For any lattice with reciprocal lattice vectors

$$\underline{b}_1 = 2\pi \frac{\underline{a}_2 \times \underline{a}_3}{\underline{a}_1 \cdot (\underline{a}_2 \times \underline{a}_3)}, \quad \underline{b}_2 = 2\pi \frac{\underline{a}_3 \times \underline{a}_1}{\underline{a}_1 \cdot (\underline{a}_2 \times \underline{a}_3)}$$

$$\underline{b}_3 = 2\pi \frac{\underline{a}_1 \times \underline{a}_2}{\underline{a}_1 \cdot (\underline{a}_2 \times \underline{a}_3)}. \quad V_c = \underline{a}_1 \cdot (\underline{a}_2 \times \underline{a}_3)$$

the reciprocal lattice of the reciprocal lattice has primitive lattice vectors

$$\underline{b}_1' = \frac{2\pi (\underline{b}_2 \times \underline{b}_3)}{\underline{b}_1 \cdot (\underline{b}_2 \times \underline{b}_3)}, \quad \underline{b}_2' = 2\pi \frac{(\underline{b}_3 \times \underline{b}_1)}{\underline{b}_1 \cdot (\underline{b}_2 \times \underline{b}_3)}$$

$$\underline{b}_3' = \frac{2\pi (\underline{b}_1 \times \underline{b}_2)}{\underline{b}_1 \cdot (\underline{b}_2 \times \underline{b}_3)}$$

$$\underline{b}_1 \times \underline{b}_2 = \frac{4\pi^2 (\underline{a}_2 \times \underline{a}_3) \times (\underline{a}_3 \times \underline{a}_1)}{\{\underline{a}_1 \cdot (\underline{a}_2 \times \underline{a}_3)\}^2}$$

$$= \frac{4\pi^2 \{\{(\underline{a}_2 \times \underline{a}_3) \cdot \underline{a}_3\} \underline{a}_1 - \{(\underline{a}_2 \times \underline{a}_3) \cdot \underline{a}_1\} \underline{a}_3\}}{\{\underline{a}_1 \cdot (\underline{a}_2 \times \underline{a}_3)\}^2}$$

$$= -\frac{4\pi^2 \underline{a}_3}{\underline{a}_1 \cdot (\underline{a}_2 \times \underline{a}_3)}$$

Similarly $\quad \underline{b}_2 \times \underline{b}_3 = -\dfrac{4\pi^2 \underline{a}_1}{\underline{a}_1 \cdot (\underline{a}_2 \times \underline{a}_3)}$

$$\underline{b}_3 \times \underline{b}_1 = -\frac{4\pi^2 \underline{a}_2}{\underline{a}_1 \cdot (\underline{a}_2 \times \underline{a}_3)}$$

$$\underline{b}_1 \cdot (\underline{b}_2 \times \underline{b}_3) = \frac{2\pi(\underline{a}_2 \times \underline{a}_3)}{\underline{a}_1 \cdot (\underline{a}_2 \times \underline{a}_3)} \cdot \left\{ \frac{-4\pi^2 \underline{a}_1}{\underline{a}_1 \cdot (\underline{a}_2 \times \underline{a}_3)} \right\}$$

$$= \frac{-8\pi^3}{\underline{a}_1 \cdot (\underline{a}_2 \times \underline{a}_3)}$$

$$\underline{b}_3' = \frac{2\pi(\underline{b}_1 \times \underline{b}_2)}{\underline{b}_1 \cdot (\underline{b}_2 \times \underline{b}_3)} = \frac{(-8\pi^3 \underline{a}_3)}{\underline{a}_1 \cdot (\underline{a}_2 \times \underline{a}_3)} \cdot \left\{ \frac{\underline{a}_1 \cdot (\underline{a}_2 \times \underline{a}_3)}{(-8\pi^3)} \right\} = \underline{a}_3$$

$$\underline{b}_2' = \frac{2\pi(\underline{b}_3 \times \underline{b}_1)}{\underline{b}_1 \cdot (\underline{b}_2 \times \underline{b}_3)} = \frac{-8\pi^2 \underline{a}_2}{\underline{a}_1 \cdot (\underline{a}_2 \times \underline{a}_3)} \cdot \left\{ \frac{\underline{a}_1 \cdot (\underline{a}_2 \times \underline{a}_3)}{-8\pi^3} \right\} = \underline{a}_2$$

$$\underline{b}_1' = \frac{2\pi(\underline{b}_2 \times \underline{b}_3)}{\underline{b}_1 \cdot (\underline{b}_2 \times \underline{b}_3)} = \frac{-8\pi^2 \underline{a}_1}{\underline{a}_1 \cdot (\underline{a}_2 \times \underline{a}_3)} \cdot \left\{ \frac{\underline{a}_1 \cdot (\underline{a}_2 \times \underline{a}_3)}{-8\pi 3} \right\} = \underline{a}_1$$

$$\Rightarrow \quad \underline{b}_1' = \underline{a}_1, \quad \underline{b}_2' = \underline{a}_2, \quad \underline{b}_3' = \underline{a}_3$$

The reciprocal lattice of a reciprocal lattice is the original direct lattice.

2) Reciprocal lattice vectors are

$$\underline{b}_1 = +2\pi \frac{(\underline{a}_2 \times \vec{\underline{z}})}{|\underline{a}_1 \times \underline{a}_2|}, \quad \underline{b}_2 = -2\pi \frac{(\underline{a}_1 \times \vec{\underline{z}})}{|\underline{a}_1 + \underline{a}_2|}$$

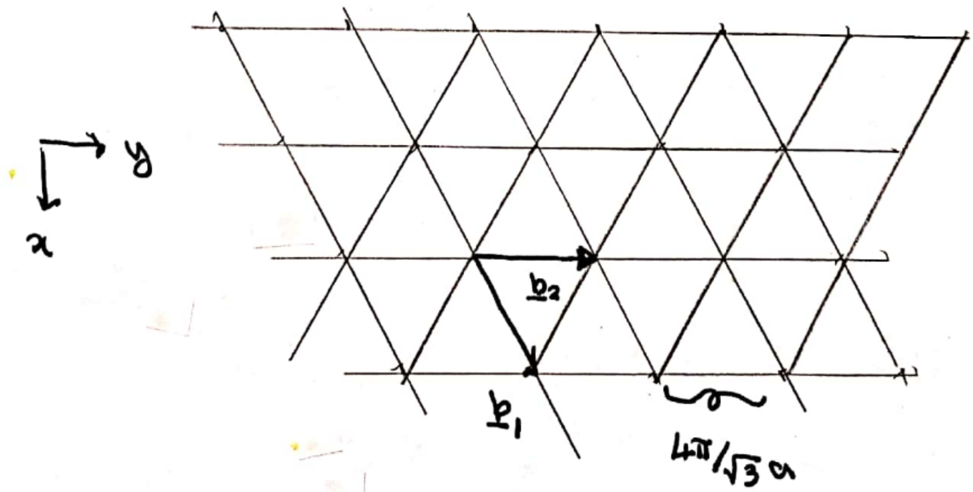
$$\underline{a}_1 \times \underline{a}_2 = \frac{\sqrt{3}\, a^2}{2} \hat{\underline{z}} \Rightarrow |\underline{a}_1 \times \underline{a}_2| = \sqrt{3}\, a^2/2$$

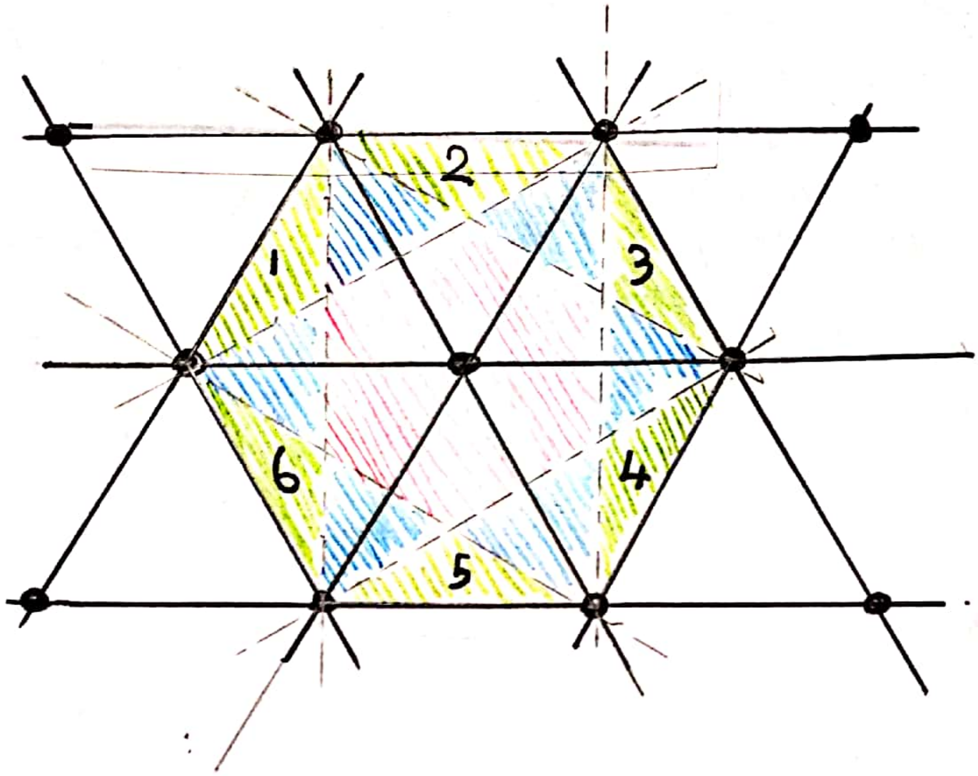
$$\underline{b}_1 = \frac{2\pi}{(\sqrt{3}\, a^2/2)} \left( \frac{a}{2} \hat{\underline{y}} + \frac{\sqrt{3}\, a}{2} \hat{\underline{x}} \right) = \frac{2\pi}{\sqrt{3}\, a} \left( \hat{\underline{y}} + \sqrt{3}\, \hat{\underline{x}} \right)$$

$$\underline{b}_2 = -\frac{2\pi}{(\sqrt{3}\, a^2/2)} \left( -a \hat{\underline{y}} \right) = \frac{4\pi\, \hat{\underline{y}}}{\sqrt{3}\, a}$$

$$\boxed{\underline{b}_1 = \frac{4\pi}{\sqrt{3}\, a} \left( \frac{\sqrt{3}}{2} \hat{\underline{x}} + \frac{1}{2} \hat{\underline{y}} \right), \quad \underline{b}_2 = \frac{4\pi}{\sqrt{3}\, a} \hat{\underline{y}}}$$

This is a hexagonal lattice, with lattice spacing $\frac{4\pi}{\sqrt{3}\, a}$





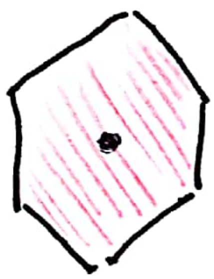

1ST Brillouin zone is red

2ND Brillouin zone is blue

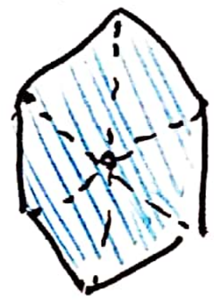
3RD Brillouin zone is green.

Perpendicular bi-sectors of lattice vectors between neighbouring points are dashed lines.

The triangles can all be assembled into primitive cells by translating them by a lattice vector (for the 3RD Brillouin zone the green triangles must also be cut in two).



1ST Brillouin zone



2ND Brillouin zone


3RD Brillouin zone.