Not due for submission, will not be marked.

The first ≈ 20 questions are repeats/twists of basic material. They should provide useful exercise or practice material for exam preparation.

The last questions require more serious calculations/understanding.

Symbols may have different meanings in different questions; do not assume continuation.

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1. The set $\{|\phi_1\rangle, |\phi_2\rangle\}$ is an orthonormal basis set for a 2-dimensional Hilbert space. The states $|W\rangle$, $|\theta_1\rangle$ and $|\theta_2\rangle$ are defined as

$$|W\rangle \; = \; \alpha\,|\phi_1\rangle + i\alpha\,|\phi_2\rangle \; ,$$
$$|\theta_1\rangle \; = \; \frac{i}{\sqrt{3}}\,|\phi_1\rangle - \frac{\sqrt{2}}{\sqrt{3}}\,|\phi_2\rangle \; , \quad |\theta_2\rangle \; = \; |\phi_1\rangle \; .$$

   (a) Find the constant α such that $|W\rangle$ is normalized. Do not assume α to be real or imaginary; keep your answer general.

   (b) Calculate the inner product $\langle\theta_1|\theta_2\rangle$.

   (c) Is the set $\{|\theta_1\rangle, |\theta_2\rangle\}$ an orthonormal set? Explain why, or why not.

   (d) Using the representation

$$|\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \qquad |\phi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} ,$$

   express $|\theta_1\rangle$ and $|\theta_2\rangle$ as column vectors. Also write down $\langle\theta_1|$ and $\langle\theta_2|$ as row vectors.

   (e) Using the same representation, express the operator $\hat{M} = |\theta_1\rangle\langle\theta_2|$ as a matrix.
   Determine whether or not $\hat{M}$ is hermitian. Make sure you learn to tell whether a matrix is hermition simply by looking at it.
   Also determine whether or not $\hat{M}$ is unitary.

   (f) Calculate the expectation value of $\hat{B} = |\phi_2\rangle\langle\phi_2|$ in the state $|\theta_1\rangle$.
   Calculate the uncertainty of the observable B (represented by operator $\hat{B}$) in the state $|\theta_1\rangle$.

   (g) Calculate the expectation value and the uncertainty of B (defined above) in the state $|\theta_2\rangle$.

2. (a) The function $\phi_1(x)$ is found to be an eigenstate of the Hamiltonian for a single particle on a line, with the corresponding energy being ϵ_1. Show that

$$f(x,t) \;=\; \phi_1(x)e^{-i\epsilon_1 t/\hbar}$$

is a solution of the corresponding time-dependent Schroedinger equation.

(b) Another eigenstate $\phi_2(x)$ corresponds to eigenenergy ϵ_2. Show that

$$g(x,t) \;=\; \phi_1(x)e^{-i\epsilon_1 t/\hbar} \;+\; \phi_2(x)e^{-i\epsilon_2 t/\hbar}$$

is a solution of the corresponding TDSE.

3. Complex numbers and functions.

(a) Assuming k_1, k_2 to be real, explain the difference between $|k_1 + ik_2|^2$ and $(k_1 + ik_2)^2$. Is there ever a situation where these two are equal? Is there any situation for which you could write $|ik_2|^2 = -k_2^2$?

(b) Express e^{ikx} in terms of $\cos(kx)$ and $\sin(kx)$.
Hint: in case this is not familiar, please look up "complex exponential" or "Eulers formula".

(c) Express $\cos(kx)$ in terms of e^{ikx} and e^{-ikx}.

(d) If k is a real positive constant, is e^{ikx} a periodic function of x? If so, what is the period?

(e) Assuming k to be a real positive constant, is e^{ikx} an odd function of x? Is it an even function of x?

(f) If λ is a real number, what is the norm/magnitude of $e^{i\lambda}$?

(g) If $\lambda = \lambda_1 + i\lambda_2$ is a complex number with real part λ_1 and imaginary part λ_2, what is the norm/magnitude of $e^{i\lambda}$?

(h) If $|\alpha|^2 = 4$, can I infer that $\alpha = 2$ necessarily? Introduce a phase factor to write the general solution for α. What is a 'phase factor'? Draw the curve in the complex α plane which represents all possible solutions for α.

4. Consider a particle of mass m on a one-dimensional line.

   (a) Write down the operators for momentum and for kinetic energy.

   (b) Is the function e^{ikx} an eigenfunction of the momentum operator? If so, what is the eigenvalue?

   Answer the same questions for the functions $-e^{ikx}$, C_1e^{ikx}, C_2e^{-ikx}, where C_1 and C_2 are real or complex constants.

   Answer the same questions for the functions $e^{ikx} + e^{-ikx}$ and $C_1e^{ikx} + C_2e^{-ikx}$. What goes wrong in these cases?

   (c) Show that the functions $\cos(kx)$ and $\sin(kx)$ are not eigenfunctions of the momentum operator.

   Find a linear combination of $\cos(kx)$ and $\sin(kx)$ which is an eigenfunction of the momentum operator.

   (d) Find out whether the functions $\cos(kx)$, $\sin(kx)$, e^{ikx}, e^{ikx} are eigenfunctions of the kinetic energy operator. If so, find the corresponding eigenvalues.

5. An operator $\hat{A}$ acts on the space of functions of a real variable, i.e., on the space of possible wavefunctions of a single particle in one dimension.

   (a) If $f(x)$ is an eigenfunction of $\hat{A}$, is $-f(x)$ also an eigenfunction? If so, does it correspond to the same eigenvalue?

   (b) If $f(x)$ is an eigenfunction of $\hat{A}$, is $Cf(x)$ also an eigenfunction? Here C is a complex constant. If $Cf(x)$ is an eigenfunction as well, does it correspond to the same eigenvalue?

   (c) If $f(x)$ and $g(x)$ are eigenfunctions of $\hat{A}$ corresponding to the *same* eigenvalue, show that any linear combination $C_1f(x) + C_2g(x)$ is also an eigenfunction of $\hat{A}$.

   (d) If $f(x)$ and $g(x)$ are eigenfunctions of $\hat{A}$ corresponding to *different* eigenvalues, find out whether any linear combination $C_1f(x)+C_2g(x)$ is also an eigenfunction of $\hat{A}$.

6. Consider a single particle in one dimension, subject to a constant (space-independent) potential $V(x) = V_0$.

   (a) Write down the time-independent Schroedinger equation.

   (b) Show that $\phi_1(x) = e^{ikx}$ is a solution to this equation. Determine the energy E corresponding to this equation, in terms of k and V_0.

   (c) Is the function $\phi_2(x) = C\phi_1(x) = Ce^{ikx}$, where C is a constant, also a solution?

   (d) What is a solution of the form e^{ikx} called? Is it right-moving or left-moving? Explain why.

   (e) Show that $\phi_3(x) = \cos(kx)$ is a solution to the time-independent Schroedinger equation. Write down the corresponding solution to the time-dependent Schroedinger equation.

   (f) Show that $\phi_4(x) = C_1\cos(kx) + C_2\sin(kx)$ is also an eigenvalue of the Hamiltonian, where C_1 and C_2 are any conplex constants.

   (g) Find out if $\phi_5(x) = C_1\cos(k_1x) + C_2\sin(k_2x)$ is an eigenvalue, if $k_1 \neq k_2$.

   (h) I have used above the phrases "solution to the time-independent Schroedinger equation" and "eigenvalue of the Hamiltonian". Explain why these mean the same thing.

7. Consider a single particle in one dimension, subject to a constant potential $V(x) = V_0$ in some region.

   (a) Write down the time-independent Schroedinger equation for this region.

   (b) Assuming $E > V_0$, write down the general solution for an energy eigenstate. If you introduce constants, relate them to E and V_0 if possible.

   (c) Write down the general solution for an energy eigenstate corresponding to $E < V_0$. Again, if any constants are related to E and V_0, write down the relationships explicity.

8. Consider a system with a 4-dimensional Hilbert space. We are given the set

$$|w_1\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad |w_2\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \qquad |w_3\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \qquad |w_4\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix},$$

(a) Is the set of four vectors $\{|w_i\rangle\}$ orthonormal?

(b) Does this set span the Hilbert space?

(c) Find the expectation value and uncertainty of the physical quantity represented by the operator $\hat{P}_1 = |w_2\rangle\langle w_2|$ in the state

$$|\theta_3\rangle = -\frac{i}{2}|w_1\rangle - \frac{1}{2}|w_1\rangle + \frac{i}{2}|w_1\rangle + \frac{1}{2}|w_4\rangle$$

Do this in two different ways: first, keeping all operators and states in terms of $|w_i\rangle$'s and $\langle w_j|$'s, and second, expressing operators/states as matrices/vectors.

Which do you find less cumbersome?

(d) In the same state $|\theta_3\rangle$, find the expectation value and then the uncertainty of

$$\hat{P}_2 = (|w_1\rangle + |w_2\rangle)(\langle w_1| + \langle w_2|)$$

9. The operator $\hat{Y}$ has eigenvalues y_n and eigenvectors $|w_n\rangle$:

$$\hat{Y}|w_n\rangle = y_n|w_n\rangle\,; \qquad n = 1, 2, 3, ..., 6$$

(a) If the corresponding observable Y is measured, what are the possible results?

(b) The system is prepared in the state

$$|\psi\rangle = \frac{1}{2}|w_1\rangle + \frac{i}{2}|w_3\rangle - \frac{i}{\sqrt{2}}|w_5\rangle$$

and a measurement of Y is performed. What are the possible results, and with what probability do they occur?

(c) What is the dimension of the Hilbert space?

10. Consider a single particle in the one-dimensional harmonic potential $V(x) = \frac{1}{2}m\omega^2 x^2$. Consider the wavefunction $\phi(x) = Axe^{-x^2/2\sigma^2}$.

    (a) Write down the time-independent Schroedinger equation.

    (b) Show that the wavefunction $\phi(x)$ is a solution of this equation, i.e., is a eigenfunction of the Hamiltonian, only if σ has a certain value. Find this value of σ.

    (c) What is the energy corresponding to this eigenfunction?

    (d) Based on the number of nodes in the wavefunction, argue whether this is the ground state, the first excited state, or the second excited state.

11. Consider a particle of mass m on a one-dimensional line. Two stationary states, $\psi_1(x)$ and $\psi_2(x)$, form an orthonormal set.

    (a) Express what orthonormal means for this set, in terms of three integral equations.

    (b) Show that, if the state $\alpha\psi_1(x)+\beta\psi_2(x)$ is normalized, then $|\alpha|^2+|\beta|^2 = 1$.

12. For a particle in one dimension, compute the commutator $[\hat{x}^2, \hat{p}^2]$. Express your result in terms of the operators $\hat{1}$, $\hat{x}$, $\hat{p}$.

    What does $\hat{1}$ mean here?

13. An operator $\hat{A}$, in an appropriate basis, is represented by the matrix $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

    (a) Show that $\hat{A}$ is hermitian.

    (b) Show that $\hat{A}$ is unitary.

    (c) Verify that the eigenvalues of $\hat{A}$ are of unit magnitude.

    (d) Find the normalized eigenvectors of $\hat{A}$.

14. Consider a particle in a one-dimensional harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2 x^2$. We denote the orthonormalized energy eigenstates of the harmonic oscillator by $|n\rangle$, on which the creation/annihilation operators act as follows:

$$\hat{a}\,|n\rangle \;=\; \sqrt{n}\,|n-1\rangle \qquad \hat{a}^\dagger\,|n\rangle \;=\; \sqrt{n+1}\,|n+1\rangle$$

The ladder operators are defined as

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \quad , \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$$

(a) What are the dimensions of $\sqrt{\hbar/mw}$? Introduce a length scale σ for the quantum harmonic oscillator. What is the appropriate momentum scale for this problem?

(b) Calculate the commutator between the ladder operators.

(c) Are the ladder operators hermitian? Explain why or why not.

(d) Express $\hat{x}$ and $\hat{p}$ in terms of the ladder operators.

(e) Show that

$$\langle m|\,\hat{a}\,|n\rangle \;=\; \sqrt{n}\,\delta_{m,n-1} \;=\; \sqrt{m+1}\,\delta_{m+1,n}$$

and derive similar results for $\langle m|\,\hat{a}^\dagger\,|n\rangle$.

(f) Show that

$$\langle m|\,\hat{x}\,|n\rangle \;=\; \sigma\left[\sqrt{\frac{n+1}{2}}\delta_{m,n+1} + \sqrt{\frac{n}{2}}\delta_{m,n-1}\right]$$

and derive a similar result for $\langle m|\,\hat{p}\,|n\rangle$.

15. Recall the commutation relations between the ladder operators for a harmonic oscillator: $[\hat{a}, \hat{a}^\dagger] = 1$. It appears that the left side of this equation is an operator, while the right side is a number. Can this be meaningful? What is really meant by 1 on the right side?

Similarly, recall the commutation relation $[\hat{x}, \hat{p}^\dagger] = i\hbar$. The left side is an operator, while the right side seems to be a complex number. How can this make sense?

16. Consider the wavefunction of a single particle

$$\phi(x) \;=\; Ne^{-|x|/\lambda} \;=\; \begin{cases} Ne^{-x/\lambda} & \text{for } x > 0 \\ Ne^{+x/\lambda} & \text{for } x < 0 \end{cases}$$

(a) Calculate the constant N so that the wavefunction is normalized.

(b) Calculate the expectation value of the position in this state. (Guess the value before performing the calculation.)

(c) Calculate the uncertainty of the position in this state. Before performing the calculation, guess the λ-dependence of the uncertainty.

(d) Calculate the expectation value of the momentum in this state. Guess the value before performing the calculation.

(e) Calculate the uncertainty of the momentum in this state. Before performing the calculation, guess the λ-dependence of the uncertainty.

(f) Calculate the probablity of finding the particle inside the region $x \in [-\lambda, \lambda]$. Calculate the probablity of finding the particle inside the region $x \in [-2\lambda, 2\lambda]$.

(g) You should recognize the wavefunction as the ground state of a particular potential. Which potential?

17. Momentum-like operators for a single particle in 1D.

(a) Show that the momentum operator is hermitian. State clearly any assumptions you might make about the space of functions on which the momentum operator acts.

(b) Show that the operator $\hat{N} = \frac{d}{dx}$ is not hermitian, and that $\hat{N}^\dagger = -N$. Do you require conditions on the space of allowed functions?

(c) Show that, for operators satisfying $\hat{A}^\dagger = -\hat{A}$ (known as anti-hermitian operators), the eigenvalues are purely imaginary.

(d) The operator $\hat{N}$ seems very similar to the momentum operator $\hat{p}$, which is hermitian and has real eigenvalues. What leads to this drastic difference between $\hat{p}$ and $\hat{N}$?

(e) Applying the operator $\hat{N}$ to the function $e^{\alpha x}$, where α is real, it might seem that this is an eigenfunction of $\hat{N}$ with a real eigenvalue. But that would contradict the previous results! Does the function $e^{\alpha x}$ violate the condition/assumption required to prove the anti-hermiticity of $\hat{N}$?

(f) Check whether the operator $\hat{N}^2 = \frac{d^2}{dx^2}$ is hermitian or not. What does this imply about the operator for kinetic energy?

18. Hermitian operators.

(a) Show that the hermitian conjugate of $\hat{A}\hat{B}$ is $\hat{B}^\dagger\hat{A}^\dagger$.

(b) Show that the sum of two hermitian operators is hermitian.

(c) If $\hat{M}$ is hermitian, show that so is $\hat{M}^2$.

(d) If $\hat{M}$ is hermitian and α is a real number, show that $\hat{P} = e^{\alpha\hat{M}}$ is also hermitian. Recall that the exponential of an operator is defined in terms of a power series.

(e) If $\hat{M}$ is hermitian and α is a real number, show that $\hat{Q} = e^{i\alpha\hat{M}}$ is unitary.

19. The generalized uncertainty relation states that, if two observables A and B are described by the operators $\hat{A}$ and $\hat{B}$, then

$$\Delta A \Delta B \;\geq\; \left|\frac{i}{2}\langle[\hat{A},\hat{B}]\rangle\right| \qquad \text{where} \begin{cases} \Delta A = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2} \\ \text{and similarly for } \Delta B \end{cases}$$

with all expectation values taken in the same state.

Let's apply this to the case of one particle in one dimension.

(a) Apply this to the case where the observables are position and momentum. You should obtain the Heisenberg uncertainty relation.

(b) Apply this to the case where the observables are position and kinetic energy. You may need to calculate the commutation relation between position and kinetic energy.

(c) Apply this to the case where the observables are momentum and kinetic energy.

20. Let's use the convention that we use symbols with hats to denote operators, symbols without hats to denote numbers, and the usual ket and bra to denote states and their duals. Which of the following statements could be meaningful, and which are nonsensical? Explain why in each case.

(a) $\hat{B} = \alpha_1 |\phi_1\rangle + \alpha_2 |\phi_2\rangle$

(b) $|\theta\rangle\langle\chi| = \hat{A}$

(c) $\langle\phi|\chi\rangle = \hat{A}$

(d) $|\theta\rangle\langle\chi| = c_1$

(e) $\langle\phi|\chi\rangle = c_2$

(f) $|\theta\rangle\langle\phi|\chi\rangle = \hat{Y}$

(g) $|\theta\rangle\langle\phi|\chi\rangle = c_3$

(h) $\hat{B} = \langle\chi|\hat{Y}|\theta\rangle$

(i) $\hat{B} = \langle\chi|\hat{B}|\theta\rangle$

(j) $\langle\chi|\hat{Y}|\theta\rangle = c_4$

(k) $|\theta\rangle = |\phi\rangle$

(l) $\langle\theta| = |\phi\rangle$

(m) $|\theta\rangle\langle\phi|\chi_1\rangle + |\chi_2\rangle = c_5 |\chi_3\rangle$

(n) $\hat{B}|\theta\rangle = \langle\chi|\hat{Y}|\theta\rangle$

(o) $\hat{B}|\theta\rangle = c_6$

(p) $\hat{B}|\theta\rangle = \langle\chi|$

(q) $\hat{B}|\theta\rangle = \hat{Y}$

(r) $\hat{B}|\theta\rangle = d_1 |\theta\rangle$

21. Consider two particles confined to a single line (one dimension), so that the wavefunctions of the system are of the form $\psi(x_A, x_B, t)$, where x_A and x_B are the coordinates of the two particles. The time-independent Schroedinger equation

$$\hat{H}\psi(x_A, x_B) = E\psi(x_A, x_B)$$

is found to have eigenstates $w_i(x_A, x_B)$ and corresponding eigenvalues E_i, with $i = 1, 2, 3, \ldots$. Show that

$$f(x,t) = C_1 w_1(x_A, x_B) e^{-iE_1 t/\hbar} + C_2 w_2(x_A, x_B) e^{-iE_2 t/\hbar}$$

is a solution of the corresponding time-dependent Schroedinger equation.

Hint: it might make sense to write down the time-dependent Schroedinger equation first.

22. Consider the wavefunction of the two-state system

$$|\psi(t)\rangle = \alpha(t)\begin{pmatrix}1\\0\end{pmatrix} + \beta(t)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\alpha(t)\\\beta(t)\end{pmatrix}.$$

If the Hamiltonian of the system is

$$\hat{H} = \begin{pmatrix} E & -\epsilon \\ -\epsilon & E \end{pmatrix}$$

use the time-dependent Schroedinger equation to find the probability amplitudes $\alpha(t)$ and $\beta(t)$ as functions of time. The initial conditions are $\alpha(0) = 1$, $\beta(0) = 0$.

Comment: a system with Hilbert space of dimension two is often called a two-state system.

23. Consider a single particle in one dimension, subject to the negative delta function potential

$$V(x) = -\lambda\delta(x) .$$

We considered this potential in Problem Set 8, and calculated the bound state. Now investigate the scattering properties of this potential, i.e., consider solutions with energy $E > 0$.

(a) Write down plane wave solutions for the left and right half-line. Identify terms representing incident, reflected and transmitted waves.

(b) Use the two boundary conditions, one for the continuity of the wavefunction and one for the discontinuity of its derivatves, derived in problem set 8.

(c) Using the two conditions, find the reflection and transmission coefficients. Check that their sum makes sense.

(d) You should be able to compare your results with that in the wikipedia page on the delta potential. The problem is probably worked out in some textbooks as well.

24. Ehrenfest's theorem. Generalized to any operator $\hat{A}$, the statement is

$$\frac{d}{dt}\langle\hat{A}\rangle = \langle[\hat{H}, \hat{A}]\rangle$$

provided that the operator $\hat{A}$ itself has no time-dependence, i.e., all the time-dependence comes from the state.

(a) Consider a single particle in one dimension, subject to no potential (free particle). Apply the relation above to the position operator. Interpret classically the equation you obtain.

(b) Now add an arbitrary potential $V(x)$ which is a function of x. Repeat the calculation to find the time derivative of $\langle \hat{x} \rangle$ in this case.

(c) Apply the relation to the momentum operator, in order to find the time derivative of $\langle \hat{p} \rangle$, Do this first for the free particle, and then for the case of an arbitrary potential $V(x)$. If you need to calculate the commutator $[V(\hat{x}), \hat{p}] = [V(x), \hat{p}]$, try applying this operator on an arbitrary function of x.

Interpret classically the equation of motion you obtain for $\langle \hat{p} \rangle$.

(d) Consider a spin-1/2 system, subject to the Hamiltonan $\hat{H} = -B\hat{S}_x$. This represents a magnetic field in the x-direction. Use the Ehrenfest relation to calculate the time dependence of $\langle \hat{S}_z \rangle$.

(e) Prove the general relation quoted at the beginning of this problem. You will get time derivatives of a ket and of a bra, which you can replace using the Schroedinger equation and its adjoint.

(f) Derive the equation of motion for $\langle \hat{x} \rangle$ that you obtained in part (24b) for a particle in one dimension, by taking the time derivative of $\int dx \psi^* x \psi$ and using the Schroedinger equation and its conjugate.

25. Consider a single particle of mass m in one dimension subject to no potential, i.e., a free particle. Imagine starting with the initial wavefunction

$$\psi(x, 0) \;=\; \frac{1}{(2\pi b^2)^{1/4}} \exp\left[-\frac{x^2}{4b^2}\right]$$

so that the initial probablity density $|\psi(x,0)|^2$ is a gaussian of width b centered around the origin.

How will this wavefunction evolve? I read somewhere:

$$\psi(x, t) \;=\; \frac{1}{(2\pi b^2)^{1/4}} \frac{1}{\sqrt{1 + i\left(\hbar t/2mb^2\right)}} \exp\left[-\frac{x^2}{4b^2\left[1 + i\left(\hbar t/2mb^2\right)\right]}\right]$$

(a) Check that the claimed $\psi(x,t)$ indeed is a solution of the TDSE of the free particle in one dimension. I may have made a mistake translating all these factors; if so, please correct.

(b) Check that the wavefunction $\psi(x,t)$ is normalized.

(c) Show that the probability density remains of gaussian form; however it's width changes. How does its width change? Plot the width as a function of time.

(d) What determines the speed at which the width changes (spreading speed)? Would an initially narrower wavepacket spread out faster or slower? Do you have an intuitive explanation?

(e) Why does the wavefunction evolve at all? Did we not start with the eigenstate of an harmonic oscillator? Don't eigenstates have simple time dependence (given by a phase factor)?

(f) Guess (explaining physically) and then calculate the expectation value of the position. How does this expectation value evolve with time?

(g) Guess (explaining physically) and then calculate the uncertainty of the position (Δx). How does this uncertainty evolve with time?

(h) Guess the expectation value and uncertainty of the momentum, up to a constant. How do you expect these to evolve with time?
Calculate these quantities.

(i) Calculate the probability density and the probability current density. Use these to check whether the continuity equation is satisfied.

(j) You can try looking up the derivation of $\psi(x,t)$. Google "gaussian wavepacket evolution" and you should find a couple of derivations. One standard way to derive this is to Fourier transform and then use the fact that each momentum eigenstate is an eigenstate of the free particle Hamiltonian.

(k) Consider giving the initial wavefunction a momentum boost $\hbar k$. What factor would you have to multiply with? You should be able to guess from physical considerations how $\psi(x,t)$ is changed when you change $\psi(x,0)$ in this way. If the wavefunction initially has a momentum, do you expect it to remain centered around the same position?

26. The generalized uncertainty relation states that, if two observables A and B are described by the operators $\hat{A}$ and $\hat{B}$, then

$$\Delta A \Delta B \;\geq\; \left| \frac{i}{2} \langle [\hat{A}, \hat{B}] \rangle \right| \qquad \text{where } \begin{cases} \Delta A = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2} \\ \text{and similarly for } \Delta B \end{cases}$$

with all expectation values taken in the same state.

Let's check this for the case of a spin-1/2 system. Recall that the spin operators can be obtained from the Pauli matrices as $\hat{S}_x = \frac{\hbar}{2}\sigma_x$, $\hat{S}_y = \frac{\hbar}{2}\sigma_y$, $\hat{S}_z = \frac{\hbar}{2}\sigma_z$.

(a) Using the commutation relation $[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$, the generalized uncertainty principle implies that we should obtain

$$\Delta S_x \Delta S_y \geq \left| \frac{i}{2} \langle \hat{S}_z \rangle \right|$$

for any state.

Is this true? Let's check for a relatively general state: $\psi_1 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, with real α and β. This is not the most general state, since we could have complex components in general, but this will be tediuos enough. Also, note that α and β are related by normalization and hence are not independent.

By calculating ΔS_x, ΔS_y and $\langle \hat{S}_z \rangle$ for this state, show that the above relation is satisfied.

(b) Next, show that the same uncertainty relation is satisfied in the state $\psi_1 = \begin{pmatrix} \alpha \\ i\beta \end{pmatrix}$, with real α and β.

(c) Do you find the *equality* to be satisfied in each case? Can you think up a state where the uncertainty product is actually larger than the minimum value imposed by the relation? Or is that not possible?

(d) Do a similar check for the uncertainty product of $\hat{S}_y$ and $\hat{S}_z$.

27. Consider the double step potential

$$V(x) = \begin{cases} 0 & \text{for } x < 0 \\ V_1 & \text{for } 0 \geq x < L \\ V_2 & \text{for } x > L \end{cases}$$

where $0 < V_1 < V_2$.

(a) Calculate the transmission and reflection coefficients, when the incident wave impinges from the left with energy $E > V_2$. Make sure you check that the sum of your calculated coefficients make sense.

(b) Why do you have to watch out for momentum factors when calculating the transmission coeffiecient, but not when you are calculating the reflection coefficients?

(c) Calculate the transmission and reflection coefficients, when the incident wave impinges from the right with energy $E > V_2$. Do you get the same coefficients as the case where incidence is from the left?

28. Angular momentum commutation relations.

From classical mechanics we know that $\vec{L} = \vec{r} \times \vec{p}$, or, in terms of components

$$L_x = yp_z - zp_y \quad \text{and similar for } L_y \text{ and } L_z.$$

The angular momentum operators in quantum mechanics are defined by using the operators for position and momentum components in the above relations:

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \quad \text{and similar for } \hat{L}_y \text{ and } \hat{L}_z.$$

(a) Write down the operators for $\hat{x}$, $\hat{y}$, $\hat{z}$, $\hat{p}_x$, $\hat{p}_y$, $\hat{p}_z$.
In class, where we focused on one dimension, we were loose with the notation for partial derivatives and full derivatives. Is there a reason to prefer one of the two, in the present situation?

(b) Calculate the commutators between $\hat{L}_x$ and $\hat{L}_y$ and express it in terms of $\hat{L}_z$. Note that x, y, z may be regarded as independent variables.

(c) Do you see a similarity with the commutation relations for a spin-1/2 system?

29. Consider a one-dimensional simple harmonic oscillator. Do the following algebraically using the ladder operators, i.e., without using wavefunctions expressed as a function of x. The eigenstates are denoted as $|n\rangle$ as usual.

(a) Construct a linear combination of $|0\rangle$ and $|1\rangle$ and tune the coefficients so that $\langle\hat{x}\rangle$ is as large as possible.

(b) Suppose the oscillator starts in the state you constructed, at time $t = 0$. What is the state at a later time t?

(c) Evaluate the expectation value $\langle\hat{x}\rangle$ as a function of time.

(d) Evaluate the uncertainty of position, Δx, as a function of time.

(e) Contrast the behavior (motion) you have found, with the behavior of a classical harmonic oscillator.

(f) Evaluate the expectation value of momentum, $\langle \hat{p} \rangle$, in the initial state and as a function of time.

Check that your $\langle \hat{x} \rangle$ and $\langle \hat{p} \rangle$ satisfy Ehrenfest's theorem. (Look up Ehrenfest's theorem if needed.)

(g) What is the maximum value of $\langle \hat{x} \rangle$ that one can get out of a linear combination of $|0\rangle$ and $|1\rangle$? Do the coefficients in the linear combination need to be complex (have nonzero imaginary parts)? If you tune the initial state to have a smaller nonzero $\langle \hat{x} \rangle$, what would the time dependence of $\langle \hat{x} \rangle$ be?