Quantum mechanics: Problem Set 3

November 1^{st}, 2011

1. Define the term observable in quantum mechanics. Define also the expectation value $\langle A \rangle_\psi$ of an observable A in a state ψ.

2. Show that expectation values $\langle A \rangle_\psi$ of observables are always real.

3. Calculate the commutator

$$[x^n, p]$$

where n is a positive integer and, as usual, $p = -i\hbar \frac{d}{dx}$.

4. The motion of a particle in one dimension is described by the Hamiltonian H whose normalized eigenstates $\psi_n(x),\ n = 0, 1, 2, \ldots$, satisfy the Schrodinger equation

$$H\psi_n(x) = E_n \psi_n(x)$$

with

$$\int_{-\infty^\infty} \overline{\psi_n(x)} \psi_m(x)\, dx = \delta_{m,n}$$

The particle is in a state represented by the wavefunction $\Psi(x)$ where

$$\Psi(x) = \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}} \right)^{n+1} \psi_n(x)$$

show that $\Psi(x)$ is normalised to unity and calculate the probability that the particle has energy E_n for a given n. *(Hint: think of something involving $\langle \Psi | \psi_n \rangle$)*

5. A quantum mechanical system has an energy eigenstate ψ_i of energy E_i so that

$$H\psi_i = E_i \psi_i$$

Show that

$$\psi_i(x, y, z, t) = e^{-iE_i t/\hbar} \psi_i(x, y, z, 0)$$

Hint: recall that $H\psi_i = i\hbar \frac{\partial \psi_i}{\partial t}$

6. Describe the Heisenberg γ-ray microscope and show that

$$\Delta x \Delta p \simeq h$$

7. Show that the harmonic oscillator Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$$

can be written in terms of certain operators a_+ and a_- as

$$H = \frac{1}{2m}a_+a_- + \frac{\hbar\omega}{2}I$$

where

$$[a_-, a_+] = 2m\omega\hbar I$$

8. With reference to the previous question use a_- to find the ground state ψ_0 and show that its energy E_0 is given by

$$E_0 = \frac{\hbar\omega}{2}$$

Construct also the first excited state $\psi_1(x)$.

9. An infinite square well has potential $V(x)$ given by

$$V(x) = \begin{cases} \infty & \text{if } -\infty < x \le 0 \\ 0 & \text{if } \quad 0 < x < a \\ \infty & \text{if } \quad a \le x < \infty \end{cases}$$

where a is some positive constant.

Write down the Hamiltonian H and find the associated energies E_n.

10. What is quantum mechanical tunnelling? A potential V is given by

$$V(x) = \begin{cases} 0 & \text{if } -\infty < x \le 0 \\ V_0 & \text{if } \quad 0 < x < a \\ 0 & \text{if } \quad a \le x < \infty \end{cases}$$

Define the tunnelling probability T, obtain an expression for T and show that it is, in general, non-zero.