CHAPTER 7: APPROXIMATION METHODS FOR TIME-DEPENDENT PROBLEMS

(From Cohen-Tannoudji, Chapter XIII)

A. STATEMENT OF THE PROBLEM

Consider a system with Hamiltonian $\hat{H}_0$; its eigenvalues and eigenvectors are

$$\hat{H}_0|\varphi_n\rangle = E_n|\varphi_n\rangle \tag{7.1}$$

($\hat{H}_0$ is discrete and non-degenerate for simplicity.)

At $t = 0$, a perturbation is applied

$$\hat{H}(t) = \hat{H}_0 + W(t) = \hat{H}_0 + \lambda\hat{W}(t) \tag{7.2}$$

where $\lambda \ll 1$, and $\hat{W}(t) = 0$ for $t < 0$:

$t < 0$	$t = 0$	$t > 0$
stationary state	$W(t)$	final state
$\|\varphi_i\rangle$	evolution starts	$\|\psi(t)\rangle$
eigenstate of $\hat{H}_0$	($\|\varphi_i\rangle$ is not eigenstate of $\hat{H}$)	

What is the probability $\mathcal{P}_{fi}(t)$ of finding the system in another eigenstate $|\varphi_f\rangle$ of $\hat{H}_0$ at time t?

Treatment: solve the Schrödinger equation (S. E.)

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle \quad = \quad \left[\hat{H}_0 + \lambda\hat{W}(t)\right]|\psi(t)\rangle \tag{7.3}$$

with the initial condition $|\psi(0)\rangle = |\varphi_i\rangle$

$$\Rightarrow \qquad \mathcal{P}_{fi}(t) = \left|\langle\varphi_f|\psi(t)\rangle\right|^2 \tag{7.4}$$

In generally this problem is not rigorously soluble!

⇒ we need APPROXIMATION METHODS

B. APPROXIMATE SOLUTION OF THE SCHRÖDINGER EQUATION

1. The Schrödinger equation in the $\{|\varphi_n\rangle\}$ representation

We will use the $\{|\varphi_n\rangle\}$ representation which is convenient as $|\varphi_i\rangle$ and $|\varphi_f\rangle$ are eigenstates of $\hat{H}_0$, and obtain the differential equations for the components of the state vector

$$|\psi(t)\rangle = \sum_n c_n(t)|\varphi_n\rangle \quad (7.5)$$

$$c_n(t) = \langle\varphi_n|\psi(t)\rangle \quad (7.6)$$

$$\hat{W}_{nk}(t) = \langle\varphi_n|\hat{W}(t)|\varphi_k\rangle \quad (7.7)$$

$$\text{and} \quad \langle\varphi_n|\hat{H}_0|\varphi_k\rangle = E_n\delta_{nk} \quad (7.8)$$

We will project both sides of S.E. onto $|\varphi_n\rangle$ (and use $\sum_k |\varphi_k\rangle\langle\varphi_k| = \hat{1}$):

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \left[\hat{H}_0 + \lambda\hat{W}(t)\right]|\psi(t)\rangle \quad (7.9)$$

$$\Rightarrow i\hbar\frac{\mathrm{d}}{\mathrm{d}t}c_n(t) = E_n c_n(t) + \sum_k \lambda\hat{W}_{nk}(t)c_k(t) \quad (7.10)$$

Changing functions

If $\lambda\hat{W}(t) = 0$ then the equations decouple

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}c_n(t) = E_n c_n(t) \tag{7.11}$$

and yield simple solution

$$c_n(t) = b_n e^{-iE_n t/\hbar} \tag{7.12}$$

where b_n is a constant depending on the initial conditions.

If $\lambda\hat{W}(t) \neq 0$ and $\lambda \ll 1$, we expect the solutions $c_n(t)$ of the full equations to be very close to the solution above (for $\lambda\hat{W}(t) = 0$), and thus if we perform the change of function

$$c_n(t) = b_n(t) e^{-iE_n t/\hbar} \tag{7.13}$$

we can predict that $b_n(t)$ will be slowly varying functions of time.

Substituted into the equation gives

$$\begin{aligned} & i\hbar e^{-iE_n t/\hbar}\frac{\mathrm{d}}{\mathrm{d}t}b_n(t) + E_n b_n(t)e^{-iE_n t/\hbar} \\ = \quad & E_n b_n(t)e^{-iE_n t/\hbar} + \sum_k \lambda \hat{W}_{nk}(t) b_k(t) e^{-iE_k t/\hbar} \end{aligned} \tag{7.14}$$

Multiplying both sides by $e^{iE_n t/\hbar}$ and introducing the Bohr frequency $\omega_{nk} = \frac{E_n - E_k}{\hbar}$ gives

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}b_n(t) = \lambda \sum_k e^{i\omega_{nk}t}\hat{W}_{nk}(t)b_k(t) \tag{7.15}$$

2. Perturbation equations

In general, the solution is not known exactly and, for $\lambda \ll 1$, we try to determine this solution in the form of a power series in λ

$$b_n(t) \quad = \quad b_n^{(0)}(t) + \lambda b_n^{(1)}(t) + \lambda^2 b_n^{(2)}(t) + \ldots \tag{7.16}$$

and substitute it into the equation, and set equal the coefficients of λ^r on both sides of the equation

$$\text{i) } r = 0 : \qquad i\hbar \frac{\mathrm{d}}{\mathrm{d}t} b_n^{(0)}(t) = 0 \tag{7.17}$$

$$\text{ii) } r \neq 0 : \qquad i\hbar \frac{\mathrm{d}}{\mathrm{d}t} b_n^{(r)}(t) = \sum_k e^{i\omega_{nk}t/\hbar} \hat{W}_{nk}(t) b_k^{(r-1)}(t) \tag{7.18}$$

RECURRENCE!

3. Solution to the first order in λ

a. The state of the system at time t

$$t < 0 : \quad |\varphi_i\rangle \text{ i.e. } b_i(t) \neq 0, b_k(t) = 0 \,\forall k \neq i \tag{7.19}$$

$$t = 0 : \quad \hat{H}_0 \to \hat{H}_0 + \lambda\hat{W} \text{ and solution of S.E. is continuous at } t = 0 \tag{7.20}$$

$$\Rightarrow b_n(t = 0) = \delta_{ni} \,\forall\lambda \tag{7.21}$$

$$\Rightarrow b_n^{(0)}(t = 0) = \delta_{ni} \tag{7.22}$$

$$\Rightarrow b_n^{(r)}(t = 0) = 0 \text{ if } r \geq 1 \tag{7.23}$$

and with $i\hbar\frac{\mathrm{d}}{\mathrm{d}t}b_n^{(0)}(t) = 0$ we get

$$0^{\text{th}}\text{-order solution: } b_n^{(0)}(t) = \delta_{ni} \text{ for all } t > 0$$

$$1^{\text{st}} - \text{order: } i\hbar\frac{\mathrm{d}}{\mathrm{d}t}b_n^{(1)}(t) = \sum_k e^{i\omega_{nk}t}\hat{W}_{nk}(t)\delta_{ki} \tag{7.24}$$

$$= e^{i\omega_{ni}t}\hat{W}_{ni}(t) \tag{7.25}$$

$$\text{By integration } b_n^{(1)}(t) = \frac{1}{i\hbar}\int_0^t e^{i\omega_{ni}t'}\hat{W}_{ni}\left(t'\right)\mathrm{d}t' \tag{7.26}$$

$$c_n(t) = b_n(t)e^{-iE_nt/\hbar} \approx \left(b_n^{(0)}(t) + \lambda b_n^{(1)}(t)\right)e^{-iE_nt/\hbar} \tag{7.27}$$

to the first order time-dependent perturbation theory we get the state of the system at time t calculated to the first order:

$$|\psi(t)\rangle \approx \sum_n c_n(t)|\varphi_n\rangle \tag{7.28}$$

b. The transition probability $\mathcal{P}_{if}(t)$

$$\left|c_f(t)\right|^2 = \left|\langle\varphi_f|\psi(t)\rangle\right|^2 = \mathcal{P}_{if}(t) \quad (7.29)$$

$$c_f(t) = b_f(t)e^{-iE_ft/\hbar} \quad (7.30)$$

$$\Rightarrow \mathcal{P}_{if}(t) = \left|b_f(t)\right|^2 \quad (7.31)$$

where $b_f(t) = b_f^{(0)}(t) + \lambda b_f^{(1)}(t) + \ldots$

Let us assume $|\varphi_i\rangle$ and $|\varphi_f\rangle$ are different (i.e. we are concerned only with transition induced by $\lambda\hat{W}$ between two distinct stationary states of $\hat{H}_0$):
$b_f^{(0)}(t) = 0$ and consequently

$$\mathcal{P}_{if}(t) = \lambda^2 \left|b_f^{(1)}(t)\right|^2 \quad (7.32)$$

and using the formula for $b_n^{(1)}(t)$ we get

$$\mathcal{P}_{if}(t) = \frac{1}{\hbar^2}\left|\int_0^t e^{i\omega_{fi}t'} \underbrace{W_{fi}\left(t'\right)}_{W(t)=\lambda\hat{W}} \mathrm{d}t'\right|^2 \qquad (7.33)$$

Consider the function $\tilde{W}_{fi}(t')$ which is zero for $t' < 0$ and $T' > t$ and is equal to $W_{fi}(t')$ for $0 \leq t' \leq t$.

$\tilde{W}_{fi}(t')$ is the matrix element of the perturbation “seen” by the system between the time $t = 0$ and the measurement time t, when we try to determine if the system is in the state $|\varphi_f\rangle$.

$\mathcal{P}_{if}(t)$ is proportional to the square of the modulus of the Fourier transform of the perturbation actually “seen” by the system, $\tilde{W}_{fi}(t)$.

C. SPECIAL CASE: A SINUSOIDAL OR CONSTANT PERTURBATION

$\hat{W}(t) = \hat{W} \sin \omega t$ or
$\hat{W}(t) = \hat{W} \cos \omega t$
$\hat{W}$ is a time independent observable and ω a constant angular frequency.

(Example: electromagnetic wave of angular frequency ω.
$\overline{\mathcal{P}_{if}(t)}$ is the probability, induced by monochromatic radiation, of a transition between the initial state $|\varphi_i\rangle$ and the final state $|\varphi_f\rangle$.)

$$\hat{W}_{fi}(t) \quad = \quad \hat{W}_{fi} \sin \omega t = \frac{\hat{W}_{fi}}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right) \tag{7.34}$$

$\hat{W}_{fi}$ is a time independent complex number and

$$b_n^{(1)}(t) \quad = \quad -\frac{\hat{W}_{ni}}{2\hbar} \int_0^t \left[e^{i(\omega_{ni}+\omega)t'} - e^{i(\omega_{ni}-\omega)t'}\right] \mathrm{d}t' \tag{7.35}$$

$$= \frac{\hat{W}_{ni}}{2i\hbar}\left[\frac{1-e^{i(\omega_{ni}+\omega)t}}{\omega_{ni}+\omega} - \frac{1-e^{i(\omega_{ni}-\omega)t}}{\omega_{ni}-\omega}\right] \tag{7.36}$$

The transition probability becomes

$$\mathcal{P}_{if}(t;\omega) = \lambda^2 \left|b_f^{(1)}(t)\right|^2 = \frac{\left|W_{fi}\right|^2}{4\hbar^2}\left|\frac{1-e^{i\left(\omega_{fi}+\omega\right)t}}{\omega_{fi}+\omega} - \frac{1-e^{i\left(\omega_{fi}-\omega\right)t}}{\omega_{fi}-\omega}\right|^2 \tag{7.37}$$

($\mathcal{P}_{if}$ depends on the frequency of the perturbation)

If $\hat{W}_{fi}(t) = \hat{W}_{fi}\cos\omega t$,

$$\mathcal{P}_{if}(t;\omega) = \frac{\left|W_{fi}\right|^2}{4\hbar^2}\left|\frac{1-e^{i\left(\omega_{fi}+\omega\right)t}}{\omega_{fi}+\omega} + \frac{1-e^{i\left(\omega_{fi}-\omega\right)t}}{\omega_{fi}-\omega}\right|^2 \tag{7.38}$$

Constant perturbation $\omega = 0$

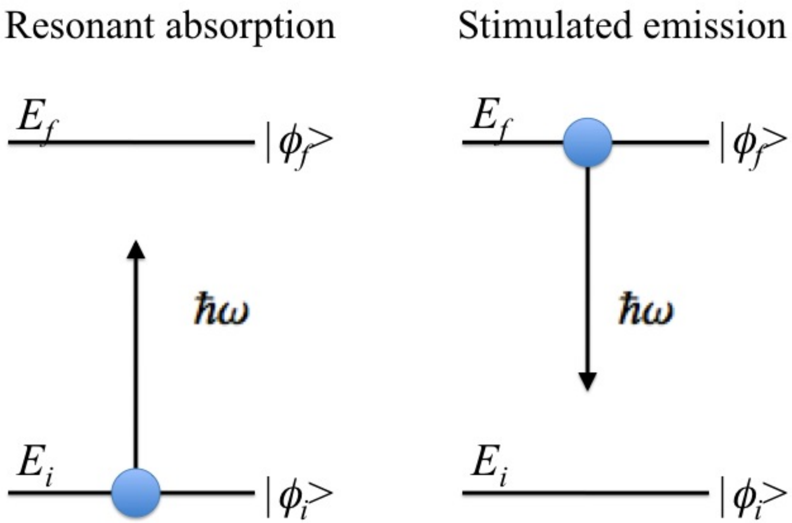
$$\mathcal{P}_{if}(t;\omega) = \frac{\left|W_{fi}\right|^2}{\hbar^2\omega_{fi}^2}\left|1 - e^{i\omega_{fi}t}\right|^2 = \frac{\left|W_{fi}\right|^2}{\hbar^2}F\left(t;\omega_{fi}\right) \tag{7.39}$$

$$F\left(t;\omega_{fi}\right) = \left[\frac{\sin\left(\omega_{fi}t/2\right)}{\omega_{fi}/2}\right]^2 \tag{7.40}$$

2. Sinusoidal perturbation which couples discrete states: resonance

a. Resonant nature of the transition probability

When t is fixed, $\mathcal{P}_{if}(t;\omega)$ is a function of one variable ω. This function has a maximum for $\omega \simeq \omega_{fi}$ or $\omega \simeq -\omega_{fi}$; this is a resonance phenomenon (choose $\omega \geq 0$)



$$\mathcal{P}_{if}(t;\omega) = \frac{\left|\hat{W}_{fi}\right|^2}{4\hbar^2}\left|\underbrace{\frac{1-e^{i\left(\omega_{fi}+\omega\right)t}}{\omega_{fi}+\omega}}_{A_+} - \underbrace{\frac{1-e^{i\left(\omega_{fi}-\omega\right)t}}{\omega_{fi}-\omega}}_{A-}\right|^2 \tag{7.41}$$

$$A_+ = -ie^{i\left(\omega_{fi}+\omega\right)t/2}\frac{\sin\left[\left(\omega_{fi}+\omega\right)t/2\right]}{\underbrace{\left(\omega_{fi}+\omega\right)/2}_{\text{goes to zero for } \omega=-\omega_{fi}}} \tag{7.42}$$

This term is anti-resonant for $\omega = \omega_{fi}$ (and resonant for $\omega = -\omega_{fi}$)

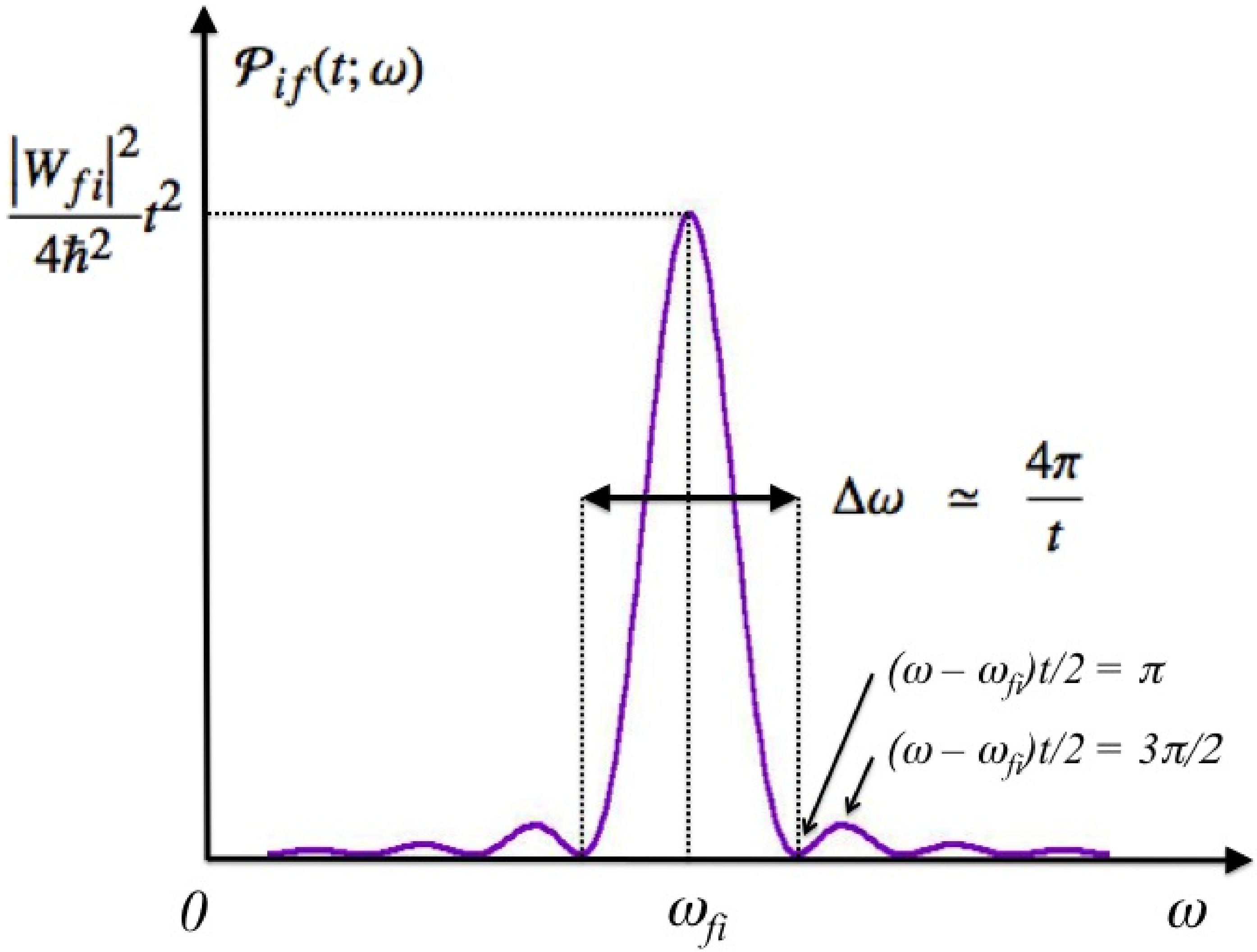
Resonant term

$$A_- = -ie^{i\left(\omega_{fi}-\omega\right)t/2}\frac{\sin\left[\left(\omega_{fi}-\omega\right)t/2\right]}{\left(\omega_{fi}-\omega\right)/2} \tag{7.43}$$

Consider the case $\left|\omega - \omega_{fi}\right| \ll \omega_{fi}$ (this is the resonant approximation):
1^{st} order transition probability:

$$\mathcal{P}_{if}(t;\omega) = \frac{\left|W_{fi}\right|^2}{4\hbar^2}F\left(t;\omega-\omega_{fi}\right) \tag{7.44}$$

$$\underbrace{F\left(t;\omega-\omega_{fi}\right)}_{\text{sinc function}} = \left\{\frac{\sin\left[\left(\omega_{fi}-\omega\right)t/2\right]}{\left(\omega_{fi}-\omega\right)/2}\right\}^2 \tag{7.45}$$



$\mathcal{P}_{if}(t;\omega)$
$\frac{|W_{fi}|^2}{4\hbar^2}t^2$
$\Delta\omega \simeq \frac{4\pi}{t}$
$(\omega - \omega_{fi})t/2 = \pi$
$(\omega - \omega_{fi})t/2 = 3\pi/2$
0
ω_{fi}
ω

b. The resonance width and time-energy uncertainty relation

The most of the resonant peak is concentrated around the resonant frequency ω_{fi}, for example at $\frac{(\omega-\omega_{fi})t}{2}=\frac{3\pi}{2}$ we get the transition probability $\frac{|W_{fi}|^2 t^2}{9\pi^2\hbar^2}$ which is approximately 5% of the transition probability at the resonance.

We can define the width of the resonant peak as the difference between the frequencies of the minima of $\mathcal{P}_{if}$ around the resonant frequency, see the figure, then

$$\Delta\omega \simeq \frac{4\pi}{t} \tag{7.46}$$

which is analogous to the time-energy uncertainty relation $\Delta E=\hbar\Delta\omega\simeq\frac{\hbar}{t}$

c. Validity of the perturbation treatment

a) Discussion of the resonant approximation

A_+ has been neglected relative to A_-:

$|A_-(\omega)|^2$ sinc function

$$|A_+(\omega)|^2 = |A_-(-\omega)|^2 \ll \left|A_-\left(\omega_{fi}\right)\right|^2 \tag{7.47}$$

The resonant approximation is justified on the condition

$$2\left|\omega_{fi}\right| >> \Delta\omega \tag{7.48}$$

that is

$$\underbrace{t}_{\text{duration of the perturbation}} >> \frac{1}{\left|\omega_{fi}\right|} \simeq \underbrace{\frac{1}{\omega}}_{\text{oscillation period}} \tag{7.49}$$

b) Limits of the first-order calculations

If t becomes too large, the first-order approximation can cease to be valid (i.e. giving infinit transition probability which is physically a nonsense):

$$\lim_{t\to\infty} \mathcal{P}_{if}\left(t;\omega=\omega_{fi}\right) = \lim_{t\to\infty} \frac{\left|W_{fi}\right|^2}{4\hbar^2} t^2 \quad = \quad \infty \tag{7.50}$$

For the first-order approximation to be valid at resonance, $\mathcal{P}_{if}(t;\omega=\omega_{fi}) \ll 1$:

$$t \quad \ll \quad \frac{\hbar}{\left|W_{fi}\right|} \tag{7.51}$$

3. Coupling with the states of the continuum

E_f belongs to a continuous part of the spectrum of $\hat{H}_0$
$\Downarrow$
We cannot measure the probability of finding the system in a well-defined state $|\varphi_f\rangle$ at time t
$\Downarrow$
We have to integrate over probability density $\left|\langle\varphi_f|\psi(t)\rangle\right|^2$ over a certain group of final states.

a. Integration over a continuum of final states; density of states

a) Example

– spinless particle of mass m

– scattering by a potential $W(\vec{r})$

$E = \vec{p}^2/2m$, $|\psi(t)\rangle$ can be expanded in terms of $|\vec{p}\rangle$

The corresponding wavefunctions are plane waves

$$\langle \vec{r}|\vec{p}\rangle = \left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{i\vec{p}\cdot\vec{r}/\hbar} \tag{7.52}$$

The probability density

$$\left|\langle \vec{p}|\psi(t)\rangle\right|^2 \tag{7.53}$$

Detector gives a signal when the particle is scattered with the momentum $\vec{p}_f$ but since it has a finite aperture it really gives the signal when the particle has momentum in a domain D_f of $\vec{p}$-space around $\vec{p}_f$ $(\delta\Omega_f, \delta E_f)$

$$\delta\mathcal{P}\left(\vec{p}_f, t\right) = \int_{\vec{p}_f \in D_f} \mathrm{d}^3\vec{p} \left|\langle\vec{p}|\psi(t)\rangle\right|^2 \tag{7.54}$$

$$\mathrm{d}^3\vec{p} = p^2\mathrm{d}p \underbrace{\mathrm{d}\Omega}_{\text{solid angle around } \vec{p}_f} = \underbrace{\rho(E)}_{\text{density of final states}} \mathrm{d}E\mathrm{d}\Omega$$

$$\rho(E) = p^2\frac{\mathrm{d}p}{\mathrm{d}E} = p^2\frac{m}{p} = m\sqrt{2mE} \tag{7.55}$$

$$\delta\mathcal{P}\left(\vec{p}_f, t\right) = \int_{\Omega\in\delta\Omega_f, E\in\delta E_f} \mathrm{d}\Omega\mathrm{d}E\,\rho(E)\left|\langle\vec{p}|\psi(t)\rangle\right|^2 \tag{7.56}$$

b) The general case

Eigenstates of $\hat{H}_0$, labeled by a continuous set of indices

$$\langle\alpha|\alpha'\rangle = \delta(\alpha-\alpha') \tag{7.57}$$

at time t: $|\psi(t)\rangle$

$$\delta\mathcal{P}\left(\alpha_f, t\right) = \int_{\alpha\in D_f} \mathrm{d}\alpha\, |\langle\alpha|\psi(t)\rangle|^2 \tag{7.58}$$

Change variables and introduce density of final states

$$\mathrm{d}\alpha = \rho(\beta, E)\mathrm{d}\beta\mathrm{d}E \tag{7.59}$$

$$\delta\mathcal{P}\left(\alpha_f, t\right) = \int_{\beta\in\delta\beta_f, E\in\delta E_f} \mathrm{d}\beta\mathrm{d}E\, \rho(\beta, E)\, |\langle\beta, E|\psi(t)\rangle|^2 \tag{7.60}$$

Fermi's Golden Rule

Let $|\psi(t)\rangle$ be the normalized state vector of the system at time t.

Consider a system which is initially in an eigenstate $|\varphi_i\rangle$ of $\hat{H}_0$ (in discrete part of spectrum)

$$\delta\mathcal{P}\left(\varphi_i, \alpha_f, t\right) = ? \tag{7.61}$$

The calculations for the case of a sinusoidal or constant perturbation remain valid when the final state of the system belongs to the continuous spectrum of $\hat{H}_0$

For W constant

$$|\langle\beta, E|\psi(t)\rangle|^2 = \frac{1}{\hbar^2}|\langle\beta, E|W|\psi(t)\rangle|^2 F\left(t; \frac{E-E_i}{\hbar}\right) \tag{7.62}$$

E – energy of the state $|\beta, E\rangle$

E_i – energy of the state $|\varphi_i\rangle$

$$\delta\mathcal{P}\left(\varphi_i, \alpha_f, t\right) = \frac{1}{\hbar^2}\int_{\beta\in\delta\beta_f, E\in\delta E_f} \mathrm{d}\beta\mathrm{d}E\,\rho(\beta, E)\,|\langle\beta, E|W|\psi(t)\rangle|^2 F\left(t; \frac{E-E_i}{\hbar}\right) \tag{7.63}$$

$F\left(t; \frac{E-E_i}{\hbar}\right)$ varies rapidly about $E = E_i$; for sufficiently large t, this function can be approximated, to within a constant factor, by the δ-fucntion $\delta\left(E - E_i\right)$:

$$\lim_{t\to\infty} F\left(t; \frac{E-E_i}{\hbar}\right) = \pi t\delta\left(\frac{E-E_i}{2\hbar}\right) = 2\pi\hbar t\delta\left(E - E_i\right) \tag{7.64}$$

The function $\rho(\beta, E)\, |\langle\beta, E|W|\psi(t)\rangle|^2$ varies much more slowly with E. We will assume that t is sufficiently large for the variation of this function over an energy interval of width $4\pi\hbar/t$ centered at $E = E_i$ to be negligible.

$\Rightarrow$ We can replace $F\left(t; \frac{E-E_i}{\hbar}\right)$ by $2\pi\hbar t\delta\,(E - E_i)$ which allows us to integrate over E immediately.

If, in addition, $\delta\beta_f$ is very small, integration over β is unnecessary and we get

(a) $E_i \in \delta E_f$

$$\delta\mathcal{P}\left(\varphi_i, \alpha_f, t\right) = \delta\beta_f \frac{2\pi}{\hbar} t \left|\langle\beta_f, E_f = E_i|W|\varphi_i\rangle\right|^2 \rho\left(\beta_f, E_f = E_i\right) \tag{7.65}$$

(b) $E_i \notin \delta E_f$

$$\delta\mathcal{P}\left(\varphi_i, \alpha_f, t\right) = 0 \tag{7.66}$$

⇒ A constant perturbation can induce transitions only between states of equal energies, and thus (b) holds.

The probability (a) increases linearly with t.
$\Rightarrow$ We can define

- transition probability per unit time $\delta\mathcal{W}\left(\varphi_i, \alpha_f\right)$

$$\delta\mathcal{W}\left(\varphi_i, \alpha_f\right) = \frac{\mathrm{d}}{\mathrm{d}t}\delta\mathcal{P}\left(\varphi_i, \alpha_f, t\right) \tag{7.67}$$

  which is time independent

- transition probability density per unit time and per unit interval of the variable β_f

$$w\left(\varphi_i, \alpha_f\right) = \frac{\delta\mathcal{W}\left(\varphi_i, \alpha_f\right)}{\delta\beta_f} \tag{7.68}$$

Fermi's Golden Rule

$$w\left(\varphi_i, \alpha_f\right) \;=\; \frac{2\pi}{\hbar}\left|\langle\beta_f, E_f = E_i|W|\varphi_i\rangle\right|^2 \rho\left(\beta_f, E_f = E_i\right) \qquad (7.69)$$

Assume that W is a sinusoidal perturbation which couples a state $|\varphi_i\rangle$ to the continuum of states $|\beta_f, E_f\rangle$ with energies E_f close to $E_i + \hbar\omega$. We can carry out the same procedure as above:

$$w\left(\varphi_i, \alpha_f\right) \;=\; \frac{\pi}{2\hbar}\left|\langle\beta_f, E_f = E_i + \hbar\omega|W|\varphi_i\rangle\right|^2 \rho\left(\beta_f, E_f = E_i + \hbar\omega\right) \qquad (7.70)$$