

MP464 – Solid State Physics

Assignment 1: The Free Electron Gas Model

Due in lecture on Friday, Mar. 13th, 2009

(Late assignments will not be accepted)

1. In lecture we calculated several important thermodynamic quantities to order T^2/T_F^2 in the low-temperature approximation. It can be shown that when you calculate to $O(T^4/T_F^4)$, the Helmholtz free energy is

$$F(T, V, N) \approx \frac{3}{5} N E_F \left[1 - \frac{5\pi^2}{12} \left(\frac{T}{T_F} \right)^2 + \frac{5\pi^4}{240} \left(\frac{T}{T_F} \right)^4 \right],$$

where the Fermi energy and temperature are given by

$$E_F := \frac{\hbar^2}{2m} \left(\frac{3\pi^2 N}{V} \right)^{2/3}, \quad T_F := \frac{E_F}{k_B}.$$

From this, find expressions for the internal energy U , the chemical potential μ , the pressure P , the entropy S and the heat capacity C_V . (Remember that $F = U - TS = -PV + \mu N$ and that $dF = -SdT - PdV + \mu dN$.)

2. (a) Calculate the Fermi temperatures of the following elements: (i) lithium; (ii) sodium; (iii) magnesium; (iv) potassium; (v) caesium; (vi) tungsten; (vii) platinum; and (viii) uranium. (It's up to you to do a bit of research and find all of the numbers of conduction electrons per atom, densities, atomic weights, etc.)
- (b) The Sommerfeld constant γ is defined as the value of c_F/T at $T = 0$, where c_F is the molar heat capacity of the electron gas model discussed in lecture. Using your result from (a), find γ for (i) lithium; (ii) sodium; (iii) potassium; (iv) caesium; and (v) platinum. (The actual experimental values are 1.63, 1.38, 2.08, 3.20 and 6.7 mJ · mol⁻¹ · K⁻² respectively.)
- (c) Lithium, sodium, magnesium, caesium, tungsten, platinum and uranium are all paramagnetic. Using your results from (a), calculate their magnetic susceptibilities for values of T near absolute zero.
Extra credit: The actual experimental values are, respectively, 1.4, 0.72, 1.2, 5.1, 6.8, 26 and 40 (all $\times 10^{-5}$); comment on the agreement (or lack thereof) between the theoretical predictions and the actual values, and offer explanations as to why they might agree or differ.
3. In the presence of a magnetic field $\vec{B} = B\hat{e}_z$, the density of states for electrons with $S_z = +\hbar/2$ (up) and $S_z = -\hbar/2$ (down) are, respectively,

$$g_{\uparrow}(E) = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} (E - \mu_B B)^{1/2}, \quad E \geq \mu_B B,$$

$$g_{\downarrow}(E) = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} (E + \mu_B B)^{1/2}, \quad E \geq -\mu_B B,$$

where μ_B is the Bohr magneton. Assume that T and $\mu_B B/k_B$ are both much less than the Fermi temperature, and show that, to first order in T/T_F and $\mu_B B/E_F$, the number of up and down electrons are approximately given by

$$\begin{aligned} N_{\uparrow} &\approx \frac{N}{2} \left(1 - \frac{3\mu_B B}{2E_F} \right), \\ N_{\downarrow} &\approx \frac{N}{2} \left(1 + \frac{3\mu_B B}{2E_F} \right), \end{aligned}$$

and that their contributions to the internal energy are

$$\begin{aligned} U_{\uparrow} &\approx \frac{3}{10} N E_F \left(1 - \frac{5\mu_B B}{6E_F} \right), \\ U_{\downarrow} &\approx \frac{3}{10} N E_F \left(1 + \frac{5\mu_B B}{6E_F} \right). \end{aligned}$$

(Note that neither the *total* electron number nor the *total* internal energy of the solid change upon application of a magnetic field.)

The following three integrals might be useful:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^x}{(e^x + 1)^2} dx &= 1, \\ \int_{-\infty}^{\infty} \frac{x e^x}{(e^x + 1)^2} dx &= 0, \\ \int_{-\infty}^{\infty} \frac{x^2 e^x}{(e^x + 1)^2} dx &= \frac{\pi^2}{3}. \end{aligned}$$

So might this Taylor series expansion for small x and any a :

$$\begin{aligned} (a + x)^m &= a^m + m a^{m-1} x + \frac{m(m-1)}{2} a^{m-2} x^2 + \frac{m(m-1)(m-2)}{6} a^{m-3} x^3 \\ &\quad + O(x^4). \end{aligned}$$