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AUTUMN REPEAT
2019–2020

MP465
Advanced Electromagnetism

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Time allowed: 2 hours

Answer **ALL** questions

All questions carry equal marks

	Yes	No	N/A
Formula and Tables book allowed (*i.e. available on request*)	✓		
Formula and Tables book required (*i.e. distributed prior to exam commencing*)	✓		
Statistics Tables and Formulae allowed (*i.e. available on request*)	✓		
Statistics Tables and Formulae required (*i.e. distributed prior to exam commencing*)		✓	
Dictionary allowed (*Supplied by the student*)			✓
Nonprogrammable calculator allowed	✓		

1. An electrostatic system has the following charge density:

$$\rho(\vec{r}) = q_1 \delta^{(3)}(\vec{r} - \vec{r}_1) + q_2 \delta^{(3)}(\vec{r} - \vec{r}_2) + q_3 \delta^{(3)}(\vec{r} - \vec{r}_3)$$

where q_i and $\vec{r}_i$ for $i = 1, 2, 3$ are constants.

   (a) Describe in words and/or pictures the particular charge configuration that gives this density.

   [5 marks]

   (b) Find the electric monopole and dipole moments if

$$q_1 = q_2 = Q,\ q_3 = -3Q,$$
$$\vec{r}_1 = a(\hat{e}_x + \hat{e}_y - \hat{e}_z),\ \vec{r}_2 = a(\hat{e}_x - \hat{e}_y + \hat{e}_z),\ \vec{r}_3 = \vec{0},$$

   where Q and a are constants, and thus determine the first two terms in the multipole expansion of the scalar potential (expressed as a function of x, y and z).

   [10 marks]

   (c) Repeat (b) for the following:

$$q_1 = Q,\ q_2 = -Q,\ q_3 = Q,$$
$$\vec{r}_1 = \frac{a}{\sqrt{3}}\hat{e}_z,\ \vec{r}_2 = \frac{a}{2}\left(\frac{1}{\sqrt{2}}\hat{e}_x + \frac{1}{\sqrt{2}}\hat{e}_y - \frac{1}{\sqrt{3}}\hat{e}_z\right), \vec{r}_3 = -\frac{a}{2}\left(\frac{1}{\sqrt{2}}\hat{e}_x + \frac{1}{\sqrt{2}}\hat{e}_y + \frac{1}{\sqrt{3}}\hat{e}_z\right).$$

   [10 marks]

2. A piece of wire is bent into a square of side length a, and a constant current of magnitude I is run through it. Let $\hat{n}$ be the unit normal vector to the plane the square is in such that when looking in the same direction as $\hat{n}$, the current flows conterclockwise.

   (a) Find the magnetic dipole moment of the square expressed in terms of $\hat{n}$.

   [10 marks]

   (b) Find the magnetic field at the centre of the square, again expressed in terms of $\hat{n}$.

   [15 marks]

3. Consider a time-dependent charge/current distribution localised near the origin. In the far-zone approximation, the magnetic and electric fields are given by

$$\vec{B}(t,\vec{r}) \approx \mathrm{Re}\left\{\frac{\mu_0\omega^2}{4\pi c}\,\frac{\vec{r}\times\tilde{\vec{p}}_0}{r^2}\,e^{-i\omega(t-r/c)}\right\},$$
$$\vec{E}(t,\vec{r}) \approx \frac{c\vec{B}(t,\vec{r})\times\vec{r}}{r},$$

where $\vec{p}(t) = \mathrm{Re}(\tilde{\vec{p}}_0 e^{-i\omega t})$ is the time-dependent electric dipole moment of the distribution.

Suppose our distribution is such that the electric dipole has constant magnitude p_0 and spins in the xz-plane with a constant angular speed ω, namely,

$$\vec{p}(t) = p_0\left(\sin\omega t\,\hat{e}_x + \cos\omega t\,\hat{e}_z\right).$$

(a) Find the complex vector $\tilde{\vec{p}}_0$ such that this can be written as $\vec{p}(t) = \mathrm{Re}(\tilde{\vec{p}}_0 e^{-i\omega t})$.

[5 marks]

(b) Show that the time-averaged power distribution of the electromagnetic radiation is, in spherical coordinates,

$$\frac{\mathrm{d}\bar{P}}{\mathrm{d}\Omega} = \frac{\mu_0\omega^4 p_0^2}{32\pi^2 c}\left(1+\sin^2\theta\sin^2\phi\right).$$

[10 marks]

(c) Find the total time-averaged power $\bar{P}$ radiated away by this dipole.

[10 marks]

4. The field strength and dual field strength of an electromagnetic field derived from the 4-potential $A^\mu = (\Phi/c, \vec{A})^{\mathrm{T}}$ are, respectively,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \star F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}.$$

For a plane wave, we know that if $\vec{k}$ is the wave vector in the direction of propagation, then $(\vec{E},\vec{B},\vec{k})$ form a right-handed triad with $\vec{B} = \vec{k}\times\vec{E}/\omega$ and $\omega = |\vec{k}|c$. Show that all of these properties may be inferred from the following two identities:

$$F_{\mu\nu}k^\nu = 0, \qquad \star F^{\mu\nu}k_\nu = 0,$$

where k^μ is the 4-vector $(\omega/c, \vec{k})^{\mathrm{T}}$.

[25 marks]

MAXWELL'S EQUATIONS

- For electric field $\vec{E}$, displacement field $\vec{D}$, magnetic field $\vec{B}$, magnetic intensity $\vec{H}$, free charge density ρ and free current density $\vec{J}$:

$$\vec{\nabla}\cdot\vec{D} = \rho, \qquad \vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t},$$
$$\vec{\nabla}\cdot\vec{B} = 0, \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$
$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

- Energy density and Poynting vector:

$$u = \frac{1}{2}\left(\vec{D}\cdot\vec{E} + \vec{H}\cdot\vec{B}\right), \qquad \vec{S} = \vec{E}\times\vec{H}.$$

VECTOR CALCULUS FORMULAE

1. Cartesian coordinates (x, y, z) with constant unit direction vectors $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$
   - position vector: $\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$
   - line element: $\mathrm{d}\vec{r} = \mathrm{d}x\,\hat{e}_x + \mathrm{d}y\,\hat{e}_y + \mathrm{d}z\,\hat{e}_z$
     surface element: $\mathrm{d}\vec{\sigma} = \mathrm{d}y\,\mathrm{d}z\,\hat{e}_x + \mathrm{d}x\,\mathrm{d}z\,\hat{e}_y + \mathrm{d}x\,\mathrm{d}y\,\hat{e}_z$
     volume element: $\mathrm{d}^3\vec{r} = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$
   - gradient of a scalar field $f(x, y, z)$:

$$\vec{\nabla}f = \frac{\partial f}{\partial x}\hat{e}_x + \frac{\partial f}{\partial y}\hat{e}_y + \frac{\partial f}{\partial z}\hat{e}_z$$

   - divergence of a vector field $\vec{A}(x, y, z) = A_x(x, y, z)\hat{e}_x + A_y(x, y, z)\hat{e}_y + A_z(x, y, z)\hat{e}_z$:

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

   - curl of a vector field $\vec{A}(x, y, z) = A_x(x, y, z)\hat{e}_x + A_y(x, y, z)\hat{e}_y + A_z(x, y, z)\hat{e}_z$:

$$\vec{\nabla}\times\vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{e}_z$$

   - Laplacian of a scalar field $f(x, y, z)$:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

2. Cylindrical coordinates (r, ϕ, z) with unit direction vectors $\hat{e}_r$, $\hat{e}_\phi$, $\hat{e}_z$

- relation to Cartesian coordinates: $x = r\cos\phi$, $y = r\sin\phi$, z unchanged
- relation to Cartesian unit vectors:

$$\left.\begin{array}{r}\hat{e}_r = \cos\phi\,\hat{e}_x + \sin\phi\,\hat{e}_y \\ \hat{e}_\phi = -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\hat{e}_x = \cos\phi\,\hat{e}_r - \sin\phi\,\hat{e}_\phi \\ \hat{e}_y = \sin\phi\,\hat{e}_r + \cos\phi\,\hat{e}_\phi\end{array}\right.$$

  with $\hat{e}_z$ the same for both systems.
- position vector: $\vec{r} = r\hat{e}_r + z\hat{e}_z$
- line element: $\mathrm{d}\vec{r} = \mathrm{d}r\,\hat{e}_r + r\mathrm{d}\phi\,\hat{e}_\phi + \mathrm{d}z\,\hat{e}_z$
  surface element: $\mathrm{d}\vec{\sigma} = r\mathrm{d}\phi\,\mathrm{d}z\,\hat{e}_r + \mathrm{d}r\,\mathrm{d}z\,\hat{e}_\phi + r\mathrm{d}r\,\mathrm{d}\phi\,\hat{e}_z$
  volume element: $\mathrm{d}^3\vec{r} = r\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z$
- gradient of a scalar field $f(r, \phi, z)$:

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{e}_\phi + \frac{\partial f}{\partial z}\hat{e}_z$$

- divergence of a vector field $\vec{A}(r,\phi,z) = A_r(r,\phi,z)\hat{e}_r + A_\phi(r,\phi,z)\hat{e}_\phi + A_z(r,\phi,z)\hat{e}_z$:

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

- curl of a vector field $\vec{A}(r,\phi,z) = A_r(r,\phi,z)\hat{e}_r + A_\phi(r,\phi,z)\hat{e}_\phi + A_z(r,\phi,z)\hat{e}_z$:

$$\vec{\nabla} \times \vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{e}_\phi + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right)\hat{e}_z$$

- Laplacian of a scalar field $f(r,\phi,z)$:

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

3. Spherical coordinates (r, θ, ϕ) with unit direction vectors $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_\phi$

   - relation to Cartesian coordinates: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$
   - relation to Cartesian unit vectors:

$$\left.\begin{array}{c} \hat{e}_r = \sin\theta\cos\phi\,\hat{e}_x + \sin\theta\sin\phi\,\hat{e}_y + \cos\theta\,\hat{e}_z \\ \hat{e}_\theta = \cos\theta\cos\phi\,\hat{e}_x + \cos\theta\sin\phi\,\hat{e}_y - \sin\theta\,\hat{e}_z \\ \hat{e}_\phi = -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} \hat{e}_x = \sin\theta\cos\phi\,\hat{e}_r + \cos\theta\cos\phi\,\hat{e}_\theta - \sin\phi\,\hat{e}_\phi \\ \hat{e}_y = \sin\theta\sin\phi\,\hat{e}_r + \cos\theta\sin\phi\,\hat{e}_\theta + \cos\phi\,\hat{e}_\phi \\ \hat{e}_z = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta \end{array}\right.$$

   - position vector: $\vec{r} = r\hat{e}_r$
   - line element: $\mathrm{d}\vec{r} = \mathrm{d}r\,\hat{e}_r + r\mathrm{d}\theta\,\hat{e}_\theta + r\sin\theta\mathrm{d}\phi\,\hat{e}_\phi$
     surface element: $\mathrm{d}\vec{\sigma} = r^2\sin\theta\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{e}_r + r\sin\theta\mathrm{d}r\,\mathrm{d}\phi\,\hat{e}_\theta + r\mathrm{d}r\,\mathrm{d}\theta\,\hat{e}_\phi$
     volume element: $\mathrm{d}^3\vec{r} = r^2\sin\theta\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$
   - gradient of a scalar field $f(r, \theta, \phi)$:

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{e}_\phi$$

   - divergence of a vector field $\vec{A}(r,\theta,\phi) = A_r(r,\theta,\phi)\hat{e}_r + A_\theta(r,\theta,\phi)\hat{e}_\theta + A_\phi(r,\theta,\phi)\hat{e}_\phi$:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

   - curl of a vector field $\vec{A}(r,\theta,\phi) = A_r(r,\theta,\phi)\hat{e}_r + A_\theta(r,\theta,\phi)\hat{e}_\theta + A_\phi(r,\theta,\phi)\hat{e}_\phi$:

$$\begin{aligned}\vec{\nabla}\times\vec{A} &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right)\hat{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right)\hat{e}_\theta \\ &\quad + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{e}_\phi\end{aligned}$$

   - Laplacian of a scalar field $f(r, \theta, \phi)$:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$