\phi$, z unchanged
   - relation to Cartesian unit vectors:

$$\left.\begin{array}{c}\hat{e}_r = \cos\phi\,\hat{e}_x + \sin\phi\,\hat{e}_y \\ \hat{e}_\phi = -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\hat{e}_x = \cos\phi\,\hat{e}_r - \sin\phi\,\hat{e}_\phi \\ \hat{e}_y = \sin\phi\,\hat{e}_r + \cos\phi\,\hat{e}_\phi\end{array}\right.$$

     with $\hat{e}_z$ the same for both systems.

- position vector: $\vec{r} = r\hat{e}_r + z\hat{e}_z$
- line element: $\mathrm{d}\vec{r} = \mathrm{d}r\,\hat{e}_r + r\mathrm{d}\phi\,\hat{e}_\phi + \mathrm{d}z\,\hat{e}_z$
  surface element: $\mathrm{d}\vec{\sigma} = r\mathrm{d}\phi\,\mathrm{d}z\,\hat{e}_r + \mathrm{d}r\,\mathrm{d}z\,\hat{e}_\phi + r\mathrm{d}r\,\mathrm{d}\phi\,\hat{e}_z$
  volume element: $\mathrm{d}^3\vec{r} = r\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z$
- gradient of a function $f(r, \phi, z)$:

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{e}_\phi + \frac{\partial f}{\partial z}\hat{e}_z$$

- divergence of a vector $\vec{A}(r, \phi, z) = A_r(r, \phi, z)\hat{e}_r + A_\phi(r, \phi, z)\hat{e}_\phi + A_z(r, \phi, z)\hat{e}_z$:

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

- curl of a vector $\vec{A}(r, \phi, z) = A_r(r, \phi, z)\hat{e}_r + A_\phi(r, \phi, z)\hat{e}_\phi + A_z(r, \phi, z)\hat{e}_z$:

$$\vec{\nabla} \times \vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{e}_\phi + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right)\hat{e}_z$$

- Laplacian of a function $f(r, \phi, z)$:

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

3. Spherical coordinates (r, θ, ϕ) with unit direction vectors $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_\phi$

- relation to Cartesian coordinates: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$
- relation to Cartesian unit vectors:

$$\left.\begin{array}{c} \hat{e}_r = \sin\theta\cos\phi\,\hat{e}_x + \sin\theta\sin\phi\,\hat{e}_y + \cos\theta\,\hat{e}_z \\ \hat{e}_\theta = \cos\theta\cos\phi\,\hat{e}_x + \cos\theta\sin\phi\,\hat{e}_y - \sin\theta\,\hat{e}_z \\ \hat{e}_\phi = -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y \end{array}\right\}$$

$$\leftrightarrow \quad \left\{\begin{array}{c} \hat{e}_x = \sin\theta\cos\phi\,\hat{e}_r + \cos\theta\cos\phi\,\hat{e}_\theta - \sin\phi\,\hat{e}_\phi \\ \hat{e}_y = \sin\theta\sin\phi\,\hat{e}_r + \cos\theta\sin\phi\,\hat{e}_\theta + \cos\phi\,\hat{e}_\phi \\ \hat{e}_z = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta \end{array}\right.$$

- position vector: $\vec{r} = r\hat{e}_r$
- line element: $\mathrm{d}\vec{r} = \mathrm{d}r\,\hat{e}_r + r\mathrm{d}\theta\,\hat{e}_\theta + r\sin\theta\mathrm{d}\phi\,\hat{e}_\phi$
  surface element: $\mathrm{d}\vec{\sigma} = r^2\sin\theta\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{e}_r + r\sin\theta\mathrm{d}r\,\mathrm{d}\phi\,\hat{e}_\theta + r\mathrm{d}r\,\mathrm{d}\theta\,\hat{e}_\phi$
  volume element: $\mathrm{d}^3\vec{r} = r^2\sin\theta\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$
- gradient of a function $f(r, \theta, \phi)$:

$$\vec{\nabla} f \;=\; \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{e}_\phi$$

- divergence of a vector $\vec{A}(r, \theta, \phi) = A_r(r, \theta, \phi)\hat{e}_r + A_\theta(r, \theta, \phi)\hat{e}_\theta + A_\phi(r, \theta, \phi)\hat{e}_\phi$:

$$\vec{\nabla}\cdot\vec{A} \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

- curl of a vector $\vec{A}(r, \theta, \phi) = A_r(r, \theta, \phi)\hat{e}_r + A_\theta(r, \theta, \phi)\hat{e}_\theta + A_\phi(r, \theta, \phi)\hat{e}_\phi$:

$$\begin{aligned} \vec{\nabla}\times\vec{A} \;=\; & \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right)\hat{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right)\hat{e}_\theta \\ & + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{e}_\phi \end{aligned}$$

- Laplacian of a function $f(r, \theta, \phi)$:

$$\nabla^2 f \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$