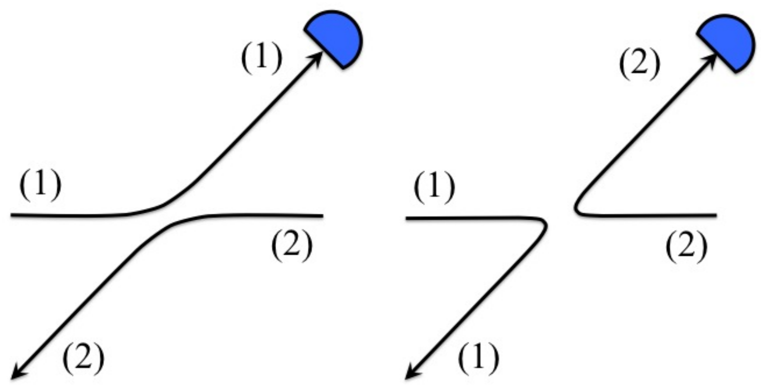
CHAPTER 8: SYSTEMS OF IDENTICAL PARTICLES

(From Cohen-Tannoudji, Chapter XIV)

Consider scattering of two quantum particles



as the particles are indistinguishable, we cannot determine the path they followed.

→ Problems: exchange degeneracy (removed by the symmetrization postulate)

Permutation operators

a) Two particle systems

$$\mathcal{E}(1) \otimes \mathcal{E}(2) \tag{8.1}$$

$$\left\{|1 : u_i; 2 : u_j\rangle\right\} \tag{8.2}$$

Note

$$|1 : u_j; 2 : u_i\rangle \neq |1 : u_i; 2 : u_j\rangle \qquad \text{if } i \neq j \tag{8.3}$$

The permutation operator

$$\hat{P}_{21}|1 : u_i; 2 : u_j\rangle = |2 : u_i; 1 : u_j\rangle \tag{8.4}$$

$$= |1 : u_j; 2 : u_i\rangle \tag{8.5}$$

The order of the vectors in a tensor product is of no importance.

b)The permutation operator P_{21}

$$\left(\hat{P}_{21}\right)^2 = \hat{1} \tag{8.6}$$

$$\hat{P}_{21}^\dagger = \hat{P}_{21} \text{ (self adjoint)} \tag{8.7}$$

$$\hat{P}_{21}^\dagger \hat{P}_{21} = \hat{P}_{21}\hat{P}_{21}^\dagger = \hat{1} \text{ (unitary)} \tag{8.8}$$

c) Symmetric and antisymmetric kets

Symmetrizer and antisymmetrizer
$\hat{P}_{21}^\dagger = \hat{P}_{21} \Rightarrow$ the eigenvalues of $\hat{P}_{21}$ must be real
$(\hat{P}_{21})^2 = \hat{1} \Rightarrow$ the eigenvalues are

$$+1 \text{ (symmetric)} \qquad -1 \text{ (antisymmetric)} \tag{8.9}$$

$$\hat{P}_{21}|\psi_S\rangle = |\psi_S\rangle \qquad \hat{P}_{21}|\psi_A\rangle = -|\psi_A\rangle \tag{8.10}$$

Consider the operators

$$\text{symmetrizer: } \hat{S} = \frac{1}{2}\left(\hat{1} + \hat{P}_{21}\right) \qquad \hat{S}^2 = \hat{S},\ \hat{S}^\dagger = \hat{S} \tag{8.11}$$

$$\text{antisymmetrizer: } \hat{A} = \frac{1}{2}\left(\hat{1} - \hat{P}_{21}\right) \qquad \hat{A}^2 = \hat{A},\ \hat{A}^\dagger = \hat{A} \tag{8.12}$$

which are projectors onto orthogonal subspaces

$$\hat{S}\hat{A} = \hat{A}\hat{S} = 0 \tag{8.13}$$

that are complementary

$$\hat{S} + \hat{A} = \hat{1} \tag{8.14}$$

If $|\psi\rangle$ is an arbitrary ket in $\mathcal{E}$,
$\hat{S}|\psi\rangle$ is a symmetric ket and
$\hat{A}|\psi\rangle$ is an antisymmetric ket

$$\hat{P}_{21}\hat{S}|\psi\rangle = \hat{S}|\psi\rangle \qquad \hat{P}_{21}\hat{A}|\psi\rangle = -\hat{A}|\psi\rangle \tag{8.15}$$

Transformation of observables by permutation

$\hat{B}(1)$ – defined in $\mathcal{E}(1)$, and extended to $\mathcal{E}$

$\{|u_i\rangle\}$ – the basis in $\mathcal{E}(1)$ from eigenvectors of $\hat{B}(1)$ (with eigenvalues b_i)

$$\hat{P}_{21}\hat{B}(1)\hat{P}_{21}^{\dagger}|1:u_i;2:u_j\rangle = \hat{P}_{21}\hat{B}(1)|1:u_j;2:u_i\rangle \tag{8.16}$$

$$= b_j\hat{P}_{21}|1:u_j;2:u_i\rangle \tag{8.17}$$

$$= b_j|1:u_i;2:u_j\rangle \tag{8.18}$$

$$\hat{P}_{21}\hat{B}(1)\hat{P}_{21}^{\dagger} = \hat{B}(2) \tag{8.19}$$

$$\hat{P}_{21}\hat{B}(2)\hat{P}_{21}^{\dagger} = \hat{B}(1) \tag{8.20}$$

$$\hat{P}_{21}\left[\hat{B}(1)+\hat{C}(2)\right]\hat{P}_{21}^{\dagger} = \hat{B}(2)+\hat{C}(1) \tag{8.21}$$

$$\hat{P}_{21}\hat{B}(1)\hat{C}(2)\hat{P}_{21}^{\dagger} = \hat{B}(2)\hat{C}(1) \tag{8.22}$$

Generalization

$$\hat{P}_{21}\hat{O}(1,2)\hat{P}_{21}^{\dagger} = \hat{O}(2,1) \quad (8.23)$$

where $\hat{O}(1,2)$ is any observable in $\mathcal{E}$ which can be expressed in terms of observables of the type $\hat{B}(1)$ and $\hat{C}(2)$.

Symmetric observables commute with the permutation operators:

$$\hat{O}_S(1,2) = \hat{O}_S(2,1) \quad (8.24)$$

$$\hat{P}_{21}\hat{O}_S(1,2) = \hat{O}_S(1,2)\hat{P}_{21} \quad (8.25)$$

$$\Rightarrow \left[\hat{O}_S(1,2), \hat{P}_{21}\right] = 0 \quad (8.26)$$

An arbitrary number of particles

Example 3 particles

$$\left\{|1:u_i;2:u_j;3:u_k\rangle\right\} \tag{8.27}$$

Six permutations

$$\hat{P}_{123},\hat{P}_{321},\hat{P}_{231},\hat{P}_{132},\hat{P}_{213},\hat{P}_{312} \tag{8.28}$$

$$\hat{P}_{npq}|1:u_i;2:u_j;3:u_k\rangle = |n:u_i;p:u_j;q:u_k\rangle \tag{8.29}$$

(N! permutation operators in a system of N particles with the same spin.)

Any permutation operator can be broken down into a product of transposition (i.e. pairwise exchange) operators, for example

$$\hat{P}_{312} = \underbrace{\hat{P}_{132}\hat{P}_{213}}_{\text{even parity of } \hat{P}_{312}} = \hat{P}_{321}\hat{P}_{132} = \ldots \tag{8.30}$$

$$\text{even:} \quad \hat{P}_{123}, \hat{P}_{321}, \hat{P}_{231} \tag{8.31}$$

$$\text{odd:} \quad \hat{P}_{132}, \hat{P}_{213}, \hat{P}_{321} \tag{8.32}$$

For any N, there are always as many even permutations as there are odd.

Permutation operators are unitary and constitute a group.

Completely symmetric or antisymmetric kets. Symmetrizer and antisymmetrizer.

Completely symmetric

$$\hat{P}_\alpha|\psi_S\rangle = |\psi_S\rangle \qquad \text{for any } \hat{P}_\alpha \tag{8.33}$$

Completely antisymmetric

$$\hat{P}_\alpha|\psi_A\rangle = \underbrace{\varepsilon_\alpha}_{+1 \text{ for even, } -1 \text{ for odd}} |\psi_A\rangle \qquad \text{for any } \hat{P}_\alpha \tag{8.34}$$

$$\text{Symmetrizer} \qquad \hat{S} = \frac{1}{N!}\sum_\alpha \hat{P}_\alpha \text{ projects onto } \mathcal{E}_S \tag{8.35}$$

$$\text{Antisymmetrizer} \qquad \hat{A} = \frac{1}{N!}\sum_\alpha \varepsilon_\alpha \hat{P}_\alpha \text{ projects onto } \mathcal{E}_A \tag{8.36}$$

The symmetrization postulate

When a system includes several identical particles, only certain kets of its state space can describe its physical states. Physical kets, depending on the nature of the identical particles, are either

completely symmetric (bosons – integral spin)
or
completely antisymmetric (fermions – half-integral spin)

with respect to permutation of these particles.

Construction of physical kets

(i) number the particles arbitrarily, and construct the ket $|u\rangle$ corresponding to the physical state considered and to the numbers given to the particles

(ii) apply $\hat{S}$ or $\hat{A}$ to $|u\rangle$, depending on whether identical particles are bosons or fermions

(iii) normalize the ket so obtained.

Example: 2 particle system

(i) $|u\rangle = |1:\varphi; 2:\chi\rangle$

(ii) If particles are bosons, symmetrize $|u\rangle$

$$\hat{S}|u\rangle = \frac{1}{2}\left[|1:\varphi; 2:\chi\rangle + |1:\chi; 2:\varphi\rangle\right] \tag{8.37}$$

If they are fermions, antisymmetrize $|u\rangle$

$$\hat{A}|u\rangle = \frac{1}{2}\left[|1:\varphi; 2:\chi\rangle - |1:\chi; 2:\varphi\rangle\right] \tag{8.38}$$

(iii) normalize

$$|\varphi;\chi\rangle = \frac{1}{\sqrt{2}}\left[|1:\varphi; 2:\chi\rangle + \epsilon|1:\chi; 2:\varphi\rangle\right] \tag{8.39}$$

$\epsilon = +1$ for bosons, -1 for fermions

Assume that the individual states $|\varphi\rangle$, $|\chi\rangle$ are identical

$$|\varphi\rangle = |\chi\rangle \tag{8.40}$$

then

$$|u\rangle = |1:\varphi; 2:\varphi\rangle \tag{8.41}$$

is already symmetric.
If the two particles are bosons, the ket $|u\rangle = |1:\varphi; 2:\varphi\rangle$ is the physical ket associated with the states in which the two bosons are in the same individual state $|\varphi\rangle$.

If the two particles are fermions,

$$\hat{A}|u\rangle = \frac{1}{2}\left[|1:\varphi;2:\varphi\rangle - |1:\varphi;2:\varphi\rangle\right] \quad = \quad 0 \tag{8.42}$$

There is no ket of $\mathcal{E}_A$ able to describe the physical state in which two fermions are in the same individual state $|\varphi\rangle$.

Pauli's exclusion principle

Two fermions cannot be in the same individual state.

Generalization to an arbitrary $N > 2$

Example $N = 3$

$$|u\rangle = |1:\varphi; 2:\chi; 3:\omega\rangle \tag{8.43}$$

α) Bosons

$$\hat{S}\,|u\rangle = \frac{1}{3!}\sum_{\alpha}\hat{P}_{\alpha}|u\rangle \tag{8.44}$$

$$= \frac{1}{6}\left[|1:\varphi; 2:\chi; 3:\omega\rangle + |1:\omega; 2:\varphi; 3:\chi\rangle\right. \tag{8.45}$$

$$+|1:\chi; 2:\omega; 3:\varphi\rangle + |1:\varphi; 2:\omega; 3:\chi\rangle \tag{8.46}$$

$$\left. + |1:\chi; 2:\varphi; 3:\omega\rangle + |1:\omega; 2:\chi; 3:\varphi\rangle\right] \tag{8.47}$$

Normalization

1) $|\varphi\rangle, |\chi\rangle, |\omega\rangle$ are orthogonal

replace $1/6$ by $1/\sqrt{6}$

2) If two states are the same and are orthogonal then

$$
\begin{aligned}
|\varphi;\varphi;\omega\rangle &= \frac{1}{\sqrt{3}}\left[|1:\varphi;2:\varphi;3:\omega\rangle + |1:\varphi;2:\omega;3:\varphi\rangle\right. && (8.48)\\
&\quad \left.+|1:\omega;2:\varphi;3:\varphi\rangle\right] && (8.49)
\end{aligned}
$$

3) If three states are the same

$$
|\varphi;\varphi;\varphi\rangle = |1:\varphi;2:\varphi;3:\varphi\rangle \qquad (8.50)
$$

β) Fermions

$$\hat{A}|u\rangle \;=\; \frac{1}{3!}\sum_{\alpha}\varepsilon_\alpha \hat{P}_\alpha |1:\varphi; 2:\chi; 3:\omega\rangle \tag{8.51}$$

The signs of the various terms are determined by the same rule as those of a 3×3 determinant

Slater determinant

$$\hat{A}|u\rangle \;=\; \frac{1}{3!}\begin{vmatrix} |1:\varphi\rangle & |1:\chi\rangle & |1:\omega\rangle \\ |2:\varphi\rangle & |2:\chi\rangle & |2:\omega\rangle \\ |3:\varphi\rangle & |3:\chi\rangle & |3:\omega\rangle \end{vmatrix} \tag{8.52}$$

Pauli exclusion principle:

$\hat{A}|u\rangle$ **is zero if two of the individual states coincide since the determinant then has two identical columns.**

Normalization:
If the three individual states are orthogonal replace $1/3!$ by $1/\sqrt{3!}$.