* The QUANTUM HARMONIC OSCILLATOR

Potential $V(x) = \frac{1}{2} m\omega^2 x^2$ $\left[= \frac{1}{2} k x^2 \text{ used in classical mech.} \right]$

Fundamental in both classical & quantum mech. For many realistic confining potentials, $\frac{1}{2} m\omega^2 x^2$ is a good approximation to the low-energy part.

Time-independent Schrödinger equation:

$$\left[\frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \right] \psi(x) = E\psi(x)$$

or: $$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + \frac{1}{2} m\omega^2 x^2 \psi = E\psi$$

* Parameters $\hbar$, m, ω can be combined to form a "length scale": $\sigma = \sqrt{\frac{\hbar}{m\omega}}$

$\boxed{\text{Dimensions: } \left[\frac{\hbar}{m\omega}\right] = \frac{ML^2T^{-1}}{M\cdot T^{-1}} = L^2}$

* We will show (partially) that the eigenvalues/functions are:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \quad [n = 0, 1, 2, 3, \ldots]$$

$$\phi_n(x) = \frac{(\pi\sigma^2)^{-1/4}}{\sqrt{2^n n!}} H_n\left(\frac{x}{\sigma}\right) e^{-x^2/2\sigma^2}$$

σ is the "width" of the gaussian.

$H_n(z)$ are the Hermite polynomials :

$$\begin{cases} H_0(z) = 1 \\ H_1(z) = 2z \\ H_2(z) = 4z^2 - 2 \\ H_3(z) = 8z^3 - 12z \\ H_4(z) = 16z^4 - 48z^2 + 12 \\ \ldots \end{cases}$$

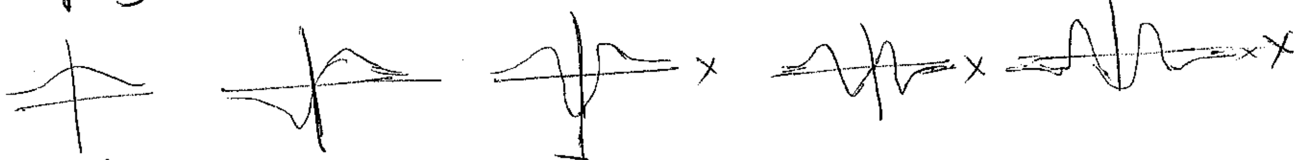
$H_n(z)$ is a polynomial of order n.

$H_0(z), H_2(z), H_4(z), \ldots$ are even functions of z : $H_{2i}(-z) = H_{2i}(z)$

$H_1(z), H_3(z), H_5(z), \ldots$ are odd functions of z.

$$\boxed{H_n(z) = (-1)^n e^{z^2/2} \frac{d^n}{dz^n}\left(e^{-z^2/2}\right)}$$

* $\Phi_n(x) = \text{const.}$ polynomial $\cdot e^{-x^2/2\sigma^2}$ decays at large $|x|$. (why?)



* [n-th order polynomial] $H_n(z)$ has n zeros, i.e., n values of x where $H_n(x/\sigma)$ is zero.

Hence $\Phi_n(x)$ has n zeros / NODES.

Higher-energy eigenfunctions $\equiv$ more nodes.

* Common notation : $|n\rangle = \Phi_n(x)$

$$\hat{H} \cdot |n\rangle = E_n |n\rangle = \left(n + \tfrac{1}{2}\right)\hbar\omega\, |n\rangle$$

$\boxed{n = 0, 1, 2, 3, \ldots \text{ 0 is included}}$

Eg., $\hat{H}|0\rangle = \frac{1}{2}\hbar\omega|0\rangle$, $\hat{H}|1\rangle = \frac{3}{2}\hbar\omega|1\rangle$

* ODE : $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2x^2\psi=E\psi$

or, with $u=\frac{x}{\sigma}=\frac{x}{\sqrt{\hbar/m\omega}}$, $\boxed{-\frac{d^2\phi}{du^2}+u^2\phi=\frac{E}{\hbar\omega/2}\phi}$

Can be solved ~~...~~ using "power series ~~...~~ method" or algebraically using operators.

* Algebraic solution, using lowering/raising operators

Define $\hat{a}=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right)$ lowering operator

$\hat{a}^\dagger=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}-\frac{i}{m\omega}\hat{p}\right)$ raising operator = adjoint/Hermitian conj of $\hat{a}$

Notation: Sometimes a^- and a^+, or a_- and a_+.

Then $\hat{x}=\frac{\sigma}{\sqrt{2}}\left(\hat{a}^\dagger+\hat{a}\right)=\sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger+\hat{a}\right)$

$\hat{p}=i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}^\dagger-\hat{a}\right)=\frac{i}{\sqrt{2}}\left(\frac{\hbar}{\sigma}\right)\left(\hat{a}^\dagger-\hat{a}\right)$

Constants look horrible but are only there for dimensional reasons. E.g., define $\hat{z}=\frac{\hat{x}}{\sigma}$, $\hat{\pi}=\frac{\hat{p}}{\hbar/\sigma}$, then $\hat{a}=\frac{1}{\sqrt{2}}\left(\hat{z}+i\hat{\pi}\right)$, $\hat{a}^\dagger=\frac{1}{\sqrt{2}}\left(\hat{z}-i\hat{\pi}\right)$

(77) * Define $\boxed{\hat{N} = a^\dagger a = \frac{\hat{p}^2}{2m\hbar\omega} + \frac{m\omega}{2\hbar}\hat{x}^2 - \frac{1}{2}}$

We can show (using $[\hat{x},\hat{p}] = i\hbar$), that

$$[a, a^\dagger] = 1, \quad [\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger, \quad [\hat{N}, \hat{a}] = -\hat{a}$$

$$\frac{1}{2\hbar}\left(i[\hat{p},\hat{x}] - i[\hat{x},\hat{p}]\right)$$

Also, since $\hat{a}^\dagger\hat{a} = \frac{1}{\hbar\omega}\left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right) - \frac{1}{2}$

$$= \frac{1}{\hbar\omega}\hat{H} - \frac{1}{2},$$

$$\boxed{\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)}$$

Also $[\hat{H}, \hat{a}^\dagger] = \hbar\omega[\hat{a}^\dagger\hat{a}, \hat{a}^\dagger] + \hbar\omega[\frac{1}{2}, \hat{a}^\dagger]$

$$= \hbar\omega\hat{a}^\dagger + 0$$

If $|n\rangle$ is an eigenstate with eigen-energy E_n,

then $\hat{H}|n\rangle = E_n|n\rangle$

$$\Rightarrow \hat{H}\hat{a}^\dagger|n\rangle = \left(\hat{H}\hat{a}^\dagger - \hat{a}^\dagger\hat{H}\right)|n\rangle + \hat{a}^\dagger\hat{H}|n\rangle$$

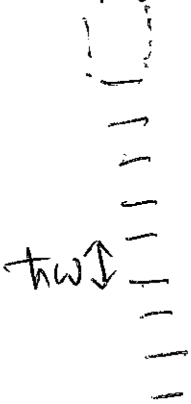
$$= \hbar\omega\hat{a}^\dagger|n\rangle + \hat{a}^\dagger E_n|n\rangle$$

$$\hat{H}(\hat{a}^\dagger|n\rangle) = (\hbar\omega + E_n)(\hat{a}^\dagger|n\rangle)$$

Thus $\hat{a}^\dagger|n\rangle$ is an eigenstate also, with eigenvalue $E_n + \hbar\omega$

Similarly, can show that $\hat{a}|n\rangle$ is an eigenstate also, with eigenvalue $E_n - \hbar\omega$.

Thus, the eigenvalues are arranged in a "ladder" with separation $\hbar\omega$.



There must be a lowest state, since $E_n < 0$ is not possible.

(We have found solution with $E < V$ in some regions, but here $E < 0$ would mean $E < V(x)$ <u>everywhere</u>.)

Let's call the lowest state $\varphi_0(x) = |0\rangle$

Since this state cannot be "lowered" to a lower-energy state, we must have $\hat{a}\,\varphi_0(x) = 0$

or $\hat{a}|0\rangle = 0$

This allows us to find E_0 and $\varphi_0(x)$:

$$\hat{H}|0\rangle = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\right)|0\rangle = \hbar\omega\,\hat{a}^\dagger\hat{a}|0\rangle + \frac{\hbar\omega}{2}|0\rangle$$

$$= 0 + \tfrac{1}{2}\hbar\omega|0\rangle$$

$$\Rightarrow \boxed{E_0 = \tfrac{1}{2}\hbar\omega}$$

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Thus the energies are: $E_0 = \frac{1}{2}\hbar\omega$

$$E_1 = E_0 + \hbar\omega = \frac{1}{2}\hbar\omega + \hbar\omega = \frac{3}{2}\hbar\omega$$

$$E_2 = E_1 + \hbar\omega = \frac{3}{2}\hbar\omega + \hbar\omega = \frac{5}{2}\hbar\omega$$

$$E_3 = \frac{7}{2}\hbar\omega = \left(3+\frac{1}{2}\right)\hbar\omega, \text{ etc.}$$

$$\boxed{E_n = \left(n+\frac{1}{2}\right)\hbar\omega \quad \text{with } n = 0, 1, 2, 3, \ldots}$$

* NUMBER OPERATOR $\hat{N} = \hat{a}^\dagger\hat{a} = \frac{1}{\hbar\omega}\hat{H} - \frac{1}{2}$

* Since $\hat{H}\varphi_n(x) = \hbar\omega\left(n+\frac{1}{2}\right)\varphi_n(x)$,

$$\hat{N}\varphi_n(x) = \left(\frac{1}{\hbar\omega}\hat{H} - \frac{1}{2}\right)\varphi_n(x) = \frac{1}{\hbar\omega}\hat{H}\varphi_n(x) - \frac{1}{2}\varphi_n(x)$$

$$= \frac{1}{\hbar\omega}\hbar\omega\left(n+\frac{1}{2}\right)\varphi_n(x) - \frac{1}{2}\varphi_n(x)$$

$$= \left(n+\frac{1}{2}\right)\varphi_n(x) - \frac{1}{2}\varphi_n(x)$$

Thus $\boxed{\hat{N}\varphi_n(x) = n\varphi_n(x), \quad \hat{N}|n\rangle = n|n\rangle}$

$\hat{N}$ measures "number of excitations" above the ground state. Hence called "number operator".

$$* \quad \hat{a}^\dagger \varphi_n(x) = c_n \varphi_{n+1}(x)$$

$$\hat{a}^\dagger |n\rangle = c_n |n+1\rangle$$

Let's Find c_n

We want $\langle n|n\rangle = \langle n+1|n+1\rangle = 1$

orthonormalized.

$$1 = \langle n+1|n+1\rangle = \frac{1}{|c_n|^2}\left(\langle n+1| c_n^*\right)\left(c_n |n+1\rangle\right)$$

$$= \frac{1}{|c_n|^2}\left(\langle n|\hat{a}\right)\left(\hat{a}^\dagger |n\rangle\right)$$

$$\left\{ \begin{array}{l} \hat{a}^\dagger |n\rangle = c_n |n\rangle \\ \Rightarrow \langle n|\hat{a} = c_n^* \langle n| \end{array} \right.$$

$$= \frac{1}{|c_n|^2} \langle n| \left(\hat{a}\hat{a}^\dagger\right) |n\rangle$$

$$= \frac{1}{|c_n|^2} \langle n| \left(\hat{N}+1\right) |n\rangle$$

$$\left\{ \begin{array}{l} \left[\hat{a}, \hat{a}^\dagger\right] = 1 \\ \hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1 \\ \qquad = \hat{N} + 1 \end{array} \right.$$

$$= \frac{1}{|c_n|^2} \left\{ \langle n|\hat{N}|n\rangle + \langle n|n\rangle \right\}$$

$$= \frac{1}{|c_n|^2}(n+1) \quad \Rightarrow |c_n| = \sqrt{n+1}$$

Choose c_n to be real:

$$\boxed{\begin{array}{l} \hat{a}^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle \\ \hat{a}^\dagger \varphi_n(x) = \sqrt{n+1}\, \varphi_{n+1}(x) \end{array}} \quad **$$

Similarly, one can show that

$$\boxed{\begin{array}{l} \hat{a}|n\rangle = \sqrt{n}\, |n-1\rangle \\ \hat{a}\varphi_n(x) = \sqrt{n}\, \varphi_{n-1}(x) \end{array}} \quad **$$

(81)

* The eigenstates

$$\hat{a}\,\varphi_0(x) = 0 \Rightarrow \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{\sigma} + \frac{\sigma}{\hbar}\, i\hat{p}\right)\varphi_0(x) = 0$$

$$\Rightarrow \frac{\hbar}{\sigma^2}\, x\,\varphi_0(x) + i(-i\hbar)\frac{d}{dx}\varphi_0(x) = 0$$

$$\Rightarrow \frac{1}{\sigma^2}x\,\varphi_0(x) + \varphi_0'(x) = 0$$

Solution? $\boxed{\varphi_0(x) = N_0\, e^{-\frac{x^2}{2\sigma^2}}}$

Imposing $\int dx\, |\varphi_0(x)|^2 = 1$, $N_0 = \frac{(\pi\sigma^2)^{-1/4}}{\sqrt{2^n n!}}$

$$\Rightarrow \boxed{\varphi_0(x) = \frac{(\pi\sigma^2)^{-1/4}}{\sqrt{2^n n!}}\, e^{-x^2/2\sigma^2}}$$

* Constructing the higher eigenstates:

$$\hat{a}^\dagger|n-1\rangle = \sqrt{(n-1)+1}\,|n\rangle = \sqrt{n}\,|n\rangle \Rightarrow |n\rangle = \frac{1}{\sqrt{n}}\hat{a}^\dagger|n-1\rangle$$

$$\Rightarrow |n\rangle = \frac{1}{\sqrt{n}}\hat{a}^\dagger|n-1\rangle = \frac{1}{\sqrt{n}}\hat{a}^\dagger\frac{1}{\sqrt{n-1}}\hat{a}^\dagger|n-2\rangle$$

$$= \frac{1}{\sqrt{n(n-1)}}(\hat{a}^\dagger)^2|n-2\rangle$$

Continuing, $|n\rangle = \frac{1}{\sqrt{n(n-1)\cdots 2\cdot 1}}(\hat{a}^\dagger)^n|0\rangle$

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle, \text{ or } \varphi_n(x) = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n\varphi_0(x)$$

$$\varphi_1(x) = \frac{1}{\sqrt{1!}}\,\hat{a}^{\dagger}\varphi_0(x) = \left(\frac{1}{\sqrt{2}\,\sigma}\hat{x} - \frac{i\sigma}{\sqrt{2}\,\hbar}\hat{p}\right)\varphi_0(x)$$

$$= \frac{1}{\sqrt{2}\,\sigma}\, x\, \varphi_0(x) - \frac{i\sigma}{\sqrt{2}\,\hbar}(-i\hbar)\frac{d\varphi_0(x)}{dx}$$

$$= \frac{N_0}{\sqrt{2}\,\sigma}\, x\, e^{-x^2/2\sigma^2} - \frac{\sigma N_0}{\sqrt{2}}\frac{d}{dx} e^{-x^2/2\sigma^2}$$

$$= \sqrt{2}\, N_0\, x\, e^{-x^2/2\sigma^2}$$

Similarly can construct $\varphi_2(x), \varphi_3(x), \ldots$

$$\varphi_n(x) = \frac{(\pi\sigma^2)^{-1/4}}{\sqrt{2^n\, n!}}\, H_n\!\left(\frac{x}{\sigma}\right) e^{-x^2/2\sigma^2}$$

$\boxed{\text{Each application of } \hat{a}^{\dagger} = \tfrac{1}{\sqrt{2}}\left(\tfrac{\hat{x}}{\sigma} - \tfrac{i\sigma}{\hbar}\hat{p}\right) \text{ gives a factor of } x\text{, hence polynomials of higher \& higher order}}$

* $\hat{x}$ and $\hat{p}$ in any eigenstate

$$\hat{x} = \frac{\sigma}{\sqrt{2}}\left(\hat{a}^{\dagger} + \hat{a}\right) \rightarrow \langle n|\hat{x}|n\rangle = \frac{\sigma}{\sqrt{2}}\langle n|(\hat{a}^{\dagger} + \hat{a})|n\rangle$$

$$= \frac{\sigma}{\sqrt{2}}\left\{\langle n|\hat{a}^{\dagger}|n\rangle + \langle n|\hat{a}|n\rangle\right\}$$

$$\hat{p} = \frac{i\hbar}{\sqrt{2}\,\sigma}\left(\hat{a}^{\dagger} - \hat{a}\right)$$

$$= \frac{\sigma}{\sqrt{2}}\left\{\langle n|n+1\rangle\sqrt{n+1} + \sqrt{n}\,\langle n|n-1\rangle\right\}$$

Similarly $\langle n|\hat{p}|n\rangle = 0$

$$= \frac{\sigma}{\sqrt{2}}(0 + 0) = 0$$

Stationary state near $x=0$: hence expected, $\langle\hat{x}\rangle = \langle\hat{p}\rangle = 0$

* Using $\hat{X}^2 = \hat{X}\hat{X} = \frac{\sigma^2}{2}(\hat{a}^\dagger+\hat{a})(\hat{a}^\dagger+\hat{a})$

$$= \frac{\sigma^2}{2}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}\right),$$

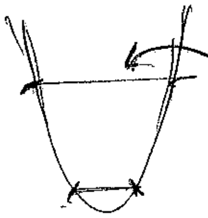
$(\Delta x)^2 = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2$ can be calculated.

In eigenstate $|n\rangle$, $(\Delta x)^2 = \frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right) = \sigma^2\left(n+\frac{1}{2}\right)$

$$\Rightarrow \quad \Delta x = \sigma\sqrt{n+\frac{1}{2}} = \frac{\sigma}{\sqrt{\hbar\omega}}\sqrt{E_n}$$

Higher eigenstates are more spread out.

If energy is higher, than the region where $E > V(x)$ is wider!



wider extent where $E-V(x)$ is positive.

Classically, $E < V(x)$ region is completely forbidden. (negative kinetic energies.) In Q.M., negative $E-V(x)$ region has finite but small probabilities.

* Similarly: $\Delta p = \frac{\hbar}{\sigma}\sqrt{n+\frac{1}{2}} \propto \sqrt{E_n}$

* Uncertainty product: $\Delta x \Delta p = \hbar\left(n+\frac{1}{2}\right) = \frac{E_n}{\omega}$

The smallest value is for the lowest state: $\Delta x \Delta p = \frac{\hbar}{2}$ for ground state.

Consistent with uncertainty principle! $\boxed{\Delta x \Delta p \geq \frac{\hbar}{2}}$

* Solving via power series (outline)

SE: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}m\omega^2x^2\psi = E\psi$

Using $u = \frac{x}{\sigma}$ and $\epsilon = \frac{E}{\hbar\omega/2}$,

$$-\frac{d^2\psi}{du^2} + u^2\psi = \epsilon\psi$$

or $\psi''(u) = (u^2-\epsilon)\psi(u)$

Consider $u \longrightarrow \infty$: $\psi''(u) \simeq \psi(u)\times u^2$

Solution $\psi(u) \approx e^{-u^2}$

Hence try $\psi(u) = s(u)\,e^{-u^2/2}$

Then $s(u)$ satisfies $s''(u) - 2us'(u) + (\epsilon-1)s(u) = 0$

Can solve by power series: $s(u) = \sum_{j=0}^{\infty} a_j u^j$

Substitute into differential equation,

obtain recursion relation: $a_{j+2} = \frac{2j+1-\epsilon}{(j+1)(j+2)}\,a_j$

which leads to the Hermite polynomials.

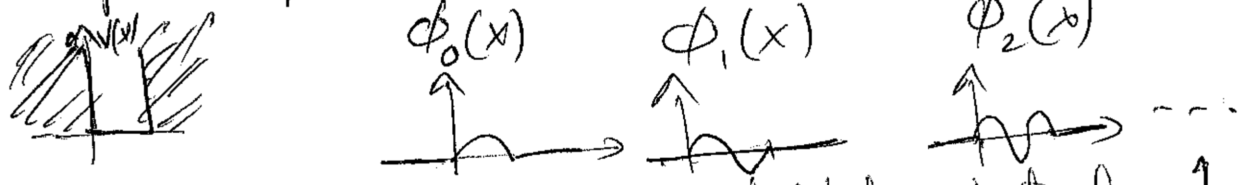
CAUTION: Nash notes defines a, a^+ with other constants	Done with the Harmonic Oscillator

* The NODE THEOREM

For one particle in 1D potentials; bond states:

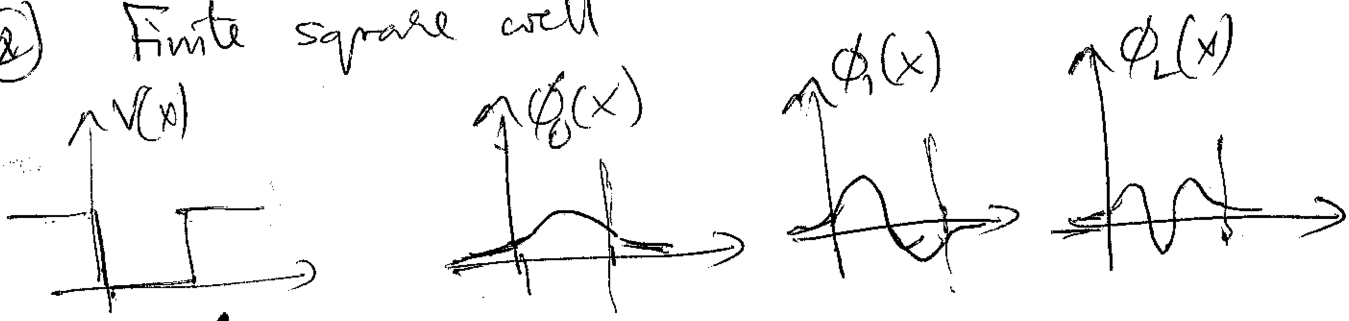
The n-th bond state has n nodes, if the grond state is labeled as $n=0$.

Ex. ① infinite square well

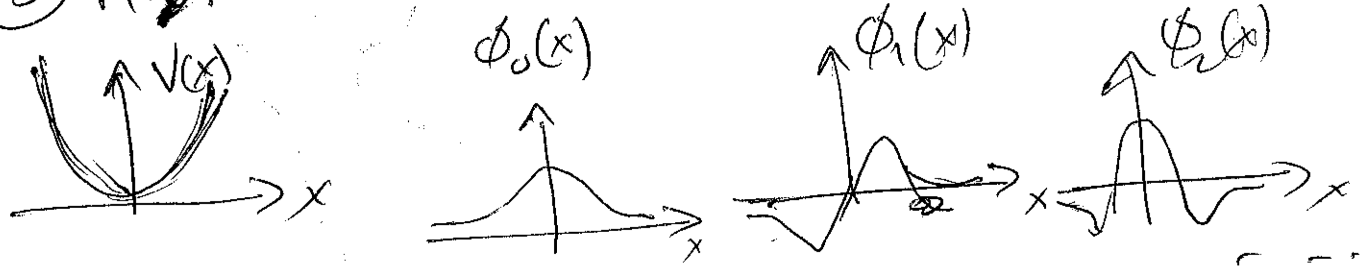


Usual labeling starts from 1.

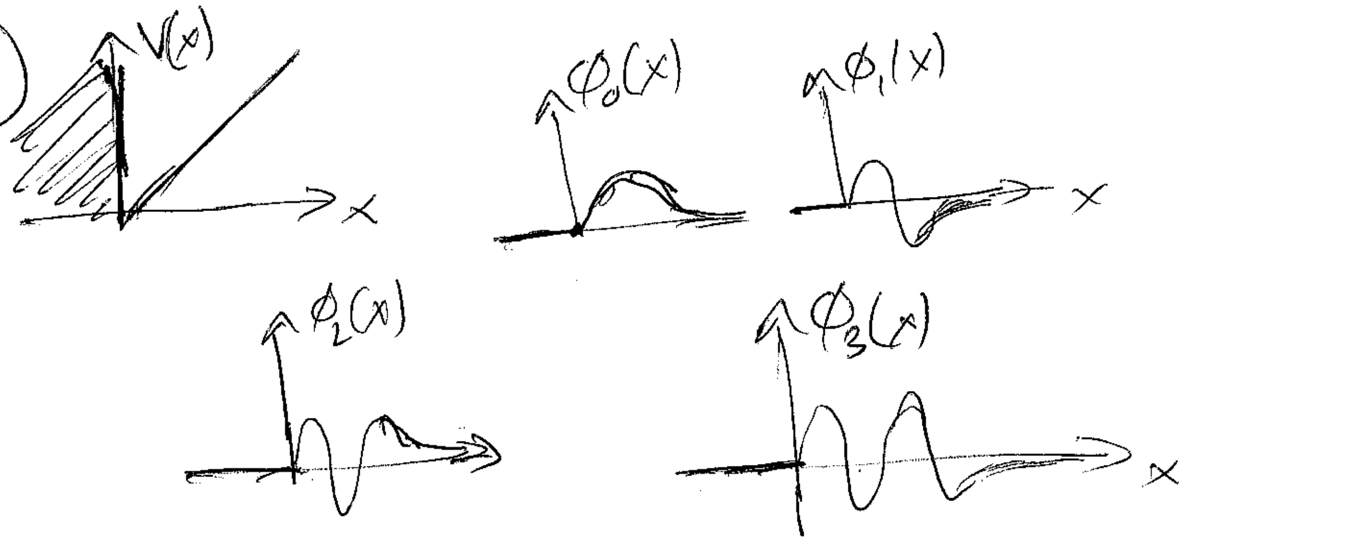
② Finite square well



③ H.O.

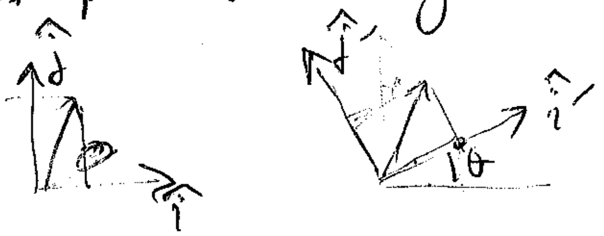


④



* Change/choice of basis ("Representation")

§' In Euclidean space: rotating axis means changing basis:



$$\vec{F} = F_x \hat{i} + F_y \hat{j} = F_x' \hat{i}' + F_y' \hat{j}'$$

$$F_x' = F_x \cos\theta + F_y \sin\theta$$
$$F_y' = -F_x \sin\theta + F_y \cos\theta$$
$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

Both (F_x, F_y) and (F_x', F_y') are valid representations of $\vec{F}$

$= (\vec{F}\cdot\hat{i}, \vec{F}\cdot\hat{j})$ $= (\vec{F}\cdot\hat{i}', \vec{F}\cdot\hat{j}')$

* Similarly, in state space (space of ~~wave functions~~ vectors, coordinates), states can be represented in terms of any complete basis.

* If $\{|\phi_n\rangle\}$ and $\{|W_n\rangle\}$ are two basis, then in the $\{|\phi_n\rangle\}$ basis, $|\psi\rangle$ can be expressed/expanded as

$$|\psi\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle + \cdots$$
$$= \langle\phi_1|\psi\rangle\, |\phi_1\rangle + \langle\phi_2|\psi\rangle\, |\phi_2\rangle + \cdots$$

or as the vector $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \langle \phi_1 | \psi \rangle \\ \langle \phi_2 | \psi \rangle \\ \vdots \end{pmatrix}$

In basis $\{|w_n\rangle\}$, $|\psi\rangle$ is represented as

$$|\psi\rangle = d_1 |w_1\rangle + d_2 |w_2\rangle + \cdots$$

$$= \langle w_1|\psi\rangle |w_1\rangle + \langle w_2|\psi\rangle |w_2\rangle + \cdots$$

or as the vector $\begin{pmatrix} d_1 \\ d_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \langle w_1 | \psi \rangle \\ \langle w_2 | \psi \rangle \\ \vdots \end{pmatrix}$

Coefficients depend on basis!

* Similarly, operators $\hat{A}$ can be represented in either basis :

In $\{|\phi_n\rangle\}$ basis

$$\hat{A} = \sum_{mn} A_{mn}^{(|\phi\rangle)} |\phi_m\rangle\langle\phi_n|$$

$$= \sum_{mn} \langle\phi_m|\hat{A}|\phi_n\rangle |\phi_m\rangle\langle\phi_n|$$

or as the matrix with elements $\langle\phi_m|\hat{A}|\phi_n\rangle$

Also $\hat{A} = \sum_{mn} A_{mn}^{(|w\rangle)} |w_m\rangle\langle w_n|$

$$= \sum_{mn} \langle w_m|\hat{A}|w_n\rangle |w_m\rangle\langle w_n|$$

Matrix elements depend on basis!

* Using $\{|\phi_n\rangle\}$ as basis means defining

$$|\phi_1\rangle = \begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix},\ |\phi_2\rangle = \begin{pmatrix}0\\1\\\vdots\end{pmatrix},\ |\phi_3\rangle = \begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix}, \ldots$$

* **Ex** — Spin-$\frac{1}{2}$ system. $\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}_z$, $\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}_z$

Matrix representation means a basis has already been chosen. Basis used is $|z;+\rangle = \begin{pmatrix}1\\0\end{pmatrix}_z$, $|z;-\rangle = \begin{pmatrix}0\\1\end{pmatrix}_z$ (eigenbasis of $\hat{S}_z$)

Couldn't we chose $|x;+\rangle = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}_z$ and $|x;-\rangle = \begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}_z$ as basis? Then

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}_x \qquad \hat{S}_z = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}_x$$

in this basis.

Elements $\langle x;+|\hat{S}_z|x;-\rangle$ etc.

→ Transformation from one basis to another occurs via <u>unitary</u> transformation.

$$\begin{pmatrix}\langle w_1|\psi\rangle\\\langle w_2|\psi\rangle\\\vdots\end{pmatrix} = \begin{pmatrix} & & \\ & \text{unitary} & \\ & & \end{pmatrix}\begin{pmatrix}\langle\phi_1|\psi\rangle\\\langle\phi_2|\psi\rangle\\\vdots\end{pmatrix}$$

* Single particle in 1D :

$\psi(x)$: what ~~basis~~ basis is this expressed in?

Express as vector ?

$-2\epsilon \quad -\epsilon \quad 0 \quad \epsilon \quad 2\epsilon$

$$\psi(x) \approx \begin{pmatrix} \vdots \\ \psi(-2\epsilon) \\ \psi(-\epsilon) \\ \psi(0) \\ \psi(\epsilon) \\ \psi(2\epsilon) \\ \vdots \end{pmatrix}$$

in the limit $\epsilon \to 0$

Basis ? Each basis vector is located at one position point. The basis ~~states~~ are $\delta(x-y)$.

[Dirac delta functions]

$$\psi(y) = \int dx \; \psi(x) \; \delta(x-y)$$

coeff's   Basis vectors

analog of $|\psi\rangle = \sum_m c_m |\phi_m\rangle$

Operators are infinite matrices.

$$\hat{x} \approx \begin{pmatrix} \ddots & & & & & & \\ & -2\epsilon & & & & 0 & \\ & & -\epsilon & & & & \\ & & & 0 & & & \\ & 0 & & & \epsilon & & \\ & & & & & 2\epsilon & \\ & & & & & & \ddots \end{pmatrix}$$

Diagonal matrix, limit $\epsilon \to 0$

* Can also find <u>countably</u> infinite basis (DISCRETE).

<u>Ex.</u> Take any harmonic potential, $V(x) = \frac{1}{2} m\omega^2 x^2$

or $V(x) = \frac{1}{2} m\omega^2 (x - x_0)^2$, any ω, any x_0.

Then the set of eigenstates $\{ |n\rangle = \varphi_n(x) \}$ can be used as a basis. The potential doesn't have to be physically present.

Ex.: $\hat{x} = \frac{\sigma}{\sqrt{2}} (\hat{a}^\dagger + \hat{a})$

$$\langle m|\hat{x}|n\rangle = \frac{\sigma}{\sqrt{2}} \left(\langle m|\hat{a}^\dagger|n\rangle + \langle m|\hat{a}|n\rangle \right)$$

$$= \frac{\sigma}{\sqrt{2}} \left(\sqrt{n+1}\, \delta_{m,n+1} + \sqrt{n}\, \delta_{m,n-1} \right)$$

Matrix

$$\hat{x} = \frac{\sigma}{\sqrt{2}} \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & \cdots \\ \sqrt{2} & 0 & \sqrt{3} & 0 & \cdots \\ 0 & \sqrt{3} & 0 & \sqrt{4} & \cdots \\ 0 & 0 & \sqrt{4} & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$$