From now on we shall restrict our attention to a linear medium, in which $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{H} = \frac{1}{\mu}\mathbf{B}$. In this case

$$\begin{aligned} \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \qquad\qquad \nabla.\mathbf{B} = 0 \\ \nabla \times \mathbf{B} - \epsilon\mu\frac{\partial \mathbf{E}}{\partial t} &= \mu\mathbf{J} \qquad\qquad \nabla.\mathbf{E} = \frac{\rho}{\epsilon}, \end{aligned} \tag{22}$$

and

$$w = \frac{1}{2}(\epsilon\mathbf{E}.\mathbf{E} + \frac{1}{\mu}\mathbf{B}.\mathbf{B}) \qquad \mathbf{S} = \frac{1}{\mu}(\mathbf{E}\times\mathbf{B}).$$

5. Plane Waves and Radiation from Simple Systems

Differentiating Maxwell's equations (22) gives

$$0 = \nabla \times \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right) = \nabla(\nabla.\mathbf{E}) - \nabla^2\mathbf{E} + \frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = \frac{1}{\epsilon}\nabla\rho - \nabla^2\mathbf{E} + \epsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2} + \mu\frac{\partial \mathbf{J}}{\partial t}$$

$$\Rightarrow \qquad \nabla^2\mathbf{E} - \epsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2} = \frac{1}{\epsilon}\nabla\rho + \mu\frac{\partial\mathbf{J}}{\partial t}$$

and

$$\mu(\nabla\times\mathbf{J}) = \nabla \times \left(\nabla \times \mathbf{B} - \mu\epsilon\frac{\partial \mathbf{E}}{\partial t}\right) = \nabla(\nabla.\mathbf{B}) - \nabla^2\mathbf{B} - \mu\epsilon\frac{\partial}{\partial t}(\nabla\times\mathbf{E}) = -\nabla^2\mathbf{B} + \epsilon\mu\frac{\partial^2\mathbf{B}}{\partial t^2}$$

$$\Rightarrow \qquad \nabla^2\mathbf{B} - \epsilon\mu\frac{\partial^2\mathbf{B}}{\partial t^2} = -\mu(\nabla\times\mathbf{J}).$$

In a charge and current free region of space, $\rho = 0$ and $\mathbf{J} = 0$, Maxwell's equations imply (but are *not* equivalent to) a set of coupled, linear, homogeneous differential equations for $\mathbf{E}$ and $\mathbf{B}$,

$$\begin{aligned} \nabla^2\mathbf{E} - \epsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2} &= 0 \\ \nabla^2\mathbf{B} - \epsilon\mu\frac{\partial^2\mathbf{B}}{\partial t^2} &= 0. \end{aligned} \tag{23}$$

These equations have wave-like solutions that move with speed $v = 1/\sqrt{\mu\epsilon}$, electromagnetic waves. To investigate this we shall adopt a complex notation and define oscillating complex electric and magnetic fields,

$$\underline{\mathcal{E}}(\mathbf{x},t) = \underline{\mathcal{E}}_0 \mathrm{e}^{i(\mathbf{k}.\mathbf{x}-\omega t)} \qquad \underline{\mathcal{B}}(\mathbf{x},t) = \underline{\mathcal{B}}_0 \mathrm{e}^{i(\mathbf{k}.\mathbf{x}-\omega t)} \tag{24}$$

where $\underline{\mathcal{E}}_0$ and $\underline{\mathcal{B}}_0$ are constant complex vectors, $\mathbf{k}$ is a real vector (the wave-vector) and $\omega > 0$ (an angular frequency). This notation is a mathematical convenience, the true physical fields are just the real part of these, $\mathbf{E} = \Re(\underline{\mathcal{E}})$ and $\mathbf{B} = \Re(\underline{\mathcal{B}})$. For example if $\underline{\mathcal{E}}_0 = \mathbf{E}_0$ and $\underline{\mathcal{B}}_0 = \mathbf{B}_0$ are real vectors then

$$\mathbf{E}(\mathbf{x},t) = \Re\big(\underline{\mathcal{E}}(\mathbf{x},t)\big) = \mathbf{E}_0 \cos(\mathbf{k}.\mathbf{x} - \omega t) \quad \text{and} \quad \mathbf{B}(\mathbf{x},t) = \Re\big(\underline{\mathcal{B}}(\mathbf{x},t)\big) = \mathbf{B}_0 \cos(\mathbf{k}.\mathbf{x} - \omega t),$$

while, if $\underline{\mathcal{E}}_0 = \mathbf{E}_0 \mathrm{e}^{i\delta}$ and $\underline{\mathcal{B}}_0 = \mathbf{B}_0 e^{i\delta}$ with $\mathbf{E}_0$ and $\mathbf{B}_0$ real vectors and δ a constant phase, then

$$\mathbf{E}(\mathbf{x},t) = \Re\big(\underline{\mathcal{E}}(\mathbf{x},t)\big) = \mathbf{E}_0 \cos(\mathbf{k}.\mathbf{x} - \omega t + \delta) \quad \text{and} \quad \mathbf{B}(\mathbf{x},t) = \Re\big(\underline{\mathcal{B}}(\mathbf{x},t)\big) = \mathbf{E}_0 \cos(\mathbf{k}.\mathbf{x} - \omega t + \delta).$$

As long as we only deal with expressions that are linear in $\mathbf{E}$ and $\mathbf{B}$ with real co-efficients, such as Maxwell's equations, then we can use this complex notation and just extract the real part at the end of the calculation.

In this notation equations (23) give

$$\begin{aligned} \nabla^2 \underline{\mathcal{E}} - \epsilon\mu \frac{\partial^2 \underline{\mathcal{E}}}{\partial t^2} &= (-\mathbf{k}.\mathbf{k} + \epsilon\mu\omega^2)\underline{\mathcal{E}} = 0 \\ \nabla^2 \underline{\mathcal{B}} - \epsilon\mu \frac{\partial^2 \underline{\mathcal{B}}}{\partial t^2} &= (-\mathbf{k}.\mathbf{k} + \epsilon\mu\omega^2)\underline{\mathcal{B}} = 0 \\ \Rightarrow \quad (-\mathbf{k}.\mathbf{k} + \epsilon\mu\omega^2) = 0 \quad &\Rightarrow \quad \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}. \end{aligned}$$

These configurations correspond to waves of oscillating electric and magnetic fields with wave-length $\lambda = 2\pi/k$ and frequency $\nu = \omega/2\pi$ moving in the direction of the unit vector $\mathbf{n} = \mathbf{k}/k$ at speed $v = \omega/k = 1/\sqrt{\mu\epsilon}$.* Thus we can relate the speed of light in a medium, such as water or glass, to ϵ and μ. For most materials $\mu \approx \mu_0$

$$\frac{v}{c} = \sqrt{\frac{\epsilon_0}{\epsilon}} = \frac{1}{\sqrt{1+\chi_e}} < 1$$

so the refractive index is

$$n = \sqrt{1+\chi_e}$$

and the speed of light in the medium is related to the electric susceptibility.†

* What we have described here is a *monochromatic* electro-magnetic wave traveling through a medium — we focused on a single frequency ω. In general a wave will consist of a superposition of many frequencies, perhaps centred around a maximum intensity of a given colour, but this can be described by adding different frequencies of different intensities — again the linearity of Maxwell's equation allows us to add solutions to get more solutions.

† The electric susceptibility can be a function of frequency: in water, for example, $\chi_e \approx 80$ for static fields but this is reduced to $\chi_e \approx 0.8$ at optical frequencies giving a refractive index of $n = 1.3$.

However this is not the whole story, equations (23) follow from, but do not imply, Maxwell's equations — information was thrown away in deriving them from (22) — to get the full picture we should substitute (24) into (22):

$$\nabla \times \underline{\mathcal{E}} + \frac{\partial \underline{\mathcal{B}}}{\partial t} = 0 \quad \Rightarrow \quad i(\mathbf{k} \times \underline{\mathcal{E}}_0) = i\omega \underline{\mathcal{B}}_0 \quad \Rightarrow \quad \underline{\mathcal{B}}_0 = \frac{1}{v}(\mathbf{n} \times \underline{\mathcal{E}}_0), \tag{25}$$

$$\nabla \times \underline{\mathcal{B}} - \mu\epsilon \frac{\partial \underline{\mathcal{E}}}{\partial t} = 0 \quad \Rightarrow \quad i(\mathbf{k} \times \underline{\mathcal{B}}_0) = -i\mu\epsilon\omega \underline{\mathcal{E}}_0 \quad \Rightarrow \quad \underline{\mathcal{E}}_0 = -v(\mathbf{n} \times \underline{\mathcal{B}}_0), \tag{26}$$

$$\nabla . \underline{\mathcal{E}} = 0 \quad \Rightarrow \quad \mathbf{k}.\underline{\mathcal{E}}_0 = 0 \quad \Rightarrow \quad \mathbf{n}.\underline{\mathcal{E}}_0 = 0, \tag{27}$$

$$\nabla . \underline{\mathcal{B}} = 0 \quad \Rightarrow \quad \mathbf{k}.\underline{\mathcal{B}}_0 = 0 \quad \Rightarrow \quad \mathbf{n}.\underline{\mathcal{B}}_0 = 0. \tag{28}$$

Thus $\mathbf{n}$, $\underline{\mathcal{E}}_0$ and $\underline{\mathcal{B}}_0$ are mutually perpendicular. Let $\mathbf{n} = \mathbf{e}_3$ and introduce a right-handed orthonormal triple, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, with $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$. Then there are two independent possibilities: either $\underline{\mathcal{E}}_0$ is proportional to $\mathbf{e}_1$,

$$\underline{\mathcal{E}}_0 = \mathcal{E}_0 \mathbf{e}_1, \qquad \underline{\mathcal{B}}_0 = \mathcal{B}_0 \mathbf{e}_2 = \frac{1}{v}\mathcal{E}_0 \mathbf{e}_2,$$

or $\underline{\mathcal{E}}_0$ is proportional to $\mathbf{e}_2$,

$$\underline{\mathcal{E}}_0 = \mathcal{E}'_0 \mathbf{e}_2, \qquad \underline{\mathcal{B}}_0 = -\mathcal{B}_0 \mathbf{e}_1 = -\frac{1}{v}\mathcal{E}'_0 \mathbf{e}_1.$$

The most general wave-like solution of Maxwell's equations is a linear combination of these two possibilities,

$$\underline{\mathcal{E}}(\mathbf{x}, t) = \big(\mathcal{E}_0 \mathbf{e}_1 + \mathcal{E}'_0 \mathbf{e}_2)\big) \mathrm{e}^{ik(\mathbf{x}.\mathbf{n} - vt)}, \qquad \underline{\mathcal{B}}(\mathbf{x}, t) = \frac{1}{v}\big(-\mathcal{E}'_0 \mathbf{e}_1 + \mathcal{E}_0 \mathbf{e}_2\big) \mathrm{e}^{ik(\mathbf{x}.\mathbf{n} - vt)}.$$

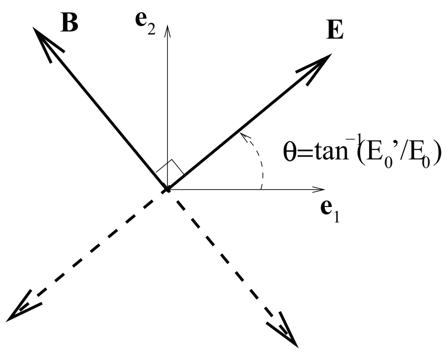
These two linearly independent possibilities are associated with the polarisation of light. If the two complex constants $\mathcal{E}_0$ and $\mathcal{E}'_0$ have the same complex phase δ, so $\mathcal{E}_0 = E_0 \mathrm{e}^{i\delta}$ and $\mathcal{E}'_0 = E'_0 \mathrm{e}^{i\delta}$ with E_0 and E'_0 real constants, and $\mathbf{n}$ is in the z-direction then $\mathbf{k}.\mathbf{n} = kz$ and the physical fields are

$$\mathbf{E}(\mathbf{x}, t) = \Re\big(\underline{\mathcal{E}}(\mathbf{x}, t)\big) = \big(E_0 \mathbf{e}_1 + E'_0 \mathbf{e}_2\big) \cos(kz - \omega t + \delta)$$

and

$$\mathbf{B}(\mathbf{x}, t) = \Re\big(\underline{\mathcal{B}}(\mathbf{x}, t)\big) = \frac{1}{v}\big(E_0 \mathbf{e}_2 - E'_0 \mathbf{e}_1\big) \cos\big(kz - \omega t + \delta\big).$$

The electric and magnetic fields therefore keep a fixed orientation in space and are at right-angles to each other, and to the direction of motion $\mathbf{n}$ of the wave, but oscillate in magnitude. This is called a *plane polarised* wave.



Plane Polarised Wave

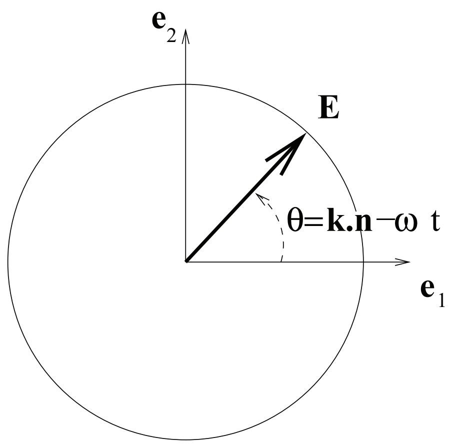
Other geometries are possible if $\underline{\mathcal{E}}_0$ and $\underline{\mathcal{E}}_0'$ have different complex phases, *e.g.* suppose $\mathcal{E}_0 = E_0$ and $\mathcal{E}_0' = iE_0'$ with E_0 and E_0' real. Then, again with $\mathbf{n}$ in the z-direction,

$$\mathbf{E}(\mathbf{x},t) = \Re\big(\underline{\mathcal{E}}(\mathbf{x},t)\big) = E_0\cos(kz-\omega t)\mathbf{e}_1 - E_0'\sin(kz-\omega t)\mathbf{e}_2$$

and

$$\mathbf{B}(\mathbf{x},t) = \Re\big(\underline{\mathcal{B}}(\mathbf{x},t)\big) = \frac{1}{v}\left\{E_0\cos(kz-\omega t)\mathbf{e}_2 + E_0'\sin(kz-\omega t)\mathbf{e}_1\right\},$$

and again $\mathbf{E}$ and $\mathbf{B}$ are always at right-angles to each other, and to $\mathbf{n}$, but this time they rotate both describing an ellipse: the wave is said to be *elliptically polarised.* If $E_0 = E_0'$ they describe a circle and the wave is *circularly polarised.* If $\underline{\mathcal{E}}_0' = -iE_0'$ the rotation is in the opposite direction (the two possible rotation directions for a circularly polarised electro-magnetic wave are called different *helicities*).



Circularly Polarised Wave

Electro-magnetic waves carry energy and we calculate the energy flux using the Poynting vector. The Poynting vector will depend on time and its average value over a cycle of oscillation is the more relevant quantity. First we must think a little about the meaning of our complex notation for quantities that are quadratic in the fields, in fact the complex notation is tailored towards calculating time-averages of quadratic quantities. To show this we shall prove a little lemma:

If $f(t) = f_0 e^{-i\omega t}$ and $g(t) = g_0 e^{-i\omega t}$, where f_0 and g_0 are independent of time t, then the time average of $\Re(f)\Re(g)$ over a complete cycle, $T = 2\pi/\omega$, is

$$\overline{fg} := \frac{1}{T}\int_0^T \Re\big(f(t)\big)\Re\big(g(t)\big)dt = \frac{1}{2}\Re(f_0^* g_0) \qquad (29)$$

where f_0^* is the complex conjugate of f_0.

To prove this let

$$f_0 = u + iv \qquad \text{and} \qquad g_0 = \zeta + i\eta$$

where u, v, ζ and η are real and independent of t. Then

$$\begin{aligned}\Re\big(f(t)\big)\Re\big(g(t)\big) &= \big(u\cos(\omega t) + v\sin(\omega t)\big)\big(\zeta\cos(\omega t) + \eta\sin(\omega t)\big)\\ &= u\zeta\cos^2(\omega t) + v\eta\cos^2(\omega t) + (u\eta + v\zeta)\cos(\omega t)\sin(\omega t)\end{aligned}$$

so

$$\int_0^{\frac{2\pi}{\omega}} \big(\Re(f)\big)\big(\Re(g)\big)dt = u\zeta\int_0^{\frac{2\pi}{\omega}} \cos^2(\omega t)dt + v\eta\int_0^{\frac{2\pi}{\omega}} \sin^2(\omega t)dt = \frac{\pi}{\omega}(u\zeta + v\eta),$$

since $\int_0^{\frac{2\pi}{\omega}} \cos(\omega t)\sin(\omega t)dt = 0$ and $\int_0^{\frac{2\pi}{\omega}} \cos^2(\omega t)dt = \int_0^{\frac{2\pi}{\omega}} \sin^2(\omega t)dt = \frac{\pi}{\omega}$. Hence the time average

$$\overline{fg} = \frac{1}{2}(u\zeta + v\eta).$$

But

$$\Re(f^* g) = \Re(f_0^* g_0) = u\zeta + v\eta,$$

which proves (29).

We can now apply this to calculate the time-average of the energy flux at a point $\mathbf{x}$ from the Poynting vector $\mathbf{S} = (\mathbf{E}\times\mathbf{B})/\mu$,

$$\overline{\mathbf{S}}(\mathbf{x}) = \frac{\omega}{2\pi\mu}\int_0^{\frac{2\pi}{\omega}} \big(\mathbf{E}(\mathbf{x},t)\times\mathbf{B}(\mathbf{x},t)\big)dt = \frac{1}{2\mu}\Re(\underline{\mathcal{E}}_0^*\times\underline{\mathcal{B}}_0) = \frac{1}{2v\mu}\underline{\mathcal{E}}_0^*.\underline{\mathcal{E}}_0\mathbf{n},$$

independent of $\mathbf{x}$ (equations (25) and (28) have been used in the last step above). This is related to the time-average of the energy density in the wave

$$\begin{aligned}\overline{w} &= \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{1}{2}\left(\epsilon\mathbf{E}(\mathbf{x},t).\mathbf{E}(\mathbf{x},t) + \frac{1}{\mu}\mathbf{B}(\mathbf{x},t).\mathbf{B}(\mathbf{x},t)\right)dt\\ &= \frac{1}{4}\left(\epsilon\underline{\mathcal{E}}_0^*.\underline{\mathcal{E}}_0 + \frac{1}{\mu}\underline{\mathcal{B}}_0^*.\underline{\mathcal{B}}_0\right) = \frac{\epsilon}{2}\underline{\mathcal{E}}_0^*.\underline{\mathcal{E}}_0.\end{aligned}$$

So, since $v = 1/\sqrt{\epsilon\mu}$,

$$\overline{\mathbf{S}} = v\,\overline{w}\,\mathbf{n},$$

a very natural result stating that the time-averaged energy-flux is in the direction $\mathbf{n}$ of the wave and has a magnitude which is just the time-averaged energy times the speed of the wave.

Electro-magnetic waves are produced by oscillating charge and current distributions and in order to describe this we shall use the potentials rather than the fields.

Vector and Scalar Potentials

Since $\nabla.\mathbf{B} = 0$ we always have $\mathbf{B} = \nabla\times\mathbf{A}$, even in the presence of matter, so

$$\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \quad\Rightarrow\quad \nabla\times\left(\mathbf{E} + \frac{\partial\mathbf{A}}{\partial t}\right) = 0.$$

Hence $\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}$ can be expressed as a gradient, $\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla\Phi$, so

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

Notice that for any twice differentiable function $\Lambda(\mathbf{r}, t)$

$$\begin{aligned}
\mathbf{A}' &:= \mathbf{A} + \nabla\Lambda & &\Rightarrow & \mathbf{B} &= \nabla \times \mathbf{A}' = \nabla \times \mathbf{A} \\
\Phi' &:= \Phi - \frac{\partial \Lambda}{\partial t} & &\Rightarrow & \mathbf{E} &= -\nabla\Phi' - \frac{\partial \mathbf{A}'}{\partial t} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}.
\end{aligned}$$

Thus Φ' and $\mathbf{A}'$ give rise to the same $\mathbf{E}$ and $\mathbf{B}$ fields as Φ and $\mathbf{A}$. The potentials Φ and $\mathbf{A}$ for an electro-magnetic field configuration are not unique, there is an ambiguity in their definition. The change

$$\begin{aligned}
\mathbf{A}' &\quad\rightarrow\quad \mathbf{A} + \nabla\Lambda \\
\Phi' &\quad\rightarrow\quad \Phi - \frac{\partial \Lambda}{\partial t}
\end{aligned} \tag{30}$$

is called a *gauge transformation*.* In the magnetostatics section we showed that $\mathbf{A}(\mathbf{r})$ arising from a given $J(\mathbf{r})$ satisfied $\nabla.\mathbf{A} = 0$, but we see now that this is not essential, if $\nabla.\mathbf{A} = 0$ then $\nabla.\mathbf{A}' \neq 0$ unless $\nabla^2\Lambda = 0$ which need not always be the case. Different choices of Λ lead to different gauges and a choice which gives $\nabla.\mathbf{A} = 0$ is called the *Coulomb gauge*, which is useful for problems in statics. For time varying fields the condition

$$\nabla.\mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$$

is often convenient, this is called the *Lorentz gauge* (obviously the Lorentz gauge reduces to the Coulomb gauge when Φ is independent of t). For any potentials $(\Phi, \mathbf{A})$ it is always possible to find a Λ so that $(\Phi', \mathbf{A}')$ satisfy the Lorentz gauge condition, since

$$\nabla.\mathbf{A}' + \frac{1}{c^2}\frac{\partial \Phi'}{\partial t} = \nabla.\mathbf{A} + \nabla^2\Lambda + \frac{1}{c^2}\frac{\partial \Phi'}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} \quad\Rightarrow\quad \nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = -\nabla.\mathbf{A} - \frac{1}{c^2}\frac{\partial \Phi'}{\partial t}.$$

The last equation here is just the inhomogeneous wave-equation for Λ, with a source $f(\mathbf{r}, t) := -\nabla.\mathbf{A} - \frac{1}{c^2}\frac{\partial \Phi'}{\partial t}$, and this equation can always be solved to find Λ so that $(\Phi', \mathbf{A}')$ satisfy the Lorentz gauge condition.

However, even the Lorentz gauge condition does not completely remove the ambiguity in $(\Phi, \mathbf{A})$, for example if $(\Phi, \mathbf{A})$ satisfy the Lorentz gauge condition then

$$\mathbf{A} \quad\rightarrow\quad \mathbf{A} + \nabla\lambda, \qquad \Phi \quad\rightarrow\quad \Phi - \frac{\partial \lambda}{\partial t}$$

* The name is historical and, from a modern perspective, is rather inappropriate, but nevertheless it has stuck.

do too, provided λ satisfies the wave equation, $-\nabla^2\lambda+\frac{1}{c^2}\frac{\partial^2\lambda}{\partial t^2}=0$. This residual ambiguity in Φ and $\mathbf{A}$ does not affect any of the following analysis.

In terms of Φ and $\mathbf{A}$ two of Maxwell's equations are automatic,

$$\mathbf{B}=\nabla\times\mathbf{A} \quad\Rightarrow\quad \nabla.\mathbf{B}=0$$
$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t} \quad\Rightarrow\quad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

so we only need worry about the equations that involve sources ρ and $\mathbf{J}$. In the vacuum with $\epsilon=\epsilon_0$ and $\mu=\mu_0$*

$$\nabla.\mathbf{E}=\frac{\rho}{\epsilon_0} \quad\Rightarrow\quad -\nabla^2\Phi-\frac{\partial(\nabla.\mathbf{A})}{\partial t}=\frac{\rho}{\epsilon_0}$$
$$\nabla\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}=\mu_0 J \quad\Rightarrow\quad \nabla(\nabla.\mathbf{A})-\nabla^2\mathbf{A}+\frac{1}{c^2}\frac{\partial(\nabla\Phi)}{\partial t}+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=\mu_0\mathbf{J}.$$

In the Lorentz gauge $\frac{1}{c^2}\frac{\partial\Phi}{\partial t}=-\nabla.\mathbf{A}$ these reduce to the inhomogeneous wave-equations

$$-\nabla^2\Phi+\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}=\frac{\rho}{\epsilon_0}, \qquad -\nabla^2\mathbf{A}+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=\mu_0\mathbf{J}.$$

In particular in a source free region of space, where $\rho=0$ and $\mathbf{J}=0$, the potentials satisfy the wave equation

$$-\nabla^2\Phi+\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}=0, \qquad -\nabla^2\mathbf{A}+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=0$$

and there will be wave-like solutions.

Radiation from Simple Systems

We shall now study the electromagnetic radiation produced by an oscillating distribution of charges and currents, using the method of Greens function. For simplicity we shall work in a vacuum and $\epsilon=\epsilon_0$ and $\mu=\mu_0$. In statics we solved

$$\nabla.\mathbf{E}=-\nabla^2\Phi=\frac{\rho}{\epsilon_0}$$
$$\nabla\times\mathbf{B}=-\nabla^2\mathbf{A}=\mu_0\mathbf{J} \qquad \text{(in the Coulomb gauge, } \nabla.\mathbf{A}=0)$$

in a volume V using Green functions which satisfy $-\nabla^2 G(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}')$. For example, if V is unbounded space, $G(\mathbf{r},\mathbf{r}')=\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$ gives

$$\Phi(\mathbf{r})=\frac{1}{\epsilon_0}\int_V \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}')dV'=\frac{1}{4\pi\epsilon_0}\int_V\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV'$$
$$\mathbf{A}(\mathbf{r})=\mu_0\int_V \mathbf{J}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')dV'=\frac{\mu_0}{4\pi}\int_V\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV'.$$

* This whole analysis works equally well in a linear medium with $\epsilon_0\to\epsilon$, $\mu_0\to\mu$ and $c\to v$ everywhere in the equations.

In a dynamical situation, using the Lorentz gauge, we must solve

$$-\nabla^2\Phi + \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = \frac{\rho}{\epsilon_0}$$
$$-\nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \mu_0\mathbf{J} \qquad (31).$$

Our strategy will again be to find suitable Green functions, but first we eliminate the time derivatives by using Fourier transforms. Define Fourier amplitudes

$$\widetilde{\Phi}(\mathbf{r},\omega) = \int_{-\infty}^{\infty}\Phi(\mathbf{r},t)e^{i\omega t}dt$$
$$\widetilde{\mathbf{A}}(\mathbf{r},\omega) = \int_{-\infty}^{\infty}\mathbf{A}(\mathbf{r},t)e^{i\omega t}dt,$$

assuming the integrals exist. Given $\widetilde{\Phi}(\mathbf{r},\omega)$ and $\widetilde{\mathbf{J}}(\mathbf{r},\omega)$ the original charge and current densities can be re-constructed using the inverse transforms

$$\Phi(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{\Phi}(\mathbf{r},\omega)e^{-i\omega t}d\omega \qquad \mathbf{A}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{\mathbf{A}}(\mathbf{r},\omega)e^{-i\omega t}d\omega.$$

Multiplying (31) by $e^{i\omega t}$, integrating over all t and equating the integrands gives

$$-\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\widetilde{\Phi} = \frac{\widetilde{\rho}}{\epsilon_0}, \qquad -\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\widetilde{\mathbf{A}} = \mu_0\widetilde{\mathbf{J}},$$

where

$$\widetilde{\rho}(\mathbf{r},\omega) = \int_{-\infty}^{\infty}\rho(\mathbf{r},t)e^{i\omega t}dt, \qquad \widetilde{\mathbf{J}}(\mathbf{r},\omega) = \int_{-\infty}^{\infty}\mathbf{J}(\mathbf{r},t)e^{i\omega t}dt,$$

are the Fourier transforms of the charge and current densities. The problem is now reduced to finding Green functions $G_k(\mathbf{r},\mathbf{r}')$ for the operator $-(\nabla^2 + k^2)$, called the *Helmholtz operator*,

$$-(\nabla^2 + k^2)G_k(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$$

where $k = \omega/c$.

If V is unbounded space we can expect, from translational invariance, that $G_k(\mathbf{r},\mathbf{r}')$ should depend only on the difference $\mathbf{R} = \mathbf{r}-\mathbf{r}'$, $G_k(\mathbf{R})$. Similarly rotational invariance implies that $G_k(\mathbf{R})$ should depend only on $R = |\mathbf{R}|$ and not on its direction, so there will be no angular dependence. Expressing ∇^2 in 3-dimensional polar co-ordinates, with the origin taken to $\mathbf{R} = 0$, we therefore have

$$\nabla^2 G_k(R) = \frac{1}{R}\left(\frac{d^2(RG_k)}{dR^2}\right) \quad\Rightarrow\quad \frac{1}{R}\left(\frac{d^2(RG_k)}{dR^2} + k^2(RG_k)\right) = \delta(R).$$

If $R \neq 0$

$$\left(\frac{d^2}{dR^2} + k^2\right)(RG_k) = 0$$

which has two linearly independent solutions which we denote by $G_k^\pm$,

$$RG_k^\pm = C_\pm e^{\pm ikR},$$

where $C_\pm$ are constants. When $k = 0$ the Helmholtz operator reduces to the Laplace operator, that we studied in the electrostatics section, with Green function $1/4\pi R$, so we can fix the normalisation

$$G_k(R) \underset{k\to 0}{\longrightarrow} G_0(R) = \frac{C_\pm}{R} = \frac{1}{4\pi R}.$$

So we choose $C_\pm = 1/4\pi$ and set*

$$G_k^\pm(R) = \frac{e^{\pm ikR}}{4\pi R} = \frac{e^{\pm i\frac{\omega}{c}R}}{4\pi R}.$$

The method of Green functions therefore leads to two linearly independent solutions in unbounded space for any given $\rho(\mathbf{r}, t$ and $\mathbf{J}(\mathbf{r}, t)$,

$$\widetilde{\Phi}^\pm(\mathbf{r}, \omega) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\widetilde{\rho}(\mathbf{r}, \omega)}{|\mathbf{r}-\mathbf{r}'|}e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}dV', \qquad \widetilde{\mathbf{A}}^\pm(\mathbf{r}, \omega) = \frac{\mu_0}{4\pi}\int_V \frac{\widetilde{\mathbf{J}}(\mathbf{r}, \omega)}{|\mathbf{r}-\mathbf{r}'|}e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}dV'.$$

These reduce to the static result when $\omega = 0$. The inverse Fourier transforms give

$$\begin{aligned}
\Phi^\pm(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_V \frac{\widetilde{\rho}(\mathbf{r}, \omega)}{|\mathbf{r}-\mathbf{r}'|}e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}dV'\right)e^{-i\omega t}d\omega \\
&= \frac{1}{4\pi\epsilon_0}\int_V \frac{1}{|\mathbf{r}-\mathbf{r}'|}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{\rho}(\mathbf{r}', \omega)e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|-i\omega t}d\omega\right)dV' \\
&= \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}', t^\pm)}{|\mathbf{r}-\mathbf{r}'|}dV', \\
\mathbf{A}^\pm(\mathbf{r}, t) &= \frac{\mu_0}{4\pi}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_V \frac{\widetilde{\mathbf{J}}(\mathbf{r}', \omega)}{|\mathbf{r}-\mathbf{r}'|}e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}dV'\right)e^{-i\omega t}d\omega \\
&= \frac{\mu_0}{4\pi}\int_V \frac{1}{|\mathbf{r}-\mathbf{r}'|}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{\mathbf{J}}(\mathbf{r}', \omega)e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|-i\omega t}d\omega\right)dV' \\
&= \frac{\mu_0}{4\pi}\int_V \frac{\widetilde{\mathbf{J}}(\mathbf{r}', t^\pm)}{|\mathbf{r}-\mathbf{r}'|}dV',
\end{aligned}$$

---

* More generally we can take any linear combination

$$G_k(R) = \frac{1}{R}\left(C_+e^{ikR} + C_-e^{-ikR}\right)$$

as a Green function, provided $C_+ + C_- = 1/4\pi$. As an exercise check, given that $\nabla^2(1/R) = -4\pi\delta(\mathbf{R})$, that $(\nabla^2 + k^2)(e^{\pm ikR}/R) = -4\pi\delta(\mathbf{R})$.

where $t^{\mp} := t \mp \frac{1}{c}|\mathbf{r}-\mathbf{r}'|$. To summarise

$$\boxed{\begin{aligned}\Phi^{\pm}(\mathbf{r},t) &= \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}',t^{\mp})}{|\mathbf{r}-\mathbf{r}'|}dV',\\ \mathbf{A}^{\pm}(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\int_V \frac{\widetilde{\mathbf{J}}(\mathbf{r}',t^{\mp})}{|\mathbf{r}-\mathbf{r}'|}dV'.\end{aligned}}$$

These formulae have a very simple physical interpretation: $\Phi^{+}(\mathbf{r},t)$ at the field point $\mathbf{r}$ depends on $\rho(\mathbf{r}',t^{-})$ at the source point $\mathbf{r}'$ not as it is at time t but as it was at time $t^{-} = t - |\mathbf{r}-\mathbf{r}'|/c$, because it takes a finite time $|\mathbf{r}-\mathbf{r}'|/c$ for information, moving at the speed of light, about the charge distribution at $\mathbf{r}'$ to reach the point $\mathbf{r}$. Φ^{+} and $\mathbf{A}^{+}$ are called *retarded* potentials, because of this time-lag. The second set of solutions, Φ^{-} and $\mathbf{J}^{-}$, correspond to the fields at $\mathbf{r}$ being influenced by what the charge and current distributions *will be* at the time $t^{+} = t + |\mathbf{r}-\mathbf{r}'|/c$ in the *future*, Φ^{-} and $\mathbf{J}^{-}$ are called *advanced* potentials. We shall restrict our attention to retarded potentials from now on.*

Multipole expansions

In principle the retarded potentials can be obtained by doing the integrals†

$$\widetilde{\Phi}(\mathbf{r},\omega) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\widetilde{\rho}(\mathbf{r}',\omega)}{|\mathbf{r}-\mathbf{r}'|}e^{ik|\mathbf{r}-\mathbf{r}'|}dV', \qquad \widetilde{\mathbf{A}}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\int_V \frac{\widetilde{\mathbf{J}}(\mathbf{r}',\omega)}{|\mathbf{r}-\mathbf{r}'|}e^{ik|\mathbf{r}-\mathbf{r}'|}dV',$$

for given ρ and $\mathbf{J}$ but, as in statics, this is often not possible analytically so we resort to a multipole approximation. We shall concentrate on a single frequency ω and consider, in complex notation, a charge and current distribution

$$\rho(\mathbf{r},t) = \widetilde{\rho}(\mathbf{r})e^{-i\omega t}, \qquad \mathbf{J}(\mathbf{r},t) = \widetilde{\mathbf{J}}(\mathbf{r})e^{-i\omega t}, \tag{32}$$

where $\widetilde{\rho}(\mathbf{r})$ and $\widetilde{\mathbf{J}}(\mathbf{r})$ are a static, possibly complex, charge and current density.‡ Their Fourier transforms are

$$\begin{aligned}\widetilde{\rho}(\mathbf{r},\omega') &= \int_{-\infty}^{\infty}\rho(\mathbf{r},t)e^{i\omega' t}dt = \int_{-\infty}^{\infty}\widetilde{\rho}(\mathbf{r})e^{i(\omega'-\omega)t}dt = 2\pi\delta(\omega-\omega')\widetilde{\rho}(\mathbf{r}),\\ \widetilde{\mathbf{J}}(\mathbf{r},\omega') &= \int_{-\infty}^{\infty}\mathbf{J}(\mathbf{r},t)e^{i\omega' t}dt = \int_{-\infty}^{\infty}\widetilde{\mathbf{J}}(\mathbf{r})e^{i(\omega'-\omega)t}dt = 2\pi\delta(\omega-\omega')\widetilde{\mathbf{J}}(\mathbf{r}),\end{aligned}$$

and for simplicity we shall omit the δ-functions and just use $\widetilde{\rho}(\mathbf{r})$ and $\widetilde{\mathbf{J}}(\mathbf{r})$ where it is understood that the angular frequency is ω. Similarly $\widetilde{\Phi}(\mathbf{r})$ and $\widetilde{\mathbf{A}}(\mathbf{r})$ are defined by

$$\widetilde{\Phi}(\mathbf{r},\omega') = 2\pi\delta(\omega-\omega')\widetilde{\Phi}(\mathbf{r}) \qquad \text{and} \qquad \widetilde{\mathbf{A}}(\mathbf{r},\omega') = 2\pi\delta(\omega-\omega')\widetilde{\mathbf{A}}(\mathbf{r})$$

---

* Advanced potentials are important in the theory of relativistic quantum mechanics, where they are related to the existence of anti-particles.

† From now on we shall only consider retarded potentials and omit the superscript $^{+}$.

‡ As before the physical charge and current densities are the real parts of these.

so

$$\widetilde{\Phi}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\widetilde{\rho}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}e^{ik|\mathbf{r}-\mathbf{r}'|}dV', \qquad \widetilde{\mathbf{A}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\widetilde{\mathbf{J}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}e^{ik|\mathbf{r}-\mathbf{r}'|}dV'. \tag{33}$$

We shall now show that, when $\omega \neq 0$, $\widetilde{\Phi}(\mathbf{r})$ and $\widetilde{\mathbf{A}}(\mathbf{r})$ are not independent — we can derive $\widetilde{\Phi}(\mathbf{r})$ from $\widetilde{\mathbf{A}}(\mathbf{r})$ using conservation of charge (17), which also follows from Maxwell's equations

$$\begin{array}{r} \nabla \times \mathbf{B} - \frac{1}{c^2}\dot{\mathbf{E}} = \mu_0 \mathbf{J} \\ \nabla.\mathbf{E} = \frac{\rho}{\epsilon_0} \end{array} \qquad \Rightarrow \qquad \nabla.\mathbf{J} = -\frac{1}{\mu_0 c^2}\nabla.\dot{\mathbf{E}} = -\dot{\rho},$$

since $\nabla.(\nabla \times \mathbf{B}) = 0$. The time dependence in (32) gives

$$\nabla.\mathbf{J} = i\omega\rho \qquad \Rightarrow \qquad \widetilde{\rho} = -\frac{i}{\omega}\nabla.\widetilde{\mathbf{J}}.$$

Hence

$$\begin{aligned}\widetilde{\Phi}(\mathbf{r},\omega) &= \frac{1}{4\pi\epsilon_0}\int_V \frac{\widetilde{\rho}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}e^{ik|\mathbf{r}-\mathbf{r}'|}dV' = -\frac{i}{4\pi\epsilon_0\omega}\int_V \frac{\left(\nabla'.\widetilde{\mathbf{J}}(\mathbf{r}')\right)}{|\mathbf{r}-\mathbf{r}'|}e^{ik|\mathbf{r}-\mathbf{r}'|}dV' \\ &= \frac{i}{4\pi\epsilon_0\omega}\int_V \widetilde{\mathbf{J}}(\mathbf{r}').\nabla'\left(\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\right)dV' = -\frac{i}{4\pi\epsilon_0\omega}\nabla.\left(\int_V \widetilde{\mathbf{J}}(\mathbf{r}')\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}dV'\right) \\ &= -\frac{ic^2}{\omega}\nabla.\mathbf{A}(\mathbf{r})\end{aligned}$$

where we have integrated by parts and assumed that there is no flux of current through the bounding surface of V. This is in fact just the Lorentz gauge condition again

$$\widetilde{\Phi}(\mathbf{r}) = -\frac{ic^2}{\omega}\nabla.\widetilde{\mathbf{A}}(\mathbf{r}) \;\Rightarrow\; \nabla.\widetilde{\mathbf{A}}(\mathbf{r}) - \frac{i\omega}{c^2}\widetilde{\Phi}(\mathbf{r}) = 0 \;\Rightarrow\; \nabla.\mathbf{A}(\mathbf{r},t) + \frac{1}{c^2}\dot{\Phi}(\mathbf{r},t) = 0.$$

The multipole expansion follows from a Taylor expansion: in Cartesian co-ordinates, x_i, expanding around $\mathbf{r}' = 0$ and using the fact that $\frac{\partial}{\partial x_i'} = -\frac{\partial}{\partial x_i}$ when acting on a function of $|\mathbf{r}-\mathbf{r}'|$,

$$
\begin{aligned}
\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} &= \frac{e^{ikr}}{r} + \sum_{i=1}^{3} x_i' \left[\frac{\partial}{\partial x_i'}\left(\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\right)\right]_{\mathbf{r}'=0} + \frac{1}{2}\sum_{i,j=1}^{3} x_i' x_j' \left[\frac{\partial^2}{\partial x_i' \partial x_j'}\left(\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\right)\right]_{\mathbf{r}'=0} + \dots \\
&= \frac{e^{ikr}}{r} - \sum_{i=1}^{3} x_i' \frac{\partial}{\partial x_i}\left(\frac{e^{ikr}}{r}\right) + \frac{1}{2}\sum_{i,j=1}^{3} x_i' x_j' \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{e^{ikr}}{r}\right) + \dots \\
&= \frac{e^{ikr}}{r} - e^{ikr}\sum_{i=1}^{3} x_i' \left(ik\frac{x_i}{r^2} - \frac{x_i}{r^3}\right) \\
&\quad + \frac{e^{ikr}}{2}\sum_{i,j=1}^{3} x_i' x_j' \left[\frac{ikx_j}{r}\left(ik\frac{x_i}{r^2} - \frac{x_i}{r^3}\right) + \delta_{ij}\left(\frac{ik}{r^2} - \frac{1}{r^3}\right) + x_i\left(\frac{3x_j}{r^5} - 2ik\frac{x_j}{r^4}\right)\right] + \dots \\
&= \frac{e^{ikr}}{r} + \frac{e^{ikr}}{r^3}(1-ikr)\sum_{i=1}^{3} x_i' x_i \\
&\qquad + \frac{e^{ikr}}{2r^5}\sum_{i,j=1}^{3}\left[x_i x_j\{3(1-ikr) - k^2r^2\} - \delta_{ij} r^2 (1-ikr)\right] x_i' x_j' + \dots
\end{aligned}
$$

Using this expansion in (33) gives

$$
\begin{aligned}
\widetilde{\Phi}(\mathbf{r}) &= \frac{e^{ikr}}{4\pi\epsilon_0}\left\{\frac{\widetilde{Q}}{r} + \frac{(1-ikr)}{r^3}(\widetilde{\mathbf{Q}}.\mathbf{r}) + \sum_{i,j=1}^{3}\frac{x_i x_j}{2r^5}\left[(1-ikr)\big(3\widetilde{q}_{ij} - \delta_{ij} Tr(\widetilde{q})\big) - k^2 r^2 \widetilde{q}_{ij}\right]\right\} + \dots \\
\widetilde{A}_i(\mathbf{r}) &= \frac{\mu_0 e^{ikr}}{4\pi}\left\{\frac{1}{r}\int_V \widetilde{J}_i(\mathbf{r}')dV' + \frac{(1-ikr)}{r^3}\sum_{j=1}^{3} x_j \left[\int_V x_j' \widetilde{J}_i(\mathbf{r}')dV'\right] + \dots\right\},
\end{aligned}
$$

where

$$
\widetilde{Q} = \int_V \widetilde{\rho}(\mathbf{r}')dV', \qquad \widetilde{\mathbf{Q}} = \int_V \mathbf{r}'\widetilde{\rho}(\mathbf{r}')dV' \qquad \text{and} \qquad \widetilde{q}_{ij} = \int_V x_i' x_j' \widetilde{\rho}(\mathbf{r}')dV'
$$

are the multipole moments and $Tr(\widetilde{q}) = \sum_{i=1}^{3} \widetilde{q}_{ii}$.* In fact conservation of charge forces $\widetilde{Q} = 0$ since

$$
\widetilde{Q} = \int_V \widetilde{\rho}(\mathbf{r}')dV' = -\frac{i}{\omega}\int_V \nabla'.\mathbf{J}(\mathbf{r}')dV' = -\frac{i}{\omega}\int_S \widetilde{\mathbf{J}}(\mathbf{r}').d\mathbf{S}' = 0
$$

if there is no flux of current through the surface S bounding V.

---

* Small $\widetilde{q}_{ij}$ is used here for the quadrupole moment because a capital Q_{ij} was used in the electrostatics section to denote the traceless part of the quadrupole moment, $\widetilde{Q}_{ij} = \frac{1}{2}\int_V (3x_i'x_j' - \delta_{ij} r'^2)\widetilde{\rho}(\mathbf{r}')dV'$.

As mentioned earlier $\widetilde{\Phi}$ and $\widetilde{\mathbf{A}}$ are not independent. From charge conservation, $\nabla.\widetilde{\mathbf{J}} = i\omega\widetilde{\rho}$, we have

$$\begin{aligned}\int_V \widetilde{J}_i(\mathbf{r}')dV' &= \sum_{j=1}^{3}\int_V \frac{\partial}{\partial x'_j}\big(x'_i\widetilde{J}_j(\mathbf{r}')\big)dV' - \int_V x'_i\big(\nabla'.\widetilde{\mathbf{J}}(\mathbf{r}')\big)dV' \\ &= -i\omega\int_V x'_i\widetilde{\rho}(\mathbf{r}')dV' = -i\omega Q_i\end{aligned}$$

where again it has been assumed that there is no flux of current through the surface bounding V, so

$$\sum_{j=1}^{3}\int_V \frac{\partial}{\partial x'_j}\big(x'_i\widetilde{J}_j(\mathbf{r}')\big)dV' = \sum_{j=1}^{3}\int_S \big(x'_i\widetilde{J}_j(\mathbf{r}')\big)dS'_j = 0$$

from the divergence theorem.

Also

$$\begin{aligned}\int_V \widetilde{x}'_j J_i(\mathbf{r}')dV' =& \sum_{k=1}^{3}\int_V \frac{\partial}{\partial x'_k}\big(x'_ix'_j\widetilde{J}_k(\mathbf{r}')\big)dV' - \int_V x'_iJ_j(\mathbf{r}')dV' - \int_V x'_ix'_j\big(\nabla'.\widetilde{\mathbf{J}}(\mathbf{r}')\big)dV' \\ =& -\int_V x'_iJ_j(\mathbf{r}')dV' - i\omega\int_V x'_ix'_j\widetilde{\rho}(\mathbf{r}')dV' = -\int_V x'_iJ_j(\mathbf{r}')dV' - i\omega\widetilde{q}_{ij} \\ \Leftrightarrow \quad \int_V \widetilde{x}'_j J_i(\mathbf{r}')dV' =& \frac{1}{2}\int_V \big(\widetilde{x}'_jJ_i(\mathbf{r}') - x'_iJ_j(\mathbf{r}')\big)dV' - \frac{i\omega}{2}\widetilde{q}_{ij}.\end{aligned}$$

The first term on the right hand side is anti-symmetric under interchange of the indices i and j and is called the magnetic dipole moment, it is equivalent to the vector

$$\widetilde{\mathbf{m}} = \frac{1}{2}\int_V \mathbf{r}' \times \widetilde{\mathbf{J}}(\mathbf{r}')dV',$$

while the second term is the electric quadrupole moment and is symmetric under interchange of i and j. Using these expressions

$$\begin{aligned}\widetilde{\Phi}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}e^{ikr}\left[\frac{(1-ikr)}{r^3}(\widetilde{\mathbf{Q}}.\mathbf{r}) + \frac{(1-ikr)}{r^5}\sum_{i,j=1}^{3}x_ix_j\left(\widetilde{Q}_{ij} - \frac{1}{2}k^2r^2\widetilde{q}_{ij}\right) + \ldots\right] \\ \widetilde{A}_i(\mathbf{r}) &= \frac{\mu_0}{4\pi}e^{ikr}\left[-\frac{i\omega}{r}\widetilde{Q}_i + \frac{(1-ikr)}{r^3}\left\{(\widetilde{\mathbf{m}}\times\mathbf{r})_i - \frac{i\omega}{2}\sum_{j=1}^{3}x_j\widetilde{q}_{ij}\right\} + \ldots\right].\end{aligned}$$

The three terms that are explicit on the right hand side of $\widetilde{\mathbf{A}}_i$ here are referred to respectively as the electric dipole term, $\widetilde{Q}_i$, the magnetic dipole term, $\widetilde{m}_i$ and the electric quadrupole term, $\widetilde{Q}_{ij}$. In a time independent situation, $\omega = 0$, $k = 0$, the electric dipole and quadrupole terms vanish leaving the familiar magnetic dipole term from statics (18). As an exercise you may wish to check that indeed $\nabla.\widetilde{\mathbf{A}}(\mathbf{r}) = i\frac{\omega}{c^2}\widetilde{\Phi}(\mathbf{r})$.

Electric Dipole Radiation

To understand how these kinds of potentials can lead to radiation we shall examine the electric dipole term as an example. So consider

$$\widetilde{\mathbf{A}}(\mathbf{r}) = -i\frac{\mu_0\omega e^{ikr}}{4\pi}\frac{\widetilde{\mathbf{Q}}}{r}.$$

Using $\nabla r = \mathbf{r}/r := \mathbf{n}$, the unit vector in the radial direction,

$$\widetilde{\mathbf{B}}(\mathbf{r}) = \nabla \times \widetilde{\mathbf{A}}(\mathbf{r}) = -i\frac{\mu_0\omega e^{ikr}}{4\pi}\left(\frac{ik\mathbf{n}}{r} - \frac{\mathbf{n}}{r^2}\right) \times \widetilde{\mathbf{Q}} = \frac{\mu_0 k^2 c}{4\pi}\frac{e^{ikr}}{r}\left(1 + \frac{i}{kr}\right)(\mathbf{n} \times \widetilde{\mathbf{Q}}).$$

The electric field can be evaluated either from $\widetilde{\mathbf{E}} = -\nabla\widetilde{\Phi} + i\omega\widetilde{\mathbf{A}}$ directly or by observing that Maxwell's equation

$$\nabla \times \mathbf{B} = \mu_0\epsilon_0\dot{\mathbf{E}} \qquad \Rightarrow \qquad \widetilde{\mathbf{E}} = \frac{ic^2}{\omega}\nabla \times \widetilde{\mathbf{B}} = \frac{ic}{k}\nabla \times \widetilde{\mathbf{B}}$$

and

$$\nabla\times\left(\frac{\mathbf{n} \times \widetilde{\mathbf{Q}}}{r}\right) = \nabla \times \left(\frac{\mathbf{r} \times \widetilde{\mathbf{Q}}}{r^2}\right) = -\frac{2\{\mathbf{n} \times (\mathbf{r} \times \widetilde{\mathbf{Q}})\}}{r^3} + \frac{1}{r^2}\nabla \times (\mathbf{r} \times \widetilde{\mathbf{Q}})$$
$$= -\frac{2\{\mathbf{n} \times (\mathbf{r} \times \widetilde{\mathbf{Q}})\}}{r^3} - \frac{(\nabla.\mathbf{r})}{r^2}\widetilde{\mathbf{Q}} + \frac{1}{r^2}(\widetilde{\mathbf{Q}}.\nabla)\mathbf{r} = -\frac{2\{\mathbf{n} \times (\mathbf{r} \times \widetilde{\mathbf{Q}})\}}{r^3} - \frac{2}{r^2}\widetilde{\mathbf{Q}} = -2\frac{\mathbf{n}(\mathbf{n}.\widetilde{\mathbf{Q}})}{r^2}$$

so

$$\widetilde{\mathbf{B}} = \frac{k^2}{4\pi\epsilon_0 c}e^{ikr}\left(1 + \frac{i}{kr}\right)\frac{(\mathbf{n} \times \widetilde{\mathbf{Q}})}{r}$$
$$\Rightarrow \quad \widetilde{\mathbf{E}} = \frac{ic}{k}\frac{k^2}{4\pi\epsilon_0 c}e^{ikr}\left[\left\{ik\left(1 + \frac{i}{kr}\right) - \frac{i}{kr^2}\right\}\frac{\mathbf{n} \times (\mathbf{n} \times \widetilde{\mathbf{Q}})}{r} - 2\left(1 + \frac{i}{kr}\right)\left(\frac{\mathbf{n}(\mathbf{n}.\widetilde{\mathbf{Q}})}{r^2}\right)\right]$$
$$= -\frac{k^2}{4\pi\epsilon_0}e^{ikr}\left[\left\{1 + \frac{i}{kr}\left(1 + \frac{i}{kr}\right)\right\}\frac{\mathbf{n} \times (\mathbf{n} \times \widetilde{\mathbf{Q}})}{r} + \frac{2i}{k}\left(1 + \frac{i}{kr}\right)\left(\frac{\mathbf{n}(\mathbf{n}.\widetilde{\mathbf{Q}})}{r^2}\right)\right]$$
$$= -\frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\left[\mathbf{n} \times (\mathbf{n} \times \widetilde{\mathbf{Q}}) + \frac{i}{kr}\left(1 + \frac{i}{kr}\right)\left\{3\mathbf{n}(\mathbf{n}.\widetilde{\mathbf{Q}}) - \widetilde{\mathbf{Q}}\right\}\right].$$

Note that $\widetilde{\mathbf{E}}.\widetilde{\mathbf{B}} = 0$. These expressions are rather involved in general and it is instructive to examine two special limits:

i) **The near zone**, $kr << 1$ so r is small,

$$\widetilde{\mathbf{B}} = \frac{ik}{4\pi\epsilon_0 c}\frac{\mathbf{n} \times \widetilde{\mathbf{Q}}}{r^2}, \qquad \widetilde{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\frac{3\mathbf{n}(\mathbf{n}.\widetilde{\mathbf{Q}}) - \widetilde{\mathbf{Q}}}{r^3},$$

where the electric field dominates.

ii) **The far zone**, $kr >> 1$ so r is large,

$$\widetilde{\mathbf{B}} = \frac{k^2}{4\pi\epsilon_0 c}\frac{e^{ikr}}{r}(\mathbf{n}\times\widetilde{\mathbf{Q}}), \qquad \widetilde{\mathbf{E}} = -c(\mathbf{n}\times\widetilde{\mathbf{B}}),$$

where the electric and magnetic fields both fall off like $1/r$. Remember that this is a multipole expansion and these expression are only accurate when r is much greater than the largest dimension of the volume containing the charges and currents.

The physical electric and magnetic fields are then the real parts

$$\mathbf{E} = \Re(\widetilde{\mathbf{E}}e^{-iwt}) \qquad \mathbf{B} = \Re(\widetilde{\mathbf{B}}e^{-iwt}).$$

The far zone is particularly important for understanding radiation a long way away from the sources where the energy flux, averaged over a cycle of period $2\pi/\omega$, is given by (29)

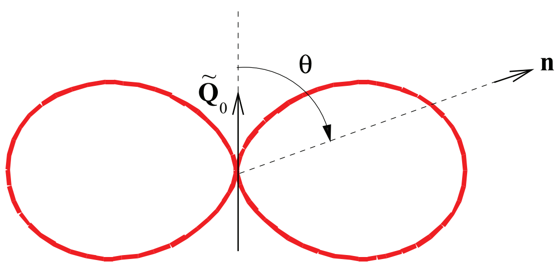
$$\begin{aligned}\bar{\mathbf{S}} &= \frac{1}{2\mu_0}\Re(\widetilde{\mathbf{E}}\times\widetilde{\mathbf{B}}^*) = \frac{1}{2\mu_0}\frac{1}{(4\pi\epsilon_0)^2}\frac{k^4}{cr^2}\{(\mathbf{n}\times\widetilde{\mathbf{Q}})\times\mathbf{n}\}\times(\mathbf{n}\times\widetilde{\mathbf{Q}}^*)\\ &= \frac{k^4 c}{2(4\pi)^2\epsilon_0}\frac{1}{r^2}\{(\widetilde{\mathbf{Q}}\times\mathbf{n}).(\widetilde{\mathbf{Q}}^*\times\mathbf{n})\}\mathbf{n}.\end{aligned}$$

The energy flux is therefore purely radial, in the $\mathbf{n}$ direction, and falls off like $1/r^2$.

Now suppose, for example, that the complex vector $\widetilde{\mathbf{Q}}$ has the same complex phase for each component, *i.e.* $\widetilde{\mathbf{Q}} = e^{i\alpha}\widetilde{\mathbf{Q}}_0$ where $\widetilde{\mathbf{Q}}_0$ is a real vector. In this particular case

$$\bar{\mathbf{S}} = \frac{k^4 c}{2(4\pi)^2\epsilon_0}\frac{\widetilde{Q}_0^2\sin^2\theta}{r^2}\mathbf{n} \tag{34}$$

where θ is the angle between $\widetilde{\mathbf{Q}}_0$ and $\mathbf{r}$ and $\widetilde{Q}_0^2 = \widetilde{\mathbf{Q}}_0.\widetilde{\mathbf{Q}}_0$. Most of the energy is radiated in the direction $\theta = \pi/2$, that is perpendicular to the direction of $\widetilde{\mathbf{Q}}_0$ and none is radiated parallel to $\widetilde{\mathbf{Q}}_0$.



The total time-averaged power radiated, $\bar{\mathcal{P}}$, is the integral of the energy flux through a sphere surrounding the dipole. Taking a sphere with large radius and using the radiation zone expressions for **E** and **B**

$$\bar{\mathcal{P}} = \frac{k^4 c\, \widetilde{Q}_0^2}{2(4\pi)^2\epsilon_0} \int_0^{2\pi} \int_0^{\pi} \left(\frac{\sin^2\theta}{r^2} \right) r^2 \sin\theta d\theta d\phi = \frac{k^4 c\, \widetilde{Q}_0^2}{2(4\pi)^2\epsilon_0} \int_0^{2\pi} \int_{-1}^{1} (1-u^2) du = \frac{k^4 c\, \widetilde{Q}_0^2}{12\pi\epsilon_0},$$

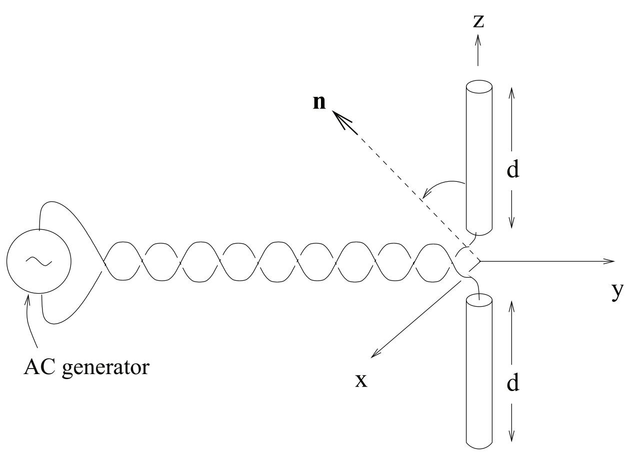
where $u = \cos\theta$. So the total power radiated through a sphere of large radius is

$$\boxed{\bar{\mathcal{P}} = \frac{\omega^4 \widetilde{Q}_0^2}{12\pi\epsilon_0 c^3} = \frac{\ddot{\widetilde{Q}}_0^2}{12\pi\epsilon_0 c^3},} \tag{35}$$

proportional to the square of the second derivative of $\widetilde{\mathbf{Q}}_0$ with respect to time.

Example: centre-fed linear antenna

A model for an antenna transmitting radio-waves is two collinear straight cylindrical rods of length d with constant circular cross-section made of some conducting material with an alternating current fed into a small gap between them (hence *centre-fed*).



We model the current as an oscillating function which decreases linearly (hence *linear*) from a maximum amplitude I_0 at the centre to zero at the end of the rods. Place the rod so as to be aligned along the z-axis with the central gap at the origin, then the physical current is the real part of

$$I(z,t) = \begin{cases} I_0 \left(1 - \frac{|z|}{d}\right) e^{-i\omega t}, & |z| \leq d \\ 0, & |z| > d. \end{cases}$$

Assuming the current density in the rods is independent of position, define J_0 by

$$I_0 = J_0 \Delta A,$$

where ΔA is the cross-sectional area of the rods. Then we define a complex current density inside the antenna

$$\mathbf{J} = \frac{I_0}{\Delta A}\left(1 - \frac{|z|}{d}\right) e^{-i\omega t}\hat{\mathbf{z}}, \qquad -d \leq z \leq d$$

while $\mathbf{J} = 0$ outside the rods. Now

$$\nabla.\mathbf{J} = \pm\frac{I_0}{\Delta A d} e^{-i\omega t}$$

and conservation of charge

$$\nabla.\mathbf{J} = -\dot{\rho}$$

then implies a charge density, $\rho(\mathbf{r}, t) = \widetilde{\rho}(\mathbf{r})e^{-i\omega t}$, with

$$\widetilde{\rho}(\mathbf{r}) = \pm\frac{iI_0}{\omega d \Delta A}$$

inside the antenna (plus for $0 < z \leq d$ and minus for $-d \leq z < 0$) while $\widetilde{\rho}$ vanishes outside the antenna. We can define a charge per unit length

$$\widetilde{\lambda} = \widetilde{\rho}\Delta A = \pm\frac{iI_0}{\omega d}$$

giving a dipole moment

$$\widetilde{Q}_z = \int_{-d}^{d} z\widetilde{\lambda}(z)dz = \frac{iI_0}{\omega d}\left(\int_0^d zdz - \int_{-d}^0 zdz\right) = \frac{2iI_0}{\omega d}\int_0^d zdz = \frac{iI_0 d}{\omega},$$

while $\widetilde{Q}_x = \widetilde{Q}_y = 0$, so

$$\widetilde{\mathbf{Q}}.\widetilde{\mathbf{Q}}^* = \widetilde{Q}_0^2 = \left(\frac{I_0 d}{\omega}\right)^2.$$

The time-averaged energy flux for $r >> d$ is now given by (34) to be

$$\bar{\mathbf{S}} = \frac{\omega^4}{2(4\pi)^2\epsilon_0 c^3}\widetilde{Q}_0^2\frac{\sin^2\theta}{r^2}\mathbf{n} = \frac{(\omega I_0 d)^2}{32\pi^2\epsilon_0 c^3}\frac{\sin^2\theta}{r^2}\mathbf{n}.$$

The time-averaged power radiated through a large sphere with the antenna at the centre and $r >> d$ is now given by (35) to be

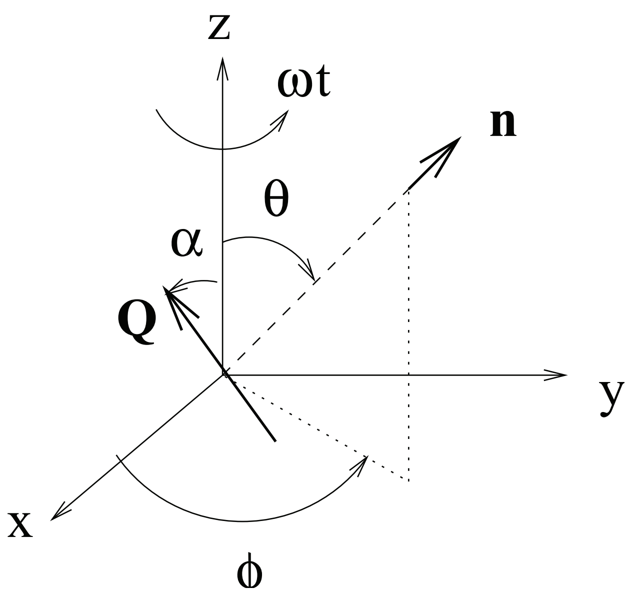
$$\bar{\mathcal{P}} = \frac{(\omega I_0 d)^2}{12\pi\epsilon_0 c^3}.$$

This is proportional to ω^2, so higher frequencies radiate more power for a given current I_0.

Example: rotating dipole

Our next example is a dipole of constant magnitude, rotating around an axis at a constant angle α to $\mathbf{Q}$. Choose the axis of rotation to be the z-axis with

$$\mathbf{Q} = Q_0 \sin\alpha\big(\cos(\omega t)\hat{\mathbf{x}} \pm \sin(\omega t)\hat{\mathbf{y}}\big) + Q_0 \cos\alpha\,\hat{\mathbf{z}} = Q_0 \sin\alpha \Re\big\{(\hat{\mathbf{x}} \mp i\hat{\mathbf{y}})e^{-i\omega t}\big\} + Q_0 \cos\alpha\,\hat{\mathbf{z}}.$$



The last term on the right hand side is independent of time and will not radiate, so we can determine the radiation by focusing on

$$\widetilde{\mathbf{Q}} = Q_0 \sin\alpha(\hat{\mathbf{x}} \mp i\hat{\mathbf{y}}).$$

Expressing the unit radial vector $\mathbf{r}/r = \mathbf{n}$ in Cartesians,

$$\mathbf{n} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}},$$

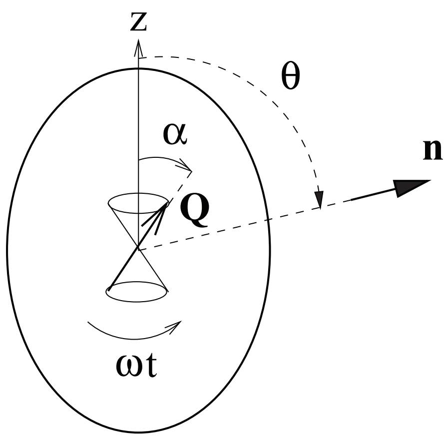
we can determine the Poynting vector from

$$\begin{aligned} &\widetilde{\mathbf{Q}} \times \mathbf{n} = Q_0 \sin\alpha\big\{(\sin\theta\sin\phi \pm i\sin\theta\cos\phi)\hat{\mathbf{z}} + (-\cos\theta)\hat{\mathbf{y}} \mp i\cos\theta\,\hat{\mathbf{x}}\big\} \\ \Rightarrow \qquad &(\widetilde{\mathbf{Q}} \times \mathbf{n}).(\widetilde{\mathbf{Q}} \times \mathbf{n})^* = (\sin^2\theta + 2\cos^2\theta)Q_0^2\sin^2\alpha = (1+\cos^2\theta)Q_0^2\sin^2\alpha \end{aligned}$$

giving

$$\bar{\mathbf{S}} = \frac{\omega^4 Q_0^2 \sin^2\alpha(1+\cos^2\theta)}{32\pi^2\epsilon_0 c^3 r^2}\mathbf{n}$$

in the radiation zone $kr >> 1$. The radiation is most intense in the direction of the axis of rotation, the z-axis when $\theta = 0$ or π, but there is still some radiation (half the intensity of that along the z-axis) in the direction perpendicular to the axis of rotation, $\theta = \pi/2$.



Time-averaged energy flux from a rotating dipole.

The time-averaged power is then

$$\bar{\mathcal{P}} = \frac{\omega^4 Q_0^2 \sin^2\alpha}{16\pi\epsilon_0 c^3}\int_0^\pi (1+\cos^2\theta)\sin^2 d\theta = \frac{\omega^4 Q_0^2 \sin^2\alpha}{16\pi\epsilon_0 c^3}\int_{-1}^1 (1+u^2)du = \frac{\omega^4 Q_0^2 \sin^2\alpha}{6\pi\epsilon_0 c^3}.$$

Thus a rotating electric dipole radiates a time-averaged power proportional to the fourth power of the frequency.

A rotating magnetic dipole with

$$\mathbf{m} = m_0(\cos\omega t\hat{\mathbf{x}} - \sin\omega t\hat{\mathbf{y}})\sin\alpha + m_0\cos\alpha\hat{\mathbf{z}},$$

so $\mathbf{m}.\mathbf{m} = m_0^2$, leads to almost the same expression, except $\epsilon_0 \to 1/\mu_0$,

$$\bar{\mathcal{P}} = \frac{\omega^4 m_0^2 \mu_0 \sin^2\alpha}{6\pi c^3}.$$

A pulsar is a rotating neutron star with a magnetic dipole that is not aligned with the axis of rotation and this expression gives the time-averaged power radiated by a pulsar in electromagnetic (radio) waves. This loss of energy makes pulsars spin down with time.

6. Relativistic Formulation of Electromagnetism

From the special theory of relativity the Lorentz transformations between two inertial co-ordinates systems* (ct, x, y, z) and (ct', x', y', z'), written in matrix form, is

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma(v) & -\gamma(v)\frac{v}{c} & 0 & 0 \\ -\gamma(v)\frac{v}{c} & \gamma(v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{36}$$

* We take the x, y and z-axis aligned with the x', y' and z'-axis respectively and the origins $(x, y, z) = (0, 0, 0)$ and $(x', y', z') = (0, 0, 0)$ co-incising at $t = t' = 0$.

where $\gamma(v) = 1/\sqrt{1-v^2/c^2}$. Equivalently, using an index notation $x^{\mu'} = (ct', x', y', z')$ and $x^{\mu} = (ct, x, y, z)$ with $\mu = 0, 1, 2, 3$,

$$x^{\mu'} = \sum_{\nu=0}^{3} L^{\mu'}{}_{\nu}(v) x^{\nu}$$

where $L^{\mu'}{}_{\nu}(v)$ are the components of the 4×4 matrix in (36). Note that, as a matrix, $L(-v) = L^{-1}(v)$. Denote four dimensional vectors (4-vectors) by $\underset{\sim}{\mathbf{U}}$, with components U^{μ} in the x^{μ} co-ordinate system and $U^{\mu'}$ in the x'^{μ} co-ordinate system so

$$U^{\mu'} = \sum_{\nu=0}^{3} L^{\mu'}{}_{\nu}(v) U^{\nu}.$$

Then an invariant "length squared" of $\underset{\sim}{\mathbf{U}}$, denoted by a dot product $\underset{\sim}{\mathbf{U}}.\underset{\sim}{\mathbf{U}}$, can be defined by first introducing a matrix

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and defining

$$U_{\mu} := \sum_{\nu=0}^{3} \eta_{\mu\nu} U^{\nu} \qquad \Rightarrow \qquad (U_0, U_1, U_2, U_3) = (-U^0, U^1, U^2, U^3).$$

Also

$$U^{\mu} = \sum_{\nu=0}^{3} \left(\eta^{-1}\right)^{\mu\nu} U_{\nu}$$

where η^{-1} is the inverse matrix to η (in fact $\eta^{-1} = \eta$ since $\eta^2 = \mathbf{1}$). With this notation

$$\underset{\sim}{\mathbf{U}}.\underset{\sim}{\mathbf{U}} := -\left(U^0\right)^2 + \left(U^1\right)^2 + \left(U^2\right)^2 + \left(U^3\right)^2 = -\left(U^0\right)^2 + \underline{\mathbf{U}}.\underline{\mathbf{U}} = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} U^{\mu} U^{\nu} = \sum_{\nu=0}^{3} U_{\nu} U^{\nu},$$

where the 3 dimensional vector (3-vector) $\underline{\mathbf{U}}$ has components (U^1, U^2, U^3) in the x^{μ} co-ordinate system and (U'^1, U'^2, U'^3) in the $x^{\mu'}$ co-ordinate system.* Note that $\underset{\sim}{\mathbf{U}}.\underset{\sim}{\mathbf{U}}$ can be positive, negative or zero depending on whether $\left(U^0\right)^2 > \underline{\mathbf{U}}.\underline{\mathbf{U}}$ (time-like vector), $\left(U^0\right)^2 < \underline{\mathbf{U}}.\underline{\mathbf{U}}$ (space-like vector) or $\left(U^0\right)^2 = \underline{\mathbf{U}}.\underline{\mathbf{U}}$. (light-like or null vector).

* Note that $\underline{\mathbf{U}}$ has no Lorentz invariant meaning, it is a different 3-vector in different reference frames. As an exercise, check that $\underset{\sim}{\mathbf{U}}.\underset{\sim}{\mathbf{U}}$ is the same in both reference frames but $\underline{\mathbf{U}}.\underline{\mathbf{U}}$ is not.

In this notation the differential form of charge conservation

$$\frac{\partial\rho}{\partial t} + \nabla.\mathbf{J} = \frac{\partial\rho}{\partial t} + \sum_{i=1}^{3} \partial_i J^i = 0,$$

where $\partial_i = \partial/\partial x^i$, can be written succinctly by defining a 4-vector $\underset{\sim}{\mathbf{J}}$, with components

$$J^\mu = (c\rho, J^1, J^2, J^3)$$

in the x^u co-ordinates, so that

$$\frac{\partial\rho}{\partial t} + \sum_{i=1}^{3} \partial_i J^i = c\left(\frac{1}{c}\frac{\partial\rho}{\partial t}\right) + \sum_{i=1}^{3} \partial_i J^i = \sum_{\mu=0}^{3} \frac{\partial J^\mu}{\partial x^\mu} = \sum_{\mu=0}^{3} \partial_\mu J^\mu = 0,$$

where $\partial_\mu = \partial/\partial x^\mu$. The 4-vector $\underset{\sim}{\mathbf{J}}$ is called the *4-current*.

Compare this with the wave equations for the potentials that follow from Maxwell's equations, with $\mu = \mu_0$, $\epsilon = \epsilon_0$ and $c^2 = 1/\epsilon_0\mu_0$ in the Lorentz gauge $\nabla.\mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0$,

$$\begin{aligned} \nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} &= -\frac{1}{\epsilon_0}\rho = -\mu_0 c^2 \rho \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} &= -\mu_0\mathbf{J}. \end{aligned}$$

Combining $c\rho$ and $\mathbf{J}$ into a 4-vector then implies that it is also natural to combine Φ/c and $\mathbf{A}$ into a 4-potential

$$A^\mu = (\Phi/c, A^1, A^2, A^3)$$

which satisfies

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\underset{\sim}{\mathbf{A}} = \sum_{\mu,\nu=0}^{3} \left(\eta^{-1}\right)^{\mu\nu} \partial_\mu\partial_\nu \underset{\sim}{\mathbf{A}} = -\mu_0\underset{\sim}{\mathbf{J}},$$

Denote by $\Box$ the second order differential operator

$$\Box = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right),$$

called the *wave operator*, or sometimes the *d'Alembertian*, then Maxwell's equations imply

$$\Box\,\underset{\sim}{\mathbf{A}} = -\mu_0\underset{\sim}{\mathbf{J}}.$$

In this notation the Lorentz gauge condition is

$$\sum_{\mu=0}^{3} \partial_\mu A^\mu = 0.$$

What about the electric and magnetic fields themselves?

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \qquad \Rightarrow \qquad E^i = -c\frac{\partial A^0}{\partial x^i} - c\frac{\partial A_i}{\partial x^0} = c\big(\partial_i A_0 - \partial_0 A_i\big)$$

(note the sign change $A^0 = -A_0$) and

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \Rightarrow \qquad B^i = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon^{ijk}\left(\frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k}\right) = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon^{ijk}\big(\partial_j A_k - \partial_k A_j\big),$$

where ϵ^{ijk} is defined to be

$$\epsilon^{ijk} = \mathbf{e}_i.(\mathbf{e}_j \times \mathbf{e}_k)$$

with $\{\mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_1\}$ a right-handed orthonormal basis.*

The 6 components of $\mathbf{E}$ and $\mathbf{B}$ can be combined into an anti-symmetric 4×4 matrix with components

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu} \tag{37}$$

with $F_{\mu\nu} = -F_{\nu\mu}$. Then $E_i/c = F_{i0}$ and $F_{jk} = \sum_{k=1}^{3} \epsilon^{ijk} B_k$ and, as a matrix,

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & B_3 & -B_2 \\ E_2/c & -B_3 & 0 & B_1 \\ E_3/c & B_2 & -B_1 & 0 \end{pmatrix}.$$

The electric and magnetic fields are different to other 3-dimensional vectors that you have met in this regard. In relativity 3-momentum $\mathbf{P}$ is combined with energy E into the the 4-momentum $\underset{\sim}{\mathbf{P}} = (E/c, \mathbf{P})$ and current density $\mathbf{J}$ is combined with the charge density ρ into the 4-current $(\rho c, \mathbf{J})$. $\mathbf{E}$ and $\mathbf{B}$ do not become 4-vectors in relativity, they are the components of the anti-symmetric matrix $F_{\mu\nu}$ which is called the *electromagnetic field tensor*. Sometimes it is convenient to 'raise' the indices on $F_{\mu\nu}$ using η^{-1} thus, using a shorthand notation $\eta^{\mu\nu} = (\eta^{-1})^{\mu\nu}$,

$$F^{\mu\nu} = \sum_{\rho,\sigma=0}^{3} \eta^{\mu\rho}\eta^{\nu\sigma} F_{\rho\sigma} = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & B_3 & -B_2 \\ -E_2/c & -B_3 & 0 & B_1 \\ -E_3/c & B_2 & -B_1 & 0 \end{pmatrix},$$

* This is shorthand way of writing the components of a vector product: there are $3^3 = 27$ different possibilities for ϵ^{ijk} but 21 of these are zero (if any two of i, j or k are the same) so i, j and k must all be different leaving 6 possibilities, $\epsilon^{123} = \epsilon^{231} = \epsilon^{312} = +1$ and $\epsilon^{213} = \epsilon^{132} = \epsilon^{321} = -1$.

or even just raise one index,

$$F^{\mu}{}_{\nu} = \sum_{\rho=0}^{3} \eta^{\mu\rho} F_{\rho\nu} = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ E_1/c & 0 & B_3 & -B_2 \\ E_2/c & -B_3 & 0 & B_1 \\ E_3/c & B_2 & -B_1 & 0 \end{pmatrix}$$

or

$$F_{\mu}{}^{\nu} = \sum_{\sigma=0}^{3} \eta^{\nu\sigma} F_{\mu\sigma} = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ -E_1/c & 0 & B_3 & -B_2 \\ -E_2/c & -B_3 & 0 & B_1 \\ -E_3/c & B_2 & -B_1 & 0. \end{pmatrix}$$

Be careful of these signs, the notation of upper and lower indices is adopted here to account for the minus signs that arise in special relativity. A zero superscript always has the opposite sign to a zero subscript but there is no practical difference between an upper 1, 2, or 3 or a lower 1,2 or 3.

Maxwell's equations are now seen to be related to

$$\sum_{\mu=0}^{3} \partial_\mu F^{\mu\nu} = \sum_{\mu=0}^{3} \partial_\mu \big(\partial^\mu A^\nu\big) - \sum_{\mu=0}^{3} \partial_\mu \big(\partial^\nu A^\mu\big) = \Box A^\nu - \partial^\nu \left(\sum_{\mu=0}^{3} \big(\partial_\mu A^\mu\big) \right) = \Box A^\nu = -\mu_0 J^\nu$$

(in the Lorentz gauge, note the sign change $\partial^\mu = \sum_{\nu=0}^{3} \eta^{\mu\nu} \partial_\nu$ so $\partial^\mu = (-\partial_0, \partial_1, \partial_2, \partial_3)$). So the two Maxwell's equations involving sources

$$\nabla \times \mathbf{B} + \frac{1}{c^2}\dot{\mathbf{E}} = \mu_0 \mathbf{J}, \qquad \nabla.\mathbf{E} = \frac{\rho}{\epsilon_0}$$

are combined in a relativistic formulation into

$$\boxed{\sum_{\mu=0}^{3} \partial_\mu F^{\mu\nu} = -\mu_0 J^\nu}$$

(4 equations, one for each value of ν).

What about the other Maxwell's equations

$$\nabla \times \mathbf{E} - \frac{\partial \mathbf{B}}{\partial t} = 0, \qquad \nabla.\mathbf{B} = 0 \ ?$$

Consider the combination

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = \frac{1}{2}\big(\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} - \partial_\mu F_{\rho\nu} - \partial_\nu F_{\mu\rho} - \partial_\rho F_{\nu\mu}\big)$$

with μ, ν and ρ all different. There are $4 \times 3 \times 2 = 24$ possibilities, but only 4 of these are independent because, up to a sign, it does not matter what order the three indices are put in. With the choice $\mu = 1$, $\nu = 2$ and $\rho = 3$ this is

$$\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = \nabla.\mathbf{B},$$

with $\mu = 0$, $\nu = 1$ and $\rho = 2$ it is

$$\frac{1}{c}\frac{\partial B_3}{\partial t} + \partial_1\left(\frac{E_2}{c}\right) + \partial_2\left(\frac{-E_1}{c}\right) = \frac{1}{c}(\nabla \times \mathbf{E})_3 + \frac{1}{c}\frac{\partial B_3}{\partial t},$$

with $\mu = 0$, $\nu = 2$ and $\rho = 3$ it is

$$\frac{1}{c}\frac{\partial B_1}{\partial t} + \partial_2\left(\frac{E_3}{c}\right) + \partial_3\left(\frac{-E_2}{c}\right) = \frac{1}{c}(\nabla \times \mathbf{E})_1 + \frac{1}{c}\frac{\partial B_1}{\partial t},$$

with $\mu = 0$, $\nu = 3$ and $\rho = 1$ it is

$$\frac{1}{c}\frac{\partial B_2}{\partial t} + \partial_3\left(\frac{E_1}{c}\right) + \partial_1\left(\frac{-E_3}{c}\right) = \frac{1}{c}(\nabla \times \mathbf{E})_2 + \frac{1}{c}\frac{\partial B_2}{\partial t}.$$

Introducing the shorthand notation

$$\partial_{[\mu}F_{\nu\rho]} := \frac{1}{3!}\left(\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} - \partial_\mu F_{\rho\nu} - \partial_\nu F_{\mu\rho} - \partial_\rho F_{\nu\mu}\right)$$

when μ, ν and ρ are all different* we have

$$\nabla \times \mathbf{E} - \frac{\partial \mathbf{B}}{\partial t} = 0, \quad \nabla.\mathbf{B} = 0 \qquad \Leftrightarrow \qquad \partial_{[\mu}F_{\nu\rho]} = 0.$$

In fact

$$\partial_{[\mu}F_{\nu\rho]} = 0$$

is an automatic consequence of the fact that $F_{\mu\nu}$ can be derived from the potential A_μ, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, provided only that A_μ is at least twice differentiable.

In summary, Maxwell's equations can be written in a relativistic formulation as

$$\boxed{\begin{aligned} \sum_{\mu=0}^{3} \partial_\mu F^{\mu\nu} &= -\mu_0 J^\nu \\ \partial_{[\mu}F_{\nu\rho]} &= 0, \end{aligned}}$$

with $J^\mu = (c\rho, \mathbf{J})$.

* The notation $[\mu\nu\rho]$ indicates that the three indices appear with all six possible permutations, with a plus sign for the three even permutations of the indices (*i.e* $\mu\nu\rho$, $\nu\rho\mu$ and $\rho\mu\nu$) and a minus sign for the three odd permutations (*i.e* $\mu\rho\nu$, $\nu\mu\rho$ and $\rho\nu\mu$). Such a linear combination is said to be *anti-symmetrised* under permutations.

Gauge invariance.

In relativistic notation the gauge transformation (30) can be written

$$A'_\mu = A_\mu + \partial_\mu \Lambda$$

where $\Lambda(x^\mu)$ is a differentiable function. Then the components of the electromagnetic field tensor are invariant

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu A_\nu + \partial_\mu \partial_\nu \Lambda - \partial_\nu A_\mu - \partial_\nu \partial_\mu \Lambda = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}.$$

This is like a 4-dimensional version of the 3-dimensional analysis for $\mathbf{B}$,

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{and} \quad \mathbf{A}' = \mathbf{A} + \nabla\Lambda \qquad \Rightarrow \qquad \mathbf{B}' = \mathbf{B} \quad \text{since} \quad \nabla \times \nabla\Lambda = 0.$$

Indeed $F_{\mu\nu}$ is like a 4-dimensional 'curl' of A_μ.

Lorentz transformations

In this section we shall discuss how $\mathbf{E}$ and $\mathbf{B}$ transform under Lorentz transformations. To simplify notation let $\beta = v/c$ and $\gamma(\beta) = 1/\sqrt{1-\beta^2}$. Then

$$x^{\mu'} = \sum_{\nu=0}^{3} L^{\mu'}{}_\nu(\beta) x^\nu$$

with

$$L^{\mu'}{}_\nu(\beta) = \begin{pmatrix} \gamma(\beta) & -\beta\gamma(\beta) & 0 & 0 \\ -\beta\gamma(\beta) & \gamma(\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Similarly J^μ are the components of a 4-vector so they transform as

$$J^{\mu'} = \sum_{\nu=0}^{3} L^{\mu'}{}_\nu(\beta) J^\nu$$

and A^μ are the components of a 4-vector so they transform as

$$A^{\mu'} = \sum_{\nu=0}^{3} L^{\mu'}{}_\nu(\beta) A^\nu$$

The 'divergence' of the 4-current

$$\sum_{\mu=0}^{3} \partial_\mu J^\mu = \underset{\sim}{\partial} . \underset{\sim}{\mathbf{J}} = 0$$

is a scalar, not a vector, and so should be *invariant*,

$$\sum_{\mu=0}^{3} \partial_\mu J^\mu = \sum_{\mu'=0}^{3} \partial_{\mu'} J^{\mu'} = 0,$$

charge is conserved in all reference frames. This dictates how ∂_μ should transform under Lorentz transformations, suppose

$$\partial_{\mu'} = \sum_{\rho=0}^{3} M^\rho{}_{\mu'}(\beta)\partial_\rho$$

for some $M^\rho{}_{\mu'}(\beta)$ then

$$\sum_{\mu'=0}^{3} \partial_{\mu'} J^{\mu'} = \sum_{\mu'=0}^{3} \left(\sum_{\rho=0}^{3} M^\rho{}_{\mu'} \partial_\rho \right) \left(\sum_{\nu=0}^{3} L^{\mu'}{}_\nu J^\nu \right) = \sum_{\nu,\rho=0}^{3} \left\{ \sum_{\mu'=0}^{3} \left(M^\rho{}_{\mu'} L^{\mu'}{}_\nu \right) \partial_\rho J^\nu \right\}$$

$$= \sum_{\nu,\rho=0}^{3} (ML)^\rho{}_\nu \partial_\rho J^\nu = \sum_{\nu=0}^{3} \partial_\nu J^\nu$$

where ML is the product of the two matrices. This can only be true for any $\underset{\sim}{\mathbf{J}}$ if ML is the identity matrix, in components $(ML)^\rho{}_\nu = \delta^\rho{}_\nu$, so $M(\beta) = L^{-1}(\beta) = L(-\beta)$. Hence

$$J^{\mu'} = \sum_{\nu=0}^{3} L^{\mu'}{}_\nu(\beta) J^\nu, \qquad \partial_{\mu'} = \sum_{\nu=0}^{3} (L^{-1})^\nu{}_{\mu'}(\beta)\partial_\nu.$$

Indeed any vector with the index as a sub-script must transform with L^{-1}, *e.g* $J_\mu = \sum_{\nu=0}^{3} \eta_{\mu\nu} J^\nu$ transforms as

$$J_{\mu'} = \sum_{\nu=0}^{3} (L^{-1})^\nu{}_{\mu'} J_\nu$$

under Lorentz transformations. Vectors that transform with L are called *contra-variant* vectors (they have sub-scripts) while vectors that transform with L^{-1} are called *co-variant* vectors (they have super-scripts). The difference again amounts to some sign differences, since $L^{-1}(\beta) = L(-\beta)$.

We can now determine how $F_{\mu\nu}$, and hence $\mathbf{E}$ and $\mathbf{B}$, transform. Since

$$A_{\mu'} = \sum_{\nu=0}^{3} \left(L^{-1}\right)^\nu{}_{\mu'} A_\nu \qquad \text{and} \qquad \partial_{\mu'} = \sum_{\nu=0}^{3} \left(L^{-1}\right)^\nu{}_{\mu'} \partial_\nu,$$

we have

$$F_{\mu'\nu'} = \partial_{\mu'} A_{\nu'} - \partial_{\nu'} A_{\mu'} = \sum_{\rho,\sigma=0}^{3} \left(L^{-1}\right)^\rho{}_{\mu'} \left(L^{-1}\right)^\sigma{}_{\mu'} \left(\partial_\rho A_\sigma - \partial_\sigma A_\rho\right)$$

$$= \sum_{\rho,\sigma=0}^{3} \left(L^{-1}\right)^\rho{}_{\mu'} \left(L^{-1}\right)^\sigma{}_{\mu'} F_{\rho\sigma.}.$$

This can be re-written using the usual rules of matrix multiplication and the fact that L is a symmetric matrix $\left(L^{-1}\right)^T = L^{-1}$, in components $\left(L^{-1}\right)^{\rho}{}_{\mu'} = \left(L^{-1}\right)_{\mu'}{}^{\rho}$, so

$$F_{\mu'\nu'} = \sum_{\rho,\sigma=0}^{3} \left(L^{-1}\right)^{\rho}{}_{\mu'} \left(L^{-1}\right)^{\sigma}{}_{\mu'} F_{\rho\sigma} = \sum_{\rho,\sigma=0}^{3} \left(L^{-1}\right)_{\mu'}{}^{\rho} F_{\rho\sigma} \left(L^{-1}\right)^{\sigma}{}_{\nu'}$$

or, in matrix notation,

$$F' = L^{-1} F L^{-1} \qquad \Leftrightarrow \qquad F = L F' L \tag{38}$$

where F is the co-variant matrix with components $F_{\mu\nu}$ and F' is the matrix with components $F_{\mu'\nu'}$.

As an illustration of (38) consider a point charge Q at rest at the origin of the $x^{\mu'}$ co-ordinate system. The electric and magnetic fields in the primed frame, with components $E_{i'}$ and $B_{i'}$, are

$$\mathbf{E}' = \frac{Q}{4\pi\epsilon_0} \frac{\mathbf{r}'}{(r')^2}, \qquad \mathbf{B} = 0 \tag{39}$$

so

$$F' = \begin{pmatrix} 0 & -E_{1'}/c & -E_{2'}/c & -E_{3'}/c \\ E_{1'}/c & 0 & 0 & 0 \\ E_{2'}/c & 0 & 0 & 0 \\ E_{3'}/c & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned}
F &= L F' L \\
&= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_{1'}/c & -E_{2'}/c & -E_{3'}/c \\ E_{1'}/c & 0 & 0 & 0 \\ E_{2'}/c & 0 & 0 & 0 \\ E_{3'}/c & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \frac{1}{c} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta\gamma E_{1'} & -\gamma E_{1'} & -E_{2'} & -E_{3'} \\ \gamma E_{1'} & -\beta\gamma E_{1'} & 0 & 0 \\ \gamma E_{2'} & -\beta\gamma E_{2'} & 0 & 0 \\ \gamma E_{3'} & -\beta\gamma E_{3'} & 0 & 0 \end{pmatrix} \\
&= \frac{1}{c} \begin{pmatrix} 0 & -(1-\beta^2)\gamma^2 E_{1'} & -\gamma E_{2'} & -\gamma E_{3'} \\ (1-\beta^2)\gamma^2 E_{1'} & 0 & \beta\gamma E_{2'} & \beta\gamma E_{3'} \\ \gamma E_{2'} & -\beta\gamma E_{2'} & 0 & 0 \\ \gamma E_{3'} & -\beta\gamma E_{3'} & 0 & 0 \end{pmatrix} \\
&= \frac{1}{c} \begin{pmatrix} 0 & -E_{1'} & -\gamma E_{1'} & -\gamma E_{2'} \\ E_{1'} & 0 & \beta\gamma E_{2'} & \beta\gamma E_{3'} \\ \gamma E_{2'} & -\beta\gamma E_{2'} & 0 & 0 \\ \gamma E_{3'} & -\beta\gamma E_{3'} & 0 & 0 \end{pmatrix}.
\end{aligned}$$

From this we can read off the components $\mathbf{E}$ and $\mathbf{B}$ in the unprimed frame and express them in terms of unprimed co-ordinates using the Lorentz transformation

$$x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z$$

,

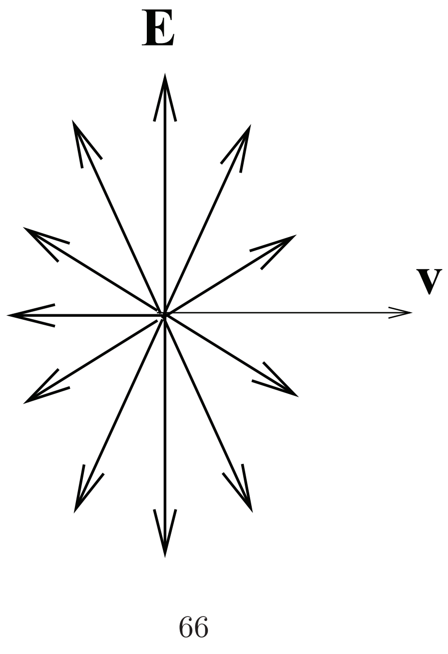
$$\begin{aligned}
E_1 = E_{1'} &= \frac{Q}{4\pi\epsilon_0}\frac{x'}{(r')^3} = \frac{Q}{4\pi\epsilon_0}\frac{\gamma(x-vt)}{\{\gamma^2(x-vt)^2+y^2+z^2\}^{3/2}} \\
E_2 = \gamma E_{2'} &= \frac{Q}{4\pi\epsilon_0}\frac{\gamma\, y'}{(r')^3} = \frac{Q}{4\pi\epsilon_0}\frac{\gamma y}{\{\gamma^2(x-vt)^2+y^2+z^2\}^{3/2}} \\
E_3 = \gamma E_{3'} &= \frac{Q}{4\pi\epsilon_0}\frac{\gamma\, z'}{(r')^3} = \frac{Q}{4\pi\epsilon_0}\frac{\gamma z}{\{\gamma^2(x-vt)^2+y^2+z^2\}^{3/2}}
\end{aligned}$$

$$\begin{aligned}
B_1 &= 0 \\
B_2 &= -\beta\gamma E_{3'} = -\frac{Qv}{4\pi\epsilon_0 c}\frac{\gamma z}{\{\gamma^2(x-vt)^2+y^2+z^2\}^{3/2}} \\
B_3 &= \beta\gamma E_{2'} = \frac{Qv}{4\pi\epsilon_0 c}\frac{\gamma y}{\{\gamma^2(x-vt)^2+y^2+z^2\}^{3/2}}.
\end{aligned}$$

Since Q is moving with velocity $\mathbf{v} = v\hat{\mathbf{x}}$ in the unprimed frame these can be more concisely written as

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0}\frac{\gamma(\mathbf{r}-\mathbf{v}t)}{\{\gamma^2(x-vt)^2+y^2+z^2\}^{3/2}}, \qquad \mathbf{B} = \frac{Q}{4\pi\epsilon_0 c}\frac{\gamma(\mathbf{v}\times\mathbf{r})}{\{\gamma^2(x-vt)^2+y^2+z^2\}^{3/2}}. \tag{40}$$

At $t = 0$ the electric field is reduced in the x-direction by a factor $1/\gamma^2$ relative to the usual spherically symmetric Coulomb field of a stationary charge and increased in the $y-z$ plane by a factor of γ,



and this picture moves to the right with constant speed v. There is a non-zero magnetic field in the unprimed frame, because Q is moving in that frame and therefore generating an electric current, which is everywhere perpendicular to $\mathbf{E}$ since $\mathbf{E}.\mathbf{B} = 0$.

Lorentz co-variance of Maxwell's equations.

Maxwell's equations are symmetric under Lorentz transformations, indeed this is how Lorentz transformations were first discovered, but nevertheless $\mathbf{E}$ and $\mathbf{B}$, and so $F_{\mu\nu}$, change — they are not invariant. Maxwell's equations are said to be *co-variant* under Lorentz transformations because their from is preserved even though the individual components change. To see what this means consider the relativistic form of Maxwell's equations in the unprimed frame

$$\sum_{\mu=0}^{3} \partial_\mu F^{\mu\nu} = -\mu_0 J^\nu, \qquad \partial_{[\mu} F_{\nu\rho]} = 0.$$

In the primed frame

$$\partial_{\mu'} = \sum_{\nu=0}^{3} \left(L^{-1}\right)^{\nu}{}_{\mu'} \partial_\nu, \qquad J^{\mu'} = \sum_{\nu=0}^{3} L^{\mu'}{}_{\nu} J^\nu, \qquad \text{and} \qquad F^{\mu'\nu'} = \sum_{\rho,\sigma=0}^{3} L^{\mu'}{}_{\rho} L^{\nu'}{}_{\sigma} F^{\rho\sigma}$$

so

$$\sum_{\mu'=0}^{3} \partial_{\mu'} F^{\mu'\nu'} = \sum_{\mu,\sigma=0}^{3} L^{\nu'}{}_{\sigma} \partial_\mu F^{\mu\sigma} = -\mu_0 \sum_{\sigma=0}^{3} L^{\nu'}{}_{\sigma} J^\sigma = -\mu_0 J^{\nu'}$$

(a factor of L has canceled a factor L^{-1} in the first equation here) and

$$\begin{aligned}\partial_{[\mu'} F_{\nu'\rho']} &= \sum_{\tau,\sigma,\rho=0}^{3} \left(L^{-1}\right)^{\tau}{}_{[\mu'} \left(L^{-1}\right)^{\sigma}{}_{\nu'} \left(L^{-1}\right)^{\lambda}{}_{\rho']} \partial_\tau F_{\sigma\rho} \\ &= \sum_{\tau,\sigma,\rho=0}^{3} \left(L^{-1}\right)^{\tau}{}_{\mu'} \left(L^{-1}\right)^{\sigma}{}_{\nu'} \left(L^{-1}\right)^{\lambda}{}_{\rho'} \partial_{[\tau} F_{\sigma\rho]} = 0.\end{aligned}$$

Hence, in the primed frame, Maxwell's equations are

$$\sum_{\mu'=0}^{3} \partial_{\mu'} F^{\mu'\nu'} = -\mu_0 J^{\nu'}, \qquad \partial_{[\mu'} F_{\nu'\rho']} = 0,$$

exactly the same from as in the unprimed frame, even though the individual components are different. This is what is meant by co-variance and the statement above that Lorentz transformations are a symmetry of Maxwell's equations.

Since the components are different in different reference frames, it can sometimes be difficult to see symmetries when the individual components are written out, as in equation (40) for example. It is often useful to construct quantities that are genuinely invariant, *i.e.* they are the same in every reference frame. Such quantities can be evaluated in any inertial reference frame and we know that we would get the same answer in any other frame and sometimes calculations are easier in one particular frame so it is clearly easiest to use that frame. One way of constructing invariants is to 'contract' indices so that there are no free indices on our expressions. For example

$$\sum_{\mu=0}^{3} \partial_\mu J^\mu = 0 = \sum_{\mu'=0}^{3} \partial_{\mu'} J^{\mu'}$$

is an invariant, it is the same in all reference frames (it happens to be zero).*

We can make an invariant out of **E** and **B** by considering the following quadratic expression in F,

$$\sum_{\mu,\nu=0}^{3} F_{\mu\nu}F^{\mu\nu} = 2\sum_{i=1}^{3} F_{0i}F^{0i} + \sum_{i,j=1}^{3} F_{ij}F^{ij} = -\frac{2}{c^2}\mathbf{E}.\mathbf{E} + \sum_{i,j,k,l=1}^{3} \big(\epsilon_{ijk}B^k\big)\big(\epsilon^{ijl}B^l\big).$$

Now $\sum_{i,j=1}^{3} \epsilon_{ijk}\epsilon^{ijl} = 2\delta_k{}^l$, so the combination

$$\frac{1}{4}\sum_{\mu\nu=0}^{3} F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\mathbf{B}.\mathbf{B} - \frac{\mathbf{E}.\mathbf{E}}{c^2}\right) = \frac{1}{2}\left(\mathbf{B}'.\mathbf{B}' - \frac{\mathbf{E}'.\mathbf{E}'}{c^2}\right)$$

is an invariant under Lorentz transformations, it is the same in all inertial references frames.† As an exercise you should check this for (39) and (40).

There is in fact a second quadratic invariant that can be constructed from $F_{\mu\nu}$. To show this we first need a 4-dimensional version of ϵ^{ijk}, which we denote by $\epsilon^{\mu\nu\rho\sigma}$. This is defined to be zero if any of the 4 indices μ, ν, ρ or σ are the same so, of the $4^4 = 256$ possibilities, 212 vanish and only $4! = 24$ are non-zero. The non-zero ones are all defined to be ± 1 and for these $\{\mu,\nu,\rho,\sigma\}$ must be some permutation of the four indices $\{0,1,2,3\}$. The permutation is called *even* if the sequence $\{\mu,\nu,\rho,\sigma\}$ can be obtained $\{0,1,2,3\}$ by an even number of interchanges of pairs and *odd* if $\{\mu,\nu,\rho,\sigma\}$ must be obtained $\{0,1,2,3\}$ by an odd number of interchanges of pairs. For example $\{0,1,2,3\}$, $\{1,0,3,2\}$, $\{0,2,3,1\}$ and $\{2,0,1,3\}$ are even permutations (there are 12 in all) while $\{1,0,2,3\}$, $\{0,1,3,2\}$, $\{2,0,3,1\}$ and $\{1,2,3,0\}$ are odd (again there are 12 of these). Equivalently one and only one index must be 0 for a non-zero value and

$$\epsilon^{0ijk} = -\epsilon^{i0jk} = \epsilon^{ij0k} = -\epsilon^{ijk0} = \epsilon^{ijk},$$

* It is crucial that one index is up and one is down here, because only then do we get a cancellation between L and L^{-1} in the primed expression $\sum_{\mu'=0}^{3} \partial_{\mu'} J^{\mu'}$. If both indices were sub-scripts, or both super-scripts, there would be no such cancellation, for example $\sum_{\mu'=0}^{3} \partial_{\mu'} J_{\mu'}$ is *not* Lorentz invariant.

† This is reminiscent of the energy density stored in the electro-magnetic field, $w = \frac{1}{2\mu_0}\left(\frac{\mathbf{E}.\mathbf{E}}{c^2} + \mathbf{B}.\mathbf{B}\right)$, but it is not the same, because of the sign difference. Energy is not Lorentz invariant.

with $i, j, k = 1, 2$ or 3, exhausts all possibilities. An important of consequence of this definition of $\epsilon^{\mu\nu\rho\sigma}$ is that it is Lorentz invariant. To see this consider the Lorentz transformed quantity

$$\epsilon^{0'1'2'3'} = \sum_{\mu,\nu,\rho,\sigma=0}^{3} L^{0'}{}_{\mu} L^{1'}{}_{\nu} L^{2'}{}_{\rho} L^{3'}{}_{\sigma} \epsilon^{\mu\nu\rho\sigma}.$$

The right hand side of this equation is nothing other than the definition of the determinant of the 4×4 matrix L, which evaluates to one

$$\epsilon^{0'1'2'3'} = \det L = 1,$$

hence

$$\epsilon^{0'1'2'3} = \epsilon^{0123}$$

and all the other components of $\epsilon^{\mu'\nu'\rho'\sigma'}$ follow from the usual properties of determinant (interchange two rows or two columns changes a sign, the determinant vanishes if any two rows or columns are identical). We conclude that

$$\epsilon^{\mu'\nu'\rho'\sigma'} = \sum_{\tau,\lambda,\eta,\zeta=0}^{3} L^{\mu'}{}_{\tau} L^{\nu'}{}_{\lambda} L^{\rho'}{}_{\eta} L^{\sigma'}{}_{\zeta} \epsilon^{\tau\lambda\eta\zeta}$$

has exactly the same components in every inertial reference frame, ± 1 or 0. Note that lowering the indices introduces minus sign, since one if them is necessarily the index 0, and $\epsilon_{0123} = -\epsilon^{0123} = -1$.

Now the combination $\sum_{\mu,\nu,\rho,\sigma=0}^{3} F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}$ has no free indices and is a Lorentz invariant, again because the four factors of L^{-1} cancel against the four factors of L in $\sum_{\mu',\nu',\rho',\sigma'=0}^{3} F_{\mu'\nu'} F_{\rho'\sigma'} \epsilon^{\mu'\nu'\rho'\sigma'}$. Expanding this in terms of $\mathbf{E}$ and $\mathbf{B}$

$$\begin{aligned} \sum_{\mu,\nu,\rho,\sigma=0}^{3} F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} &= 4 \sum_{i,j,k=1}^{3} F_{0i} F_{jk} \epsilon^{ijk} = -\frac{4}{c} \sum_{i,j,k=1}^{3} E^i \Big(\sum_{l=1}^{3} \epsilon_{jkl} B^l \Big) \epsilon^{ijk} \\ &= -\frac{4}{c} \sum_{i,l=1}^{3} E^i B^l \big(2\delta^i{}_l \big) = -\frac{8}{c} \sum_{i=1}^{3} E^i B^i = -\frac{8}{c} \mathbf{E}.\mathbf{B}. \end{aligned}$$

So

$$-\frac{1}{8} \sum_{\mu,\nu,\rho,\sigma=0}^{3} F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} = \frac{\mathbf{E}.\mathbf{B}}{c}$$

has the same value in all inertial reference frames.

It is convenient to define

$$\widetilde{F}^{\mu\nu} := \frac{1}{2} \sum_{\rho,\sigma=0}^{3} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma},$$

called the *dual* of $F_{\mu\nu}$, which has components

$$\widetilde{F}^{\mu\nu} = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & -E_3/c & E_2/c \\ -B_2 & E_3/c & 0 & -E_1/c \\ -B_3 & -E_2/c & E_1/c & 0 \end{pmatrix},$$

so $E_i/c \to B_i$ and $B_i \to -E_i/c$, the operation of taking the dual essentially interchanges $\mathbf{E}$ and $\mathbf{B}$. In terms of the dual

$$-\frac{1}{4}\sum_{\mu\nu=0}^{3} F_{\mu\nu}\widetilde{F}^{\mu\nu} = \frac{\mathbf{E.B}}{c}$$

and

$$\sum_{\nu=0}^{3}\partial_\mu \widetilde{F}^{\mu\nu} = \frac{1}{2}\sum_{\mu,\rho,\sigma=0}^{3}\epsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\rho\sigma} = \frac{1}{2}\sum_{\mu,\rho,\sigma=0}^{3}\epsilon^{\mu\nu\rho\sigma}\partial_{[\mu}F_{\rho\sigma]} = 0.$$

Maxwell's equations are now succinctly written as

$$\boxed{\sum_{\nu=0}^{3}\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu, \qquad \sum_{\nu=0}^{3}\partial_\mu \widetilde{F}^{\mu\nu} = 0.}$$

When $J^\mu = 0$ Maxwell's equations are symmetric under the interchange

$$\widetilde{F}^{\mu\nu} \quad \leftrightarrow \quad F^{\mu\nu},$$

and in modern attempts to unify the fundamental forces of nature, such as string theory, this kind of duality symmetry plays a very important rôle. The symmetry is not there when $J^\mu \neq 0$ but it can be re-instated by postulating a dual current $\widetilde{J}^\mu$ such that

$$\sum_{\nu=0}^{3}\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu, \qquad \sum_{\nu=0}^{3}\partial_\mu \widetilde{F}^{\mu\nu} = -\mu_0 \widetilde{J}^\nu.$$

Since the duality operation interchanges electric and magnetic fields and J^μ is a current arising from electric charges $\widetilde{J}^\mu$ is a current arising from *magnetic* charges — re-instating full duality symmetry necessitates introducing magnetic monopoles. Such particles have never been observed, if they exist they must be both very rare, because we do not see any that may have been produced in high energy astrophysical processes, and very heavy, because we have not been able to produce any in the laboratory. If magnetic monopoles exist they may be as heavy as 10^{16} times the mass of a proton.