MP465 – Advanced Electromagnetism

Problem Set 4

Due by 5pm on Friday, 24 April 2020

Note: even though this assignment has only two problems, each of the questions will require a bit of work and so the assignment will possibly take as long as the usual four-question problem set. Please take this into account when budgeting your time.

1. In lecture, we showed that an EM plane wave travelling in the positive z-direction has the form

$$\vec{E} = |\mathcal{E}_x|\,\hat{e}_x\cos\theta + |\mathcal{E}_y|\,\hat{e}_y\cos(\theta-\delta),$$

where $\mathcal{E}_x$ and $\mathcal{E}_y$ are the complex amplitudes of the x- and y-components of the electric field, δ their phase difference and $\theta = kz - \omega t + \alpha_x$.

(a) Show that the x- and y-components of $\vec{E}$ satisfy the equation

$$\frac{E_x^2}{A^2} - \frac{2E_xE_y\cos\delta}{AB} + \frac{E_y^2}{B^2} = \sin^2\delta,$$

where $A = |\mathcal{E}_x|$ and $B = |\mathcal{E}_y|$ and explain why, if $\sin\delta \neq 0$, this describes an ellipse in the E_xE_y-plane. (This why we say that, in general, an EM plane wave is "elliptically polarised".)

(b) It is often convenient to describe the polarisation of such a wave by its "ellipticity" ε, defined as the length of its semiminor (smaller) axis divided by the length of its semimajor (greater) one. This means that $0 \le \varepsilon \le 1$, with the extreme values 0 and 1 corresponding to linear and circular polarisation respectively. Show that

$$\varepsilon = \frac{A^2 + B^2 - \sqrt{A^4 + 2A^2B^2\cos 2\delta + B^4}}{2AB\,|\sin\delta|},$$

and verify that this gives the correct values of the ellipticity for the linearly ($\delta = 0$ or π) and circularly ($A = B$ and $\delta = \pm\pi/2$) polarised cases. (**Note:** In deriving the above expression, you may assume without loss of generality that $A \ge B$.)

Note: depending on your mathematical background – specifically, your knowledge of conic sections – this problem may require a bit of research on your part on how ellipses are described in Cartesian coordinates. If you do come across any formulae that prove useful, cite your source(s); just pulling a formula out of nowhere without stating where you got it will result in not getting full marks.

2. In lecture, when we examined the reflection and transmission of an EM plane wave incident on a boundary, we did so for the case where the incident electric field vector $\vec{E}_I$ was linearly-polarised in the same plane as the three wave vectors $\vec{k}_I$, $\vec{k}_R$ and $\vec{k}_T$.

   Now consider the case where the incident electric field is linearly-polarised to be *normal* to the plane of the three wave vectors: take $z < 0$ to be a linear medium with magnetic permeability of $\mu_1 = \mu_0$ and index of refraction n_1, and $z > 0$ a linear medium with permeability $\mu_2 = \mu_0$ and index of refraction n_2. The boundary is hit by an incident electric field of the form

$$\vec{E}_I = \mathrm{Re}\left[\mathcal{E}_I e^{i(\vec{k}_\mathrm{I}\cdot\vec{r}-\omega t)}\right] \hat{e}_y,$$

   where $\mathcal{E}_I$ is the field's complex amplitude and the incident wave vector is

$$\vec{k}_I = \frac{n_1\omega}{c}\left(\sin\theta_I\,\hat{e}_x + \cos\theta_I\,\hat{e}_z\right).$$

   (a) Find the reflected and transmitted electric fields $\vec{E}_R$ and $\vec{E}_T$. (You may assume the same things we proved in lecture, namely, that $\vec{k}_I$, $\vec{k}_R$ and $\vec{k}_T$ all lie in the xz-plane, $\theta_R = \theta_I$, $n_1 \sin\theta_I = n_2 \sin\theta_T$ and all waves have the same frequency ω,)

   (b) Compute the reflection and transmission coefficients and confirm that $R + T = 1$.

   Comment: although only a pure vacuum has permeability exactly equal to μ_0, most nonferromagnetic materials have magnetic susceptibilities so small that their permeabilities can be well-approximated by μ_0, which is what we do here. As an added bonus, it also makes the calculations a bit less tedious. However, you should still be able to do this problem for arbitrary μ_1 and μ_2 (much as we did with the example in lecture).

VECTOR CALCULUS FORMULAE

1. Cartesian coordinates (x, y, z) with constant unit direction vectors $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$

   - position vector: $\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$
   - line element: $\mathrm{d}\vec{r} = \mathrm{d}x\,\hat{e}_x + \mathrm{d}y\,\hat{e}_y + \mathrm{d}z\,\hat{e}_z$
     surface element: $\mathrm{d}\vec{\sigma} = \mathrm{d}y\,\mathrm{d}z\,\hat{e}_x + \mathrm{d}x\,\mathrm{d}z\,\hat{e}_y + \mathrm{d}x\,\mathrm{d}y\,\hat{e}_z$
     volume element: $\mathrm{d}^3\vec{r} = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$
   - gradient of a function $f(x, y, z)$:

   $$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{e}_x + \frac{\partial f}{\partial y}\hat{e}_y + \frac{\partial f}{\partial z}\hat{e}_z$$

   - divergence of a vector $\vec{A}(x, y, z) = A_x(x, y, z)\hat{e}_x + A_y(x, y, z)\hat{e}_y + A_z(x, y, z)\hat{e}_z$:

   $$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

   - curl of a vector $\vec{A}(x, y, z) = A_x(x, y, z)\hat{e}_x + A_y(x, y, z)\hat{e}_y + A_z(x, y, z)\hat{e}_z$:

   $$\vec{\nabla} \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{e}_z$$

   - Laplacian of a function $f(x, y, z)$:

   $$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

2. Cylindrical coordinates (r, ϕ, z) with unit direction vectors $\hat{e}_r$, $\hat{e}_\phi$, $\hat{e}_z$

   - relation to Cartesian coordinates: $x = r\cos\phi$, $y = r\sin\phi$, z unchanged
   - relation to Cartesian unit vectors:

   $$\left.\begin{array}{c} \hat{e}_r = \cos\phi\,\hat{e}_x + \sin\phi\,\hat{e}_y \\ \hat{e}_\phi = -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} \hat{e}_x = \cos\phi\,\hat{e}_r - \sin\phi\,\hat{e}_\phi \\ \hat{e}_y = \sin\phi\,\hat{e}_r + \cos\phi\,\hat{e}_\phi \end{array}\right.$$

   with $\hat{e}_z$ the same for both systems.

- position vector: $\vec{r} = r\hat{e}_r + z\hat{e}_z$
- line element: $\mathrm{d}\vec{r} = \mathrm{d}r\,\hat{e}_r + r\mathrm{d}\phi\,\hat{e}_\phi + \mathrm{d}z\,\hat{e}_z$
  surface element: $\mathrm{d}\vec{\sigma} = r\mathrm{d}\phi\,\mathrm{d}z\,\hat{e}_r + \mathrm{d}r\,\mathrm{d}z\,\hat{e}_\phi + r\mathrm{d}r\,\mathrm{d}\phi\,\hat{e}_z$
  volume element: $\mathrm{d}^3\vec{r} = r\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z$
- gradient of a function $f(r, \phi, z)$:

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{e}_\phi + \frac{\partial f}{\partial z}\hat{e}_z$$

- divergence of a vector $\vec{A}(r, \phi, z) = A_r(r, \phi, z)\hat{e}_r + A_\phi(r, \phi, z)\hat{e}_\phi + A_z(r, \phi, z)\hat{e}_z$:

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

- curl of a vector $\vec{A}(r, \phi, z) = A_r(r, \phi, z)\hat{e}_r + A_\phi(r, \phi, z)\hat{e}_\phi + A_z(r, \phi, z)\hat{e}_z$:

$$\vec{\nabla} \times \vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{e}_\phi + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right)\hat{e}_z$$

- Laplacian of a function $f(r, \phi, z)$:

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

3. Spherical coordinates (r, θ, ϕ) with unit direction vectors $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_\phi$

- relation to Cartesian coordinates: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$
- relation to Cartesian unit vectors:

$$\left.\begin{array}{c} \hat{e}_r = \sin\theta\cos\phi\,\hat{e}_x + \sin\theta\sin\phi\,\hat{e}_y + \cos\theta\,\hat{e}_z \\ \hat{e}_\theta = \cos\theta\cos\phi\,\hat{e}_x + \cos\theta\sin\phi\,\hat{e}_y - \sin\theta\,\hat{e}_z \\ \hat{e}_\phi = -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y \end{array}\right\}$$

$$\leftrightarrow \quad \left\{\begin{array}{c} \hat{e}_x = \sin\theta\cos\phi\,\hat{e}_r + \cos\theta\cos\phi\,\hat{e}_\theta - \sin\phi\,\hat{e}_\phi \\ \hat{e}_y = \sin\theta\sin\phi\,\hat{e}_r + \cos\theta\sin\phi\,\hat{e}_\theta + \cos\phi\,\hat{e}_\phi \\ \hat{e}_z = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta \end{array}\right.$$

- position vector: $\vec{r} = r\hat{e}_r$
- line element: $\mathrm{d}\vec{r} = \mathrm{d}r\,\hat{e}_r + r\mathrm{d}\theta\,\hat{e}_\theta + r\sin\theta\mathrm{d}\phi\,\hat{e}_\phi$
  surface element: $\mathrm{d}\vec{\sigma} = r^2\sin\theta\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{e}_r + r\sin\theta\mathrm{d}r\,\mathrm{d}\phi\,\hat{e}_\theta + r\mathrm{d}r\,\mathrm{d}\theta\,\hat{e}_\phi$
  volume element: $\mathrm{d}^3\vec{r} = r^2\sin\theta\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$
- gradient of a function $f(r, \theta, \phi)$:

$$\vec{\nabla} f \;=\; \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{e}_\phi$$

- divergence of a vector $\vec{A}(r, \theta, \phi) = A_r(r, \theta, \phi)\hat{e}_r + A_\theta(r, \theta, \phi)\hat{e}_\theta + A_\phi(r, \theta, \phi)\hat{e}_\phi$:

$$\vec{\nabla}\cdot\vec{A} \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

- curl of a vector $\vec{A}(r, \theta, \phi) = A_r(r, \theta, \phi)\hat{e}_r + A_\theta(r, \theta, \phi)\hat{e}_\theta + A_\phi(r, \theta, \phi)\hat{e}_\phi$:

$$\begin{aligned} \vec{\nabla}\times\vec{A} \;=\; & \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right)\hat{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right)\hat{e}_\theta \\ & + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{e}_\phi \end{aligned}$$

- Laplacian of a function $f(r, \theta, \phi)$:

$$\nabla^2 f \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$