CHAPTER VI

Fourier series

§ 1. Introduction and recapitulation

First let us recall that a periodic function $f(x)$, of period 2π, defined on the interval $[-\pi,\pi]$ can be expanded in the well known Fourier series giving us

$$\begin{aligned} f(x) = a_0 + a_1\cos(x) + a_2\cos(2x) + \cdots + a_n\cos(nx) + \cdots \\ + b_1\sin(x) + b_2\sin(2x) + \cdots + b_n\sin(nx) + \cdots \end{aligned} \tag{6.1}$$

The formula for the coefficients a_n and b_n is that

$$\begin{aligned} a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx,\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx,\ n = 1,2,\ldots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx,\ n = 1,2,\ldots \end{aligned} \tag{6.2}$$

If $f(x)$ is not periodic then it can be redefined to be periodic outside the interval $[-\pi,\pi]$ and so the same formula still applies and we can still write (more concisely this time)

$$f(x) = \sum_{n=0}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx) \tag{6.3}$$

with the same formula for the coefficients, i.e.

$$\begin{aligned} a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx,\ n = 1,2,\ldots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx,\ n = 1,2,\ldots \end{aligned} \tag{6.4}$$

This expansion technique is not limited to functions defined on an interval $[-\pi,\pi]$—it can be applied to functions defined on *any* interval $[a,b]$. In this latter case the formulae above become

$$f(x) = \sum_{n=0}^{\infty} a_n\cos\left(\frac{2\pi nx}{b-a}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{2\pi nx}{b-a}\right) \tag{6.5}$$

and the new coefficients are given by

$$\begin{aligned} a_0 = \frac{1}{b-a}\int_a^b f(x)dx,\ a_n &= \frac{2}{b-a}\int_a^b f(x)\cos\left(\frac{2\pi nx}{b-a}\right)dx,\ n=1,2,\ldots \\ b_n &= \frac{2}{b-a}\int_a^b f(x)\sin\left(\frac{2\pi nx}{b-a}\right)dx,\ n=1,2,\ldots \end{aligned} \tag{6.6}$$

It can readily be verified that these new coefficient formulae revert to the old ones when $[a,b]=[-\pi,\pi]$.

§ 2. Convergence properties of Fourier series

In general the convergence of Fourier series is quite fast and the rate of convergence is related to how *smooth* the function $f(x)$ is; by this we mean how many derivatives of $f(x)$ exist.

We begin our study of convergence by supposing that the first k derivatives $f(x)$ exist—we also denote this more briefly by saying f is C^k. In addition, without loss of generality, we can just deal with the case where $[a,b]=[-\pi,\pi]$.

Now we take the expression for the coefficients a_n and integrate by parts giving us

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \\ &= \frac{1}{\pi}\left[f(x)\frac{\sin(nx)}{n}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\frac{\sin(nx)}{n}dx \\ &= 0 - \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\frac{\sin(nx)}{n}dx \end{aligned} \tag{6.7}$$

But we can repeatedly integrate by parts, *as long as the derivatives of $f(x)$ exist.* So let us do it once more giving

$$\begin{aligned} a_n &= \frac{1}{\pi}\left[f'(x)\frac{\cos(nx)}{n^2}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} f''(x)\frac{\cos(nx)}{n^2}dx \\ &= 0 - \frac{1}{\pi}\int_{-\pi}^{\pi} f''(x)\frac{\cos(nx)}{n^2}dx \end{aligned} \tag{6.8}$$

where the vanishing of the surface term requires a little more thought this time: it vanishes because $f'(x)$, being the derivative of a periodic function is itself periodic and so $f'(-\pi)=f'(\pi)$ hence we have

$$\begin{aligned} \left[f'(x)\frac{\cos(nx)}{n^2}\right]_{-\pi}^{\pi} &= \left\{f'(\pi)\frac{\cos(n\pi)}{n^2} - f'(-\pi)\frac{\cos(-n\pi)}{n^2}\right\} \\ &= \frac{f'(\pi)}{n^2}\left\{\cos(n\pi)-\cos(-n\pi)\right\} \\ &= 0, \quad \text{since } \cos(n\pi)=\cos(-n\pi) \end{aligned} \tag{6.9}$$

To see that $f(x)$ periodic $\Rightarrow f'(x)$ periodic one just differentiates both sides of the Fourier series for $f(x)$, this gives one a Fourier series for $f'(x)$ proving that $f'(x)$ is also periodic. Now we just repeat the integration by parts as many times as we are allowed to do and that is k times if f is C^k. The surface terms will always vanish, as we have just seen, because all derivatives of $f(x)$ are also periodic. Hence after k integrations by parts we find that

$$\begin{gathered} a_n = -\frac{1}{\pi}\int_{-\pi}^{\pi} f^{(k)}(x)\frac{T_k(nx)}{n^k}dx, \quad \text{when } f \text{ is } C^k \\ \text{where } T_k(nx) = \begin{cases} (-1)^{k/2} & \cos(nx), \qquad \text{if } k \text{ is even} \\ (-1)^{(k+1)/2} & \sin(nx), \qquad \text{if } k \text{ is odd} \end{cases} \end{gathered} \tag{6.10}$$

Finally we want to estimate the size of the integral giving a_n. To do this we make use of the fact that for any function $g(x)$ it is clear that

$$\left|\int_a^b g(x)dx\right| \le \int_a^b |g(x)|\,dx \tag{6.11}$$

So, taking $g(x) = f^{(k)}(x)T_k(nx)$, and using the further fact that

$$|f^{(k)}(x)T_k(nx)| \le |f^{(k)}(x)|, \quad \text{because } |T_k(x)| \le 1, \quad (\textit{see its definition}) \tag{6.12}$$

we have the inequality

$$|a_n| \le \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{|f^{(k)}(x)|}{n^k}dx, \quad \text{when } f \text{ is } C^k \tag{6.13}$$

The point now is that if we write

$$C_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |f^{(k)}(x)|dx \tag{6.14}$$

then we have shown that

$$|a_n| \le \frac{C_k}{n^k}, \; n = 1, 2, \ldots \tag{6.15}$$

An almost identical calculation shows that

$$|b_n| \le \frac{C_k}{n^k}, \; n = 1, 2, \ldots \tag{6.16}$$

i.e. the Fourier coefficients a_n and b_n *decrease at least as fast as* n^{-k} *as* $n \to \infty$; this provides excellent and rapid convergence of the Fourier series as long as $k \ge 2$ which is usually the case.

§ 3. Convergence and the Gibbs' phenomenon

There is a very interesting phenomenon of Fourier series which applies to the case where $f(x)$ has a jump discontinuity. This is known as the Gibbs' phenomenon and we shall now give a short account of it.

We begin with a certain function* ϕ which has the simple definition

$$\phi(x) = \begin{cases} -(\pi + x)/2, & -\pi \le x < 0 \\ 0, & x = 0 \\ (\pi - x)/2, & 0 < x \le \pi \end{cases} \tag{6.17}$$

We draw attention to the fact that $\phi(x)$ is discontinuous at $x = 0$ and jumps there from $-\pi$ to π; the graph of $\phi(x)$ is displayed in fig. 1 below.

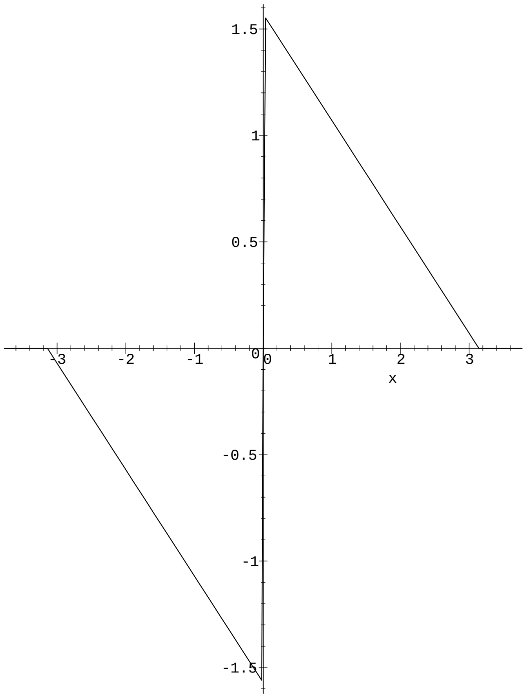

Fig. 1: The function $\phi(x)$

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[*] We shall only give the details for this function $\phi(x)$. It is easy to see that this discussion of $\phi(x)$ also applies to a discontinuity of an arbitrary function $f(x)$. We shall show why this is so at the end.

Now it straightforward to compute the Fourier series for $\phi(x)$ (but we simply quote it here) the result is that

$$\phi(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k} \tag{6.18}$$

Consider the *partial sums* of this series i.e. replace the infinite sum for $\phi(x)$ by the finite sum $S_n(x)$ where

$$S_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k} \tag{6.19}$$

The Gibbs' phenomenon require us to develop a rather special formula for $S_n(x)$ which we at once set about doing.

The next few algebraic steps are somewhat out of the blue but their purpose is to obtain a simple form for $S_n(x)$ and so the reader must be a little patient.

Adding $x/2$ to $S_n(x)$ gives

$$\begin{aligned} \frac{x}{2} + S_n(x) &= \frac{x}{2} + \sum_{k=1}^{n} \frac{\sin(kx)}{k} \\ &= \int_0^x \left[\frac{1}{2} + \sum_{k=1}^{n} \cos(kt) \right] dt \qquad (6.20) \\ \Rightarrow S_n(x) &= -\frac{x}{2} + \int_0^x \left[\frac{1}{2} + \sum_{k=1}^{n} \cos(kt) \right] dt \end{aligned}$$

Leaving this expression 6.20 temporarily we now quote the elementary trigonometrical fact (following from the usual formula for $\sin(A) - \sin(B)$) that

$$\sin(k + 1/2)s - sin(k - 1/2)s = 2\sin(s/2)\cos(ks) \tag{6.21}$$

Now summing both sides 6.21 from $k = 1$ to $k = n$ we find that all except two terms on the LHS mutually cancel (the reader should write out the first few terms to verify this) and we obtain

$$\sin(n + 1/2)s - sin(1/2)s = 2\sin(s/2)\sum_{k=1}^{n}\cos(ks) \tag{6.22}$$

This we rearrange to give

$$\frac{1}{2} + \sum_{k=1}^{n} \cos(ks) = \frac{\sin(n + 1/2)s}{2\sin(s/2)} \tag{6.23}$$

But we can now use 6.23 in 6.20 to give

$$S_n(x) = -\frac{x}{2} + \frac{1}{2}\int_0^x \frac{\sin(n+1/2)t}{\sin(t/2)}\,dt \tag{6.24}$$

Next we add and subtract the quantity $\sin(n+1/2)t/(t/2)$ under the integral and write

$$\begin{aligned}\frac{1}{2}\int_0^x \frac{\sin(n+1/2)t}{\sin(t/2)}\,dt &= \frac{1}{2}\int_0^x \left\{\frac{\sin(n+1/2)t}{(t/2)} + \frac{\sin(n+1/2)t}{\sin(t/2)} - \frac{\sin(n+1/2)t}{(t/2)}\right\}dt \\ &= \frac{1}{2}\left[\int_0^x \frac{\sin(n+1/2)t}{t}\,dt + \int_0^x \left\{\frac{1}{\sin(t/2)} - \frac{1}{(t/2)}\right\}\sin(n+1/2)t\,dt\right]\end{aligned} \tag{6.25}$$

Now we substitute $u = (n+1/2)t$ in the first integral above and the second we simply define to be the function $F(n,x)$. In this way we obtain

$$\begin{aligned}\frac{1}{2}\int_0^x \frac{\sin(n+1/2)t}{(t/2)}\,dt &= \int_0^{(n+1/2)x} \frac{\sin u}{u}\,du \\ F(n,x) &= \frac{1}{2}\int_0^x \left\{\frac{1}{\sin(t/2)} - \frac{1}{(t/2)}\right\}\sin(n+1/2)t\,dt\end{aligned} \tag{6.26}$$

So we have finally,

$$S_n(x) = -\frac{x}{2} + \int_0^{(n+1/2)x} \frac{\sin t}{t}\,dt + F(n,x) \tag{6.27}$$

We are now ready to examine what happens at the point of discontinuity, namely at $x = 0$. We make x tend to zero by the special choice

$$x = \frac{h}{n}, \quad \text{where } h \text{ is a small positive constant} \tag{6.28}$$

Thus $x \to 0$ as $n \to \infty$. With this value of x we have

$$\begin{aligned}S_n(\frac{h}{n}) &= -\frac{h}{2n} + \int_0^{(n+1/2)h/n} \frac{\sin t}{t}\,dt + F(n,\frac{h}{n}) \\ &= -\frac{h}{2n} + \int_0^{(h+h/2n)} \frac{\sin t}{t}\,dt + F(n,\frac{h}{n})\end{aligned} \tag{6.29}$$

We shall now *assume* (though it is true but we do not require the proof here) that

$$\lim_{n\to\infty} F(n,\frac{h}{n}) = 0, \text{ uniformly in } n \tag{6.30}$$

(The reader need not worry about the precise meaning of the term uniform here; the main point for us is that the limit contains no term in h.) Performing the limit on the partial sum $S_n(h/n)$ itself we find that

$$\begin{aligned}\lim_{n\to\infty} S_n(\frac{h}{n}) &= \int_0^h \frac{\sin t}{t}\,dt \\ &= A(h), \quad \text{say}\end{aligned} \tag{6.31}$$

The first point to note is that $A(h)$ which represents the value of $\phi(x)$ at $x = 0$ is not unique but depends on h. This is not too surprising since it is precisely at $x = 0$ that $\phi(x)$ has a discontinuity causing it to jump by the amount π. The Gibbs' phenomenon is the fact the Fourier series for $\phi(x)$ jumps by *more* than $\phi(x)$ does itself.

Let us now see how to establish this: We have seen that the Fourier series for $\phi(x)$ has the value $A(h)$ at $x = 0$ and this depends on h. Hence we define the jump in the Fourier series to be the difference between the *maximum* and *minimum* values of $A(h)$ as h varies. It is convenient to define what is called the Gibbs' interval I by writing

The Gibbs' interval

$$\begin{aligned}&I = [\min A, \max A] \\ &\text{where} \begin{cases}\min A = & \text{minimum value of A(h)} \\ \max A = & \text{maximum value of A(h)}\end{cases}\end{aligned} \tag{6.32}$$

Routine calculus will provide us with the numbers $\min A$ and $\max A$: we have

$$\begin{aligned}A(h) &= \int_0^h \frac{\sin t}{t}\,dt \\ \Rightarrow \frac{d}{dh}A(h) &= \frac{\sin h}{h} \\ \text{and } \frac{\sin h}{h} &= 0 \Rightarrow h = \mp n\pi \\ \text{One then checks that if} &\begin{cases} h = \pi & \text{we have a maximum} \\ h = -\pi & \text{we have a minimum}\end{cases}\end{aligned} \tag{6.33}$$

Hence

$$\begin{aligned}I &= [\int_0^{-\pi} \frac{\sin t}{t}\,dt, \int_0^{\pi} \frac{\sin t}{t}\,dt] \\ &= [-\int_0^{\pi} \frac{\sin t}{t}\,dt, \int_0^{\pi} \frac{\sin t}{t}\,dt]\end{aligned} \tag{6.34}$$

But we can numerically evaluate the integral above and, when we do so we find that

$$\int_0^{\pi} \frac{\sin t}{t}\,dt = \frac{\pi}{2}(1.1789\ \ldots) \tag{6.35}$$

where we have deliberately stated the answer with a factor of $\pi/2$ for convenience below.

Now the function ϕ has a jump at zero of size π and so it is customary to define the quantity G by

$$G = \frac{\text{length of the Gibbs' interval } I}{\text{size of jump in } \phi} \tag{6.36}$$

But the length of I is clearly $2\int_0^h \sin t/t, dt$ so we find that

$$G = \frac{2\int_0^\pi \frac{\sin t}{t}\,dt}{\pi} = 1.1780\ldots \tag{6.37}$$

The relevant point is that $G > 1$ and so the Fourier series at the discontinuity jumps more, or overshoots, at the discontinuity than the function ϕ itself.

At the beginning of this section we said in a footnote that we would show why our discussion of the discontinuity of the rather special function $\phi(x)$ also applies to a discontinuity of any function $f(x)$. Let us now present the argument.

Suppose $f(x)$ is a function on $[-\pi, \pi]$ which has a jump discontinuity at some point x_0, say of size J. Then we can define a new function $g(x)$ by writing

$$f(x) = g(x) + \frac{J}{\pi}\phi(x - x_0) \tag{6.38}$$

The reader can see that we have isolated the discontinuity of $f(x)$ in the second term on the RHS of 6.38; but this means that the function $g(x)$ is *continuous* and therefore the Fourier series of $g(x)$ will have no Gibbs' phenomenon. This in turn means that the only part of $f(x)$ that needs to be expanded in a Fourier series (to understand its Gibbs' phenomenon) is the term $J\phi(x - x_0)/\pi$; but this only involves the function ϕ whose Gibbs' phenomenon we have already analysed.

We finish by showing some plots of $S_n(x)$ and $\phi(x)$ which illustrate well the Gibbs' phenomenon.

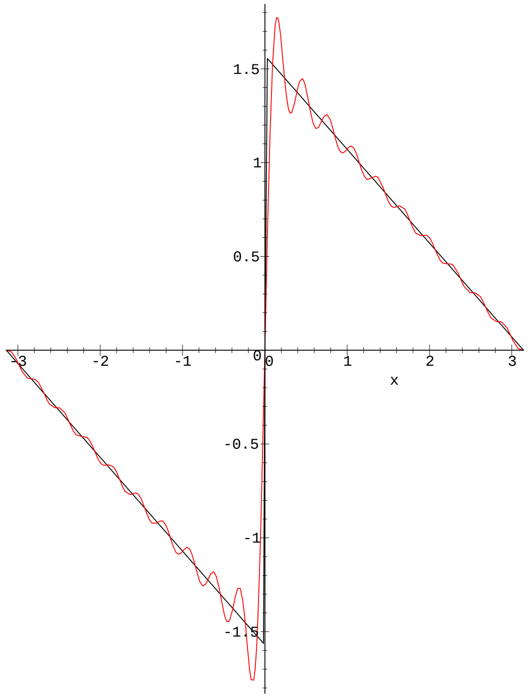

Fig. 2. The partial sum $S_{20}(x)$ and the function $\phi(x)$.

Note that already with only 20 terms of the Fourier series included the function $S_n(x)$ follows the graph of $\phi(x)$ quite well except near $x = 0$.

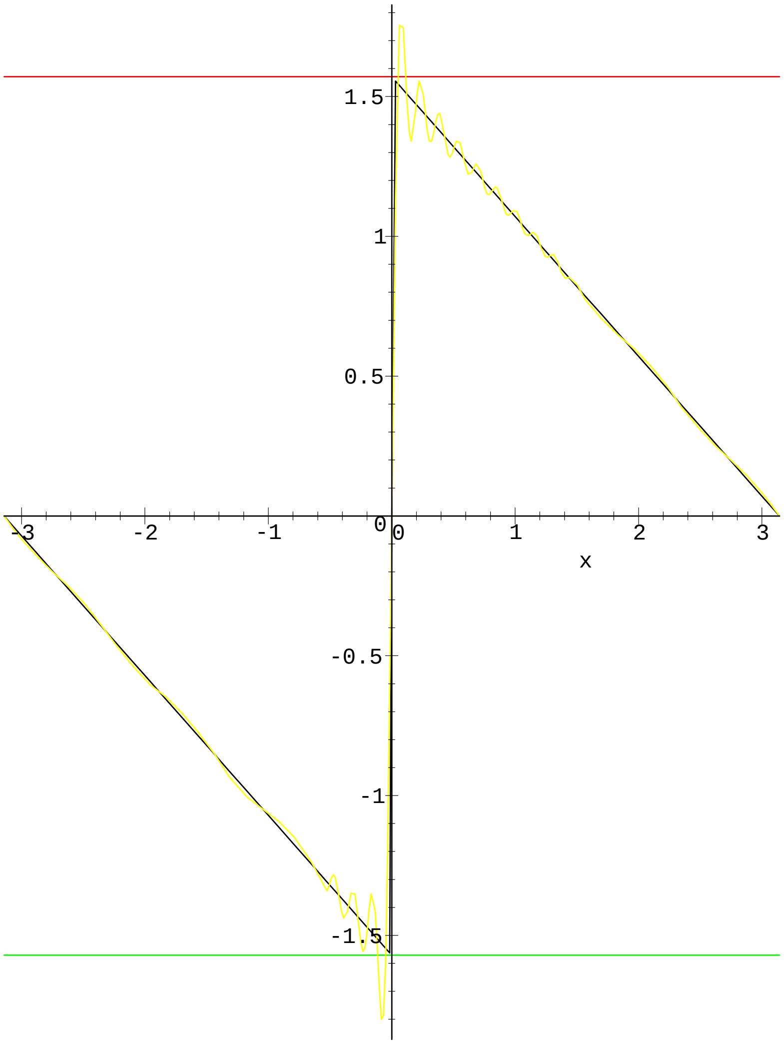

Fig. 3. The partial sum $S_{40}(x)$ and the function $\phi(x)$.

Here that we have included two horizontal lines at $x = \mp\pi/2$; this allows one to see that the Gibbs' overshooting at $x = 0$ is quite plain and is consistent in size with the value given in 6.37.

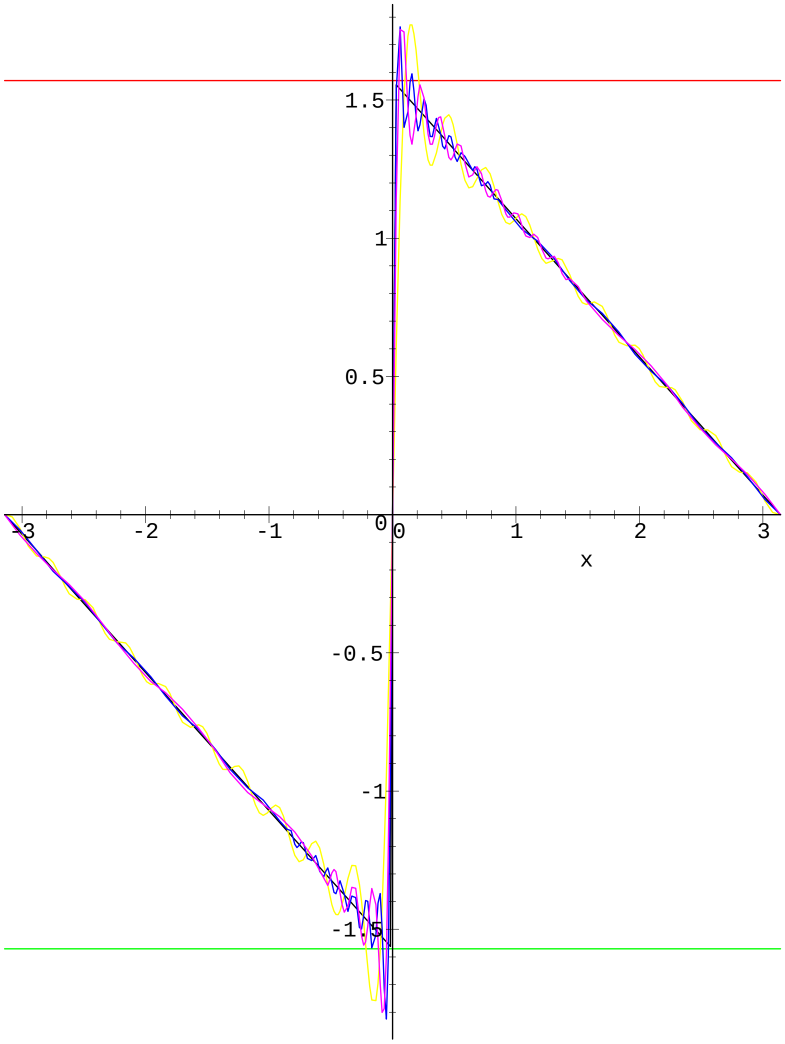

Fig. 4. The function $\phi(x)$ and the partial sums $S_{20}(x)$, $S_{40}(x)$, and $S_{60}(x)$.

In this figure we show a simultaneous plot of several partial sums together with $\phi(x)$. We can see from the graph how the accuracy with which $S_n(x)$ represents $\phi(x)$ improves as n increases; indeed it is very good when $n = 60$. We see too how the Gibbs' phenomenon does not fade away when n increases but persists and gets bigger, tending towards the value given in 6.37.

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