OLLSCOIL NA hÉIREANN MÁ NUAD
THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

MATHEMATICAL PHYSICS

Year 4

SEMESTER 2
2009-2010

Statistical Mechanics
MP461

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Time allowed: 1 $\frac{1}{2}$ hours
Answer 2 questions
All questions carry equal marks

Tip: *Take 5 minutes to read through all questions before you begin and select the ones which most suit you.*

1. (a) For a system in thermal equilibrium with a heat reservoir at constant temperature T, show that the variance of the energy E of the system is related to the heat capacity C_V by the following equation,

$$C_V = \frac{\left\langle (E - \langle E \rangle)^2 \right\rangle}{kT^2}$$

You may assume that the system does not exchange particles with the reservoir.

(b) State the equipartition theorem for the internal energy of a system with N degrees of freedom. Then use it to derive Dulong and Petit's law, which states that, at sufficiently high temperatures, the specific heat of a monatomic crystalline solid, per unit mass, will be given by

$$c_V = \frac{3k}{m_p},$$

where m_p is the mass of one of the atoms of which the solid consists.

(c) Use the results of parts **b.** and **c.** to argue that for a solid consisting of N atoms, at high temperatures, we have, approximately

$$\frac{\sqrt{\left\langle (E - \langle E \rangle)^2 \right\rangle}}{\langle E \rangle} = \sqrt{\frac{1}{3N}}$$

2. (a) For a system of N particles with energy E volume V, give expressions for the pressure p, temperature T, and chemical potential μ in terms of partial derivatives of the entropy with respect to E, V and N.

(b) Two systems at equal positive temperature T and pressure p but with different values μ_1 and μ_2 of the chemical potential are brought into contact. Argue that particles will flow from the system at higher chemical potential to the system at lower chemical potential as equilibrium is approached. You may assume that the two systems taken together are isolated from the environment and that no particles are created or destroyed.

(c) Now consider a system at fixed volume V and in contact with a heat and particle bath which remains at constant temperature T and chemical potential μ. Derive the Boltzmann distribution, that is, show that the probability $P(E, N)$ of a microstate of the system with energy E and N particles is given by

$$P(E, N) = \frac{e^{-\beta(E - \mu N)}}{Z}$$

where $\beta = \frac{1}{kT}$ and Z does not depend on E and N.
Also give an expression for Z.

3. (a) Describe the ground state of a system of N non-interacting bosons in terms of the occupation numbers of single particle states.
   Do the same for a system of N non-interacting fermions.
   In the fermionic case, give an argument that, at $T = 0$, we must have $\mu = \epsilon_F$, where ϵ_F is the energy of the highest occupied state.

   (b) A particle in two dimensions is confined to a square box of side L. Its energy levels are given by $E(\vec{n}) = \frac{h^2 n^2}{8mL^2}$, where $\vec{n} = (n_1, n_2)$ and n_1 and n_2 are positive integers. Show that the density of states $g(E)$ for this system is independent of E and given by
$$g(E) = \frac{2\pi m L^2}{h^2}.$$

   (c) Calculate or argue that, for a two dimensional gas of identical non-interacting fermions confined to a square box, the average energy per particle in the ground state, E_0 is equal to $\frac{1}{2}\epsilon_F$

   (d) Show that the chemical potential for this two dimensional gas of identical non-interacting fermions confined to a square box, at arbitrary temperature T, is given by
$$\mu = \frac{1}{\beta}\log(e^{\frac{h^2\beta N}{2\pi m L^2}} - 1)$$
   with $\beta = \frac{1}{kT}$

4. Consider a paramagnet consisting of N spin $\frac{1}{2}$ particles at fixed positions, subject to a magnetic field of magnitude B pointing in the z-direction.
   The system has energy $U = -\mu B(N_\uparrow - N_\downarrow)$, where $N_\uparrow$ and $N_\downarrow$ are the number of particles with positive and negative spin in the z-direction and μ is the magnitude of the magnetic moment of the particles.

   (a) Show that the number of microstates $\Omega(U)$ corresponding to a macrostate of energy U is
$$\Omega(U) = \frac{N!}{(\frac{\mu BN+U}{2\mu B})!(\frac{\mu BN-U}{2\mu B})!}$$

   (b) Show that, at large N, the temperature of the system is given, to good approximation, by
$$T = \frac{2\mu B}{k\log\left(\frac{\mu BN-U}{\mu BN+U}\right)}$$

   (c) This system can have a negative temperature. Indicate for which range of values of U this happens. Argue that macrostates with negative temperatures must occur in all thermodynamic systems with a finite number of microstates and a continuous density of states.