Statistical Mechanics (MP461) Assignment 1

Please hand in your solutions no later than Monday, October 14, 14:05 pm. If you have questions about this assignment, please ask your lecturer,
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Please hand in Ex. 2.1 from this sheet and Ex. 2.18 and 2.22 from the book by Schroeder. Ex. 2.2 is included for practice.

Ex. 2.1: The Otto Cycle

An internal combustion engine (e.g. a car engine) has an idealized description as a heat engine. A single cycle of such an engine consists of four steps.

1. A mixture of air and fuel vapor, at initial temperature T_1 is adiabatically compressed (in a cylinder) from an initial volume V_1 to a final volume $V_2 < V_1$.
2. The mixture is ignited, which raises its temperature to T_2, at constant volume. We can think of T_2 as the temperature of the hot reservoir (even though there is no hot reservoir, strictly speaking). We also ignore the chemical changes in the gas.
3. The mixture is allowed to expand adiabatically until it reaches the initial volume V_1
4. Finally, the gas is replaced by a new fuel/air mixture at the initial temperature T_1 and the same volume V_1. This can be modeled as if the system was simply cooling down at constant volume, in contact with a cold reservoir at temperature T_1.

This cycle is called the Otto cycle, after Nikolaus August Otto (1832-1891), inventor of the four stroke engine.

a. Draw the idealized cycle in a (p, V) diagram.

b. Assume that throughout the process, the gas in the cylinder always has a constant specific heat c_V and energy $U = c_V nT$. Calculate the change in temperature in each of the four steps of the cycle in terms of T_1, T_2, V_1 and V_2.

c. Express the efficiency of this cycle in terms of the compression ratio $\frac{V_1}{V_2}$

Ex. 2.2: Interacting paramagnets

A two-state paramagnet is composed of N elementary magnetic dipoles with dipole moments $\pm\mu$ in the z-direction. The total magnetization M is $M = (N_\uparrow - N_\downarrow)\mu$, where $N_\uparrow$ is the number of dipoles with positive (upward pointing) dipole moment, $N_\downarrow$ is the number of dipoles with negative (downward pointing) dipole moment and $N_\uparrow + N_\downarrow = N$. if we apply a magnetic field of strength B in the positive z-direction, the energy $U(B)$ of the paramagnet is given by $U = -MB$.

a. What is the total number of microstates of this paramagnet? What is the number of microstates with given magnetisation $M = M_0$?

b. What is the lowest energy state of the parmagnet if $B > 0$? What if $B < 0$? What is the entropy of a macrostate with given energy $U = U_0$?

Suppose we have two two-state paramagnets composed of N_1 and N_2 elementary dipoles of the same strength μ.

c. What is the total number of microstates of this system of two paramagnets? What is the number of microstates in which the first paramagnet has magnetisation $M = M_1$ and the second has magnetisation $M = M_2$?

d. If the first paramagnet is in a macrostate A with entropy $S(A)$ and the second is in a macrostate B with entropy $S(B)$, what is the entropy of the two paramagnets combined?

In any state of the combined system, the total energy U equals $U_1 + U_2$, where U_1 and U_2 are the energies of the individual paramagnets. We assume the two paramagnets are exposed to the same magnetic field B, which fixes the total energy of the system U.

d. Consider the case where $N_1 = 5$ and $N_2 = 3$. Calculate the total number of microstates with $U = -2B\mu$. If $U = -2B\mu$, what are the possible values of M_1 and U_1? How many microstate are there for each of the possible values of M_1 ?

e. Now consider the case where both N_1 and N_2 are large. Show that the entropy of the system is given, to good approximation, by

$$S_{total} = -N_1(x\log(x) + (1-x)\log(1-x)) - N_2(y\log(y) + (1-y)\log(1-y)),$$

where $x = \frac{1}{2} - \frac{U_1}{2N_1\mu B}$ and $y = \frac{1}{2} - \frac{U_2}{2N_2\mu B}$.

f. Show that, for the given total energy U, the entropy S_{total} is maximized when $U_1 = \frac{N_1}{N}U$ and $U_2 = \frac{N_2}{N}U$.