MP465 – Advanced Electromagnetism

Problem Set 2

Due by 5pm on Thursday, 26 March 2020

1. The dipole contribution to the scalar potential is

$$\Phi_1(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$$

where $r = |\vec{r}|$. Show from this that the dipole contribution to the electric field is thus

$$\vec{E}_1(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{3(\vec{p}\cdot\vec{r})\,\vec{r} - r^2\vec{p}}{r^5}.$$

2. A thin spherical shell of radius R has a charge density given in spherical coordinates (with the origin at the shell's centre) by $\rho(r,\theta,\phi) = \sigma_0\sin(2\phi)\delta(r-R)$, where σ_0 is a constant.

   (a) Show that the electric monopole and dipole moments of the shell are zero.

   (b) Show that the only nonzero elements of the electric quadrupole moment are Q_{xy} and Q_{yx} and calculate them.

   (c) Using your answer from (b), find the quadrupole contributions to the scalar potential and the electric field.

   (You'll find this problem easier to do if you utilise the fact that, for nonnegative integers m and n, the integral of $\sin^m\phi\cos^n\phi$ from $\phi = 0$ to 2π vanishes unless m and n are both even integers.)

3. A uniformly-charged spherical shell spinning at a constant angular frequency $\vec{\omega}$ has two remarkable properties which we'll explore in this problem and the next. First off, if the sphere has radius a and a total charge q, then its current charge density is given by

$$\vec{J}(r,\theta,\phi) = K\sin\theta\,\delta(r-a)\hat{e}_\phi$$

in a spherical coordinate system with an origin at the sphere's centre and with the z-axis along the sphere's axis of spin (i.e. $\vec{\omega} = \omega \hat{e}_z$). Here, K is a constant (called the "line current density") equal to $q\omega/4\pi a$.

The first remarkable property is that the magnetic field inside the sphere is *constant*. We will not prove this here; instead we'll support this claim by showing that the magnetic field along the z-axis is constant. Do this, with the following hints:

- Remember that in spherical coordinates, the positive z-axis is defined by $\theta = 0$ and the negative by $\theta = \pi$, and we're only considering the $0 \leq r < a$ case (i.e. inside the sphere);
- Make sure you use the fact that $\sqrt{s^2} = |s|$ (and not just s) for any real number s;
- Use the symmetry of the system to simplify any computations;
- In computing the field, you should be able to do the r' and ϕ' integrals easily, but the θ' integral may look impossible. However, it can be done if you make the change of variables $\mu = \cos\theta'$ and use the integrals

$$\begin{aligned} \int \frac{1}{(\alpha - \beta\mu)^{3/2}}\,\mathrm{d}\mu &= \frac{2}{\beta\sqrt{\alpha - \beta\mu}}, \\ \int \frac{\mu^2}{(\alpha - \beta\mu)^{3/2}}\,\mathrm{d}\mu &= \frac{2}{\beta^3}\left[\frac{\alpha^2}{\sqrt{\alpha - \beta\mu}} + 2\alpha\sqrt{\alpha - \beta\mu} - \frac{1}{3}(\alpha - \beta\mu)^{3/2}\right]. \end{aligned}$$

  (You should be able to prove both of these, but you don't have to.)

Using these hints, you should be able to find $\vec{B}(r, \theta, \phi)$ along the z-axis inside the sphere and show that it's constant.

4. Now we look *outside* the sphere, i.e. $r > a$. The second remarkable property of this system is that the magnetic field in this region is *exactly*, not approximately, that of a magnetic dipole $\vec{m}$ located at the origin. We want to support this claim by once again finding the magnetic field on the z-axis.

   (a) Using the same hints as in Problem 1, but keeping in mind we now have $r > a$, find the magnetic field along the z-axis.

(b) Calculate the shell's magnetic dipole moment and show that your answer in (a) exactly agrees with

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \vec{r})\vec{r} - r^2 \vec{m}}{r^5}.$$

when $\vec{r}$ is a point on the z-axis.

VECTOR CALCULUS FORMULAE

1. Cartesian coordinates (x, y, z) with constant unit direction vectors $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$

   - position vector: $\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$
   - line element: $\mathrm{d}\vec{r} = \mathrm{d}x\,\hat{e}_x + \mathrm{d}y\,\hat{e}_y + \mathrm{d}z\,\hat{e}_z$
     surface element: $\mathrm{d}\vec{\sigma} = \mathrm{d}y\,\mathrm{d}z\,\hat{e}_x + \mathrm{d}x\,\mathrm{d}z\,\hat{e}_y + \mathrm{d}x\,\mathrm{d}y\,\hat{e}_z$
     volume element: $\mathrm{d}^3\vec{r} = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$
   - gradient of a function $f(x, y, z)$:
     $$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{e}_x + \frac{\partial f}{\partial y}\hat{e}_y + \frac{\partial f}{\partial z}\hat{e}_z$$
   - divergence of a vector $\vec{A}(x, y, z) = A_x(x, y, z)\hat{e}_x + A_y(x, y, z)\hat{e}_y + A_z(x, y, z)\hat{e}_z$:
     $$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$
   - curl of a vector $\vec{A}(x, y, z) = A_x(x, y, z)\hat{e}_x + A_y(x, y, z)\hat{e}_y + A_z(x, y, z)\hat{e}_z$:
     $$\vec{\nabla} \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{e}_z$$
   - Laplacian of a function $f(x, y, z)$:
     $$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

2. Cylindrical coordinates (r, ϕ, z) with unit direction vectors $\hat{e}_r$, $\hat{e}_\phi$, $\hat{e}_z$

   - relation to Cartesian coordinates: $x = r\cos\phi$, $y = r\sin\phi$, z unchanged
   - relation to Cartesian unit vectors:
     $$\left.\begin{array}{c} \hat{e}_r = \cos\phi\,\hat{e}_x + \sin\phi\,\hat{e}_y \\ \hat{e}_\phi = -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y \end{array}\right\} \leftrightarrow \left\{\begin{array}{l} \hat{e}_x = \cos\phi\,\hat{e}_r - \sin\phi\,\hat{e}_\phi \\ \hat{e}_y = \sin\phi\,\hat{e}_r + \cos\phi\,\hat{e}_\phi \end{array}\right.$$
     with $\hat{e}_z$ the same for both systems.

- position vector: $\vec{r} = r\hat{e}_r + z\hat{e}_z$
- line element: $\mathrm{d}\vec{r} = \mathrm{d}r\,\hat{e}_r + r\mathrm{d}\phi\,\hat{e}_\phi + \mathrm{d}z\,\hat{e}_z$
  surface element: $\mathrm{d}\vec{\sigma} = r\mathrm{d}\phi\,\mathrm{d}z\,\hat{e}_r + \mathrm{d}r\,\mathrm{d}z\,\hat{e}_\phi + r\mathrm{d}r\,\mathrm{d}\phi\,\hat{e}_z$
  volume element: $\mathrm{d}^3\vec{r} = r\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z$
- gradient of a function $f(r,\phi,z)$:

$$\vec{\nabla} f \;=\; \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{e}_\phi + \frac{\partial f}{\partial z}\hat{e}_z$$

- divergence of a vector $\vec{A}(r,\phi,z) = A_r(r,\phi,z)\hat{e}_r + A_\phi(r,\phi,z)\hat{e}_\phi + A_z(r,\phi,z)\hat{e}_z$:

$$\vec{\nabla}\cdot\vec{A} \;=\; \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

- curl of a vector $\vec{A}(r,\phi,z) = A_r(r,\phi,z)\hat{e}_r + A_\phi(r,\phi,z)\hat{e}_\phi + A_z(r,\phi,z)\hat{e}_z$:

$$\vec{\nabla}\times\vec{A} \;=\; \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{e}_\phi + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right)\hat{e}_z$$

- Laplacian of a function $f(r,\phi,z)$:

$$\nabla^2 f \;=\; \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

3. Spherical coordinates (r, θ, ϕ) with unit direction vectors $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_\phi$

   - relation to Cartesian coordinates: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$
   - relation to Cartesian unit vectors:

$$\left.\begin{array}{c} \hat{e}_r = \sin\theta\cos\phi\,\hat{e}_x + \sin\theta\sin\phi\,\hat{e}_y + \cos\theta\,\hat{e}_z \\ \hat{e}_\theta = \cos\theta\cos\phi\,\hat{e}_x + \cos\theta\sin\phi\,\hat{e}_y - \sin\theta\,\hat{e}_z \\ \hat{e}_\phi = -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y \end{array}\right\}$$

$$\leftrightarrow \quad \left\{\begin{array}{c} \hat{e}_x = \sin\theta\cos\phi\,\hat{e}_r + \cos\theta\cos\phi\,\hat{e}_\theta - \sin\phi\,\hat{e}_\phi \\ \hat{e}_y = \sin\theta\sin\phi\,\hat{e}_r + \cos\theta\sin\phi\,\hat{e}_\theta + \cos\phi\,\hat{e}_\phi \\ \hat{e}_z = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta \end{array}\right.$$

   - position vector: $\vec{r} = r\hat{e}_r$
   - line element: $\mathrm{d}\vec{r} = \mathrm{d}r\,\hat{e}_r + r\mathrm{d}\theta\,\hat{e}_\theta + r\sin\theta\,\mathrm{d}\phi\,\hat{e}_\phi$
     surface element: $\mathrm{d}\vec{\sigma} = r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{e}_r + r\sin\theta\,\mathrm{d}r\,\mathrm{d}\phi\,\hat{e}_\theta + r\mathrm{d}r\,\mathrm{d}\theta\,\hat{e}_\phi$
     volume element: $\mathrm{d}^3\vec{r} = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$
   - gradient of a function $f(r, \theta, \phi)$:

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{e}_\phi$$

   - divergence of a vector $\vec{A}(r,\theta,\phi) = A_r(r,\theta,\phi)\hat{e}_r + A_\theta(r,\theta,\phi)\hat{e}_\theta + A_\phi(r,\theta,\phi)\hat{e}_\phi$:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

   - curl of a vector $\vec{A}(r,\theta,\phi) = A_r(r,\theta,\phi)\hat{e}_r + A_\theta(r,\theta,\phi)\hat{e}_\theta + A_\phi(r,\theta,\phi)\hat{e}_\phi$:

$$\begin{aligned}\vec{\nabla}\times\vec{A} &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right)\hat{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right)\hat{e}_\theta \\ &\quad + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{e}_\phi\end{aligned}$$

   - Laplacian of a function $f(r,\theta,\phi)$:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$