Linear models: initial value problem

(1) **Spring-mass problem: free undamped motion**

Newton's law

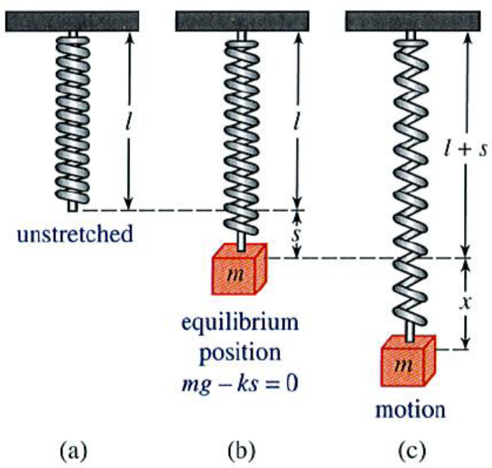
$$F = ma = m\frac{\mathrm{d}v}{\mathrm{d}t} = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

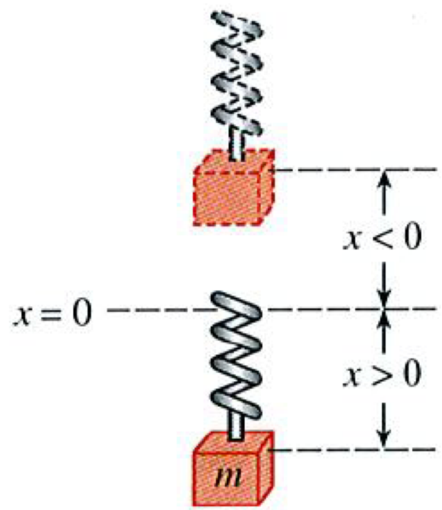
Hook's law

$$F = -kx$$

By putting these two laws together we get the desired ODE

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + kx = 0$$





If we divide the equation by mass m and introduce the angular frequency $\omega = \sqrt{k/m}$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2 x = 0$$

we have a homogeneous linear second-order which describes **simple harmonic motion** or **free undamped motion**.

The initial conditions associated with the DE above are the amount of initial displacement $x(0) = x_0$, and the initial velocity of the mass $x'(0) = x_1$.

To solve the equation, we note that the auxiliary equation $m^2 + \omega^2 = 0$ has two complex roots $m_1 = i\omega$ and $m_2 = -i\omega$, so the general solution is to be

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

We determine c_1 and c_2 from the initial condition and obtain the **equation of motion**.

Example: The equation of motion

$$x(t) = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t$$

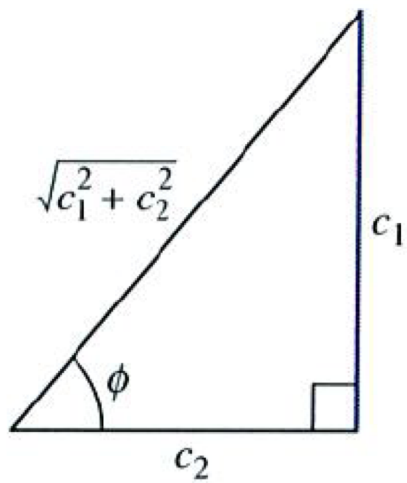
Angular frequency: $\omega = 8$
Period: $T = 2\pi/\omega = 2\pi/8 = \pi/4$
Frequency: $f = 1/T = 4/\pi$

Alternative form of $x(t)$:

$$x(t) = A sin(\omega t + \phi)$$

where $A = \sqrt{c_1 + c_2}$ is the **amplitude** of free vibrations, and ϕ is the **phase angle** defined by

$$\sin\phi = \frac{c_1}{A}, \qquad \cos\phi = \frac{c_2}{A}, \qquad \Rightarrow \qquad \tan\phi = \frac{c_1}{c_2}$$

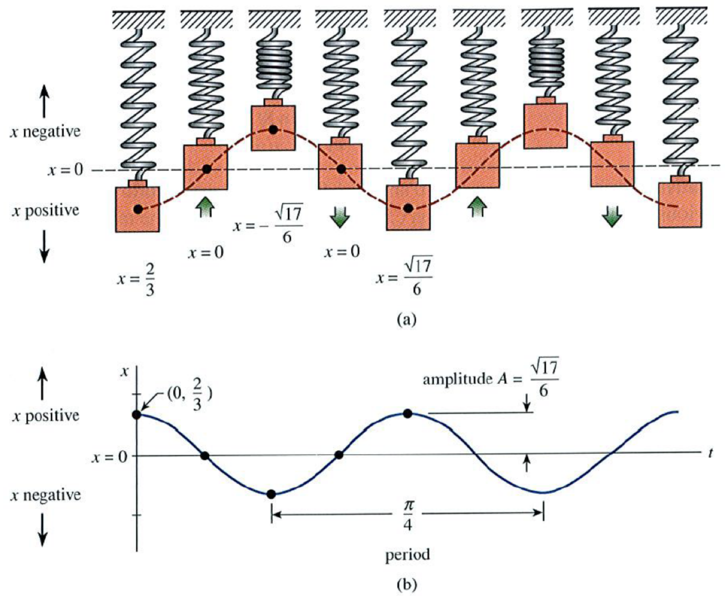


To see the relation between the original solution and its alternative form, we use trigonometry

$$
\begin{aligned}
x(t) &= c_1 \cos \omega t + c_2 \sin \omega t \\
&= A \sin \phi \cos \omega t + A \cos \phi \sin \omega t \\
&= A \sin(\omega t + \phi)
\end{aligned}
$$

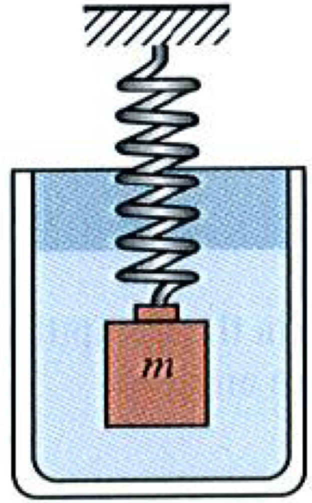
In our specific example, we get

$$
\begin{aligned}
x(t) &= \frac{2}{3} \cos 8t - \frac{1}{6} \sin 8t \\
&= \frac{\sqrt{17}}{6} \sin(8t + 1.816)
\end{aligned}
$$



(2) **Spring-mass problem: free damped motion**

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx - \beta\frac{\mathrm{d}x}{\mathrm{d}t}$$



By dividing by the mass m we get the DE of **free damped motion**:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\beta}{m}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{k}{m}x = 0$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2 x = 0$$

The corresponding auxiliary equation $m^2 + 2\gamma m + \omega^2 = 0$ has the roots

$$m_1 = -\gamma + \sqrt{\gamma^2 - \omega^2} \quad \text{and} \quad m_2 = -\gamma - \sqrt{\gamma^2 - \omega^2}$$

Each solution will contain the **damping factor** $e^{-\gamma t}$, $\gamma > 0$ and thus the displacements of the mass become negligible over time.

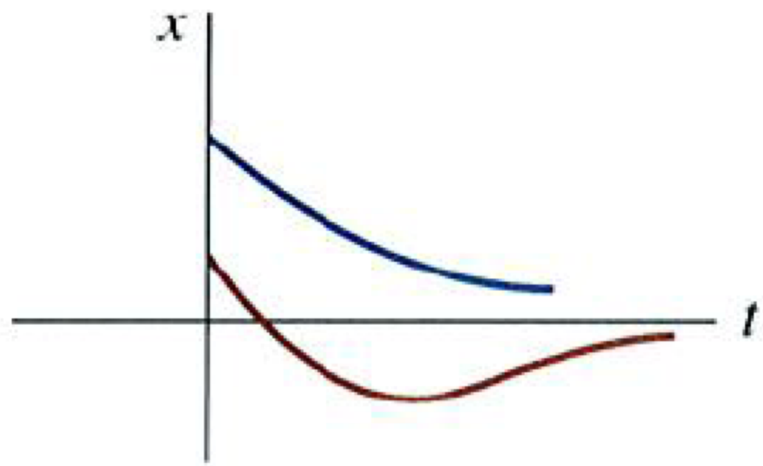
Depending on the algebraic sign of $\gamma^2 - \omega^2$, we distinguish three cases:

- **Case I:** $\gamma^2 - \omega^2 > 0$

  In this case the system is **overdamped**, as the damping coefficient β is large compared to the spring constant k.

  The corresponding solution $x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$ is

$$x(t) = e^{-\gamma t}\left(c_1 e^{\sqrt{\gamma^2-\omega^2}t} + c_2 e^{-\sqrt{\gamma^2-\omega^2}t}\right)$$

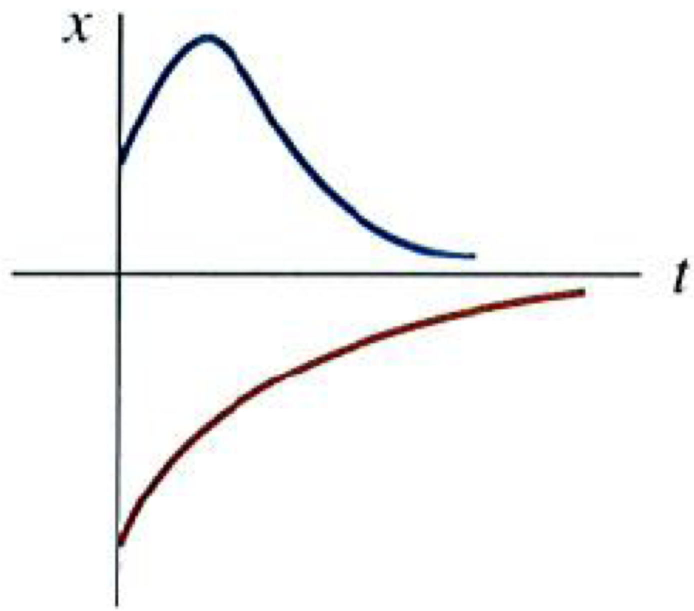


- **Case II:** $\gamma^2 - \omega^2 = 0$

  In this case the system is **critically damped**, because a slight decrease of the damping would result in oscillatory motion.

  The general solution $x(t) = c_1 e^{m_1 t} + c_2 t e^{m_2 t}$ is

$$x(t) = e^{-\gamma t}(c_1 + c_2 t)$$



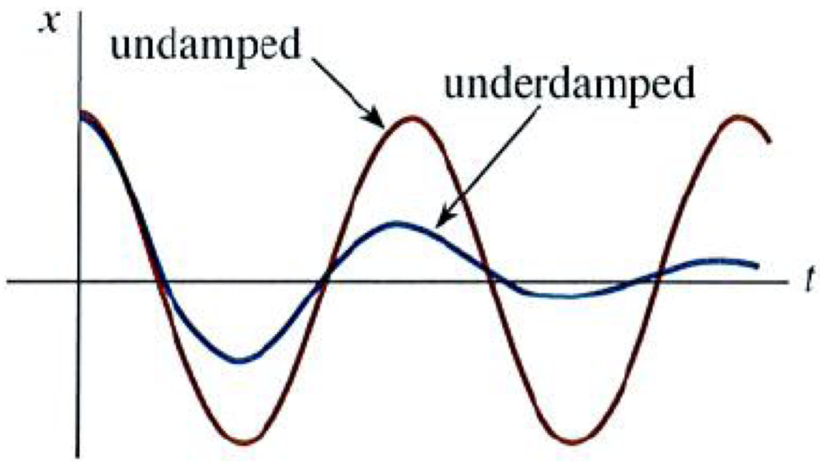
- **Case III:** $\gamma^2 - \omega^2 < 0$

  In this case the system is **underdamped**, as the damping coefficient is small compared to the spring constant. The roots of the auxiliary equation are now complex:

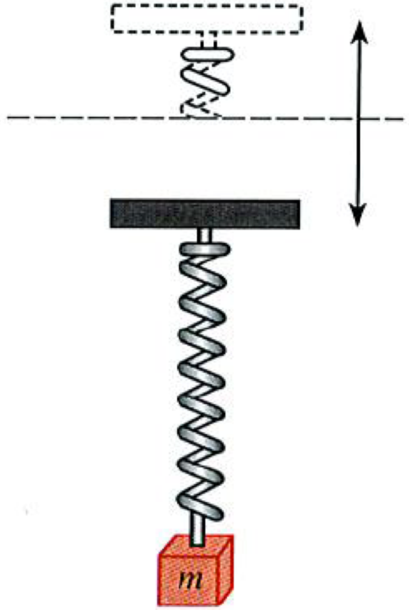
  $$m_1 = -\gamma + i\sqrt{\omega^2 - \gamma^2} \quad \text{and} \quad m_2 = -\gamma - i\sqrt{\omega^2 - \gamma^2}$$

  and thus the general solution is

  $$x(t) = e^{-\gamma t}\left(c_1 \cos\sqrt{\omega^2 - \gamma^2}\, t + c_2 \sin\sqrt{\omega^2 - \gamma^2}\, t\right)$$



(3) **Spring-mass problem: driven motion**



$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t)$$

By dividing by the mass m we get the DE of **driven motion**:

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega^2 x = F(t)$$

which is a nonhomogeneous differential equation whose solution can be obtained either using

- the *method of undetermined coefficients*, or
- the *method of variation of parameters*.

Example: Transient/Steady-state solutions

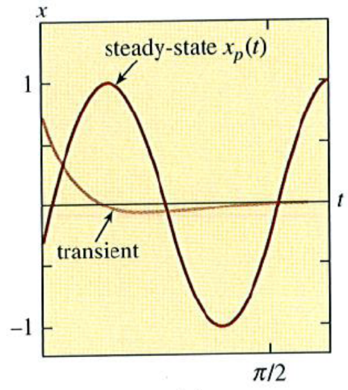
The solution of the IVP

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 10x = 25\cos 4t, \qquad x(0) = \frac{1}{2}, \quad x'(0) = 0$$

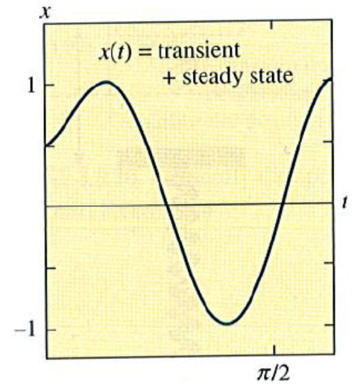
is given by

$$x(t) = x_c + x_p = e^{-3t}\left(\frac{38}{51}\cos t - \frac{86}{51}\sin t\right) - \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t$$

where the first term represents the **transient** solution and the remaining two terms are the **steady state** solution of the IVP.



(a)



(b)

Example: Undamped forced motion

Consider the IVP

$$\frac{d^2x}{dt^2} + \omega_0^2 x = F_0 \sin \omega t, \qquad x(0) = 0, \quad x'(0) = 0$$

the complementary solution is $x_c(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$. We assume the particular solution in the form $x_p = A \cos \omega t + B \sin \omega t$, so that

$$x_p'' + \omega_0^2 x_p = A\left(\omega_0^2 - \omega^2\right) \cos \omega t + B\left(\omega_0^2 - \omega^2\right) \sin \omega t = F_0 \sin \omega t$$

Equating coefficients gives $A = 0$ and $B = F_0/\left(\omega_0^2 - \omega^2\right)$, and thus the general solution is

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{\left(\omega_0^2 - \omega^2\right)} \sin \omega t$$

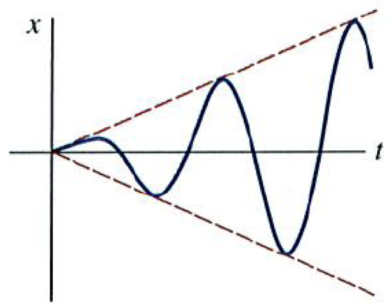
The initial conditions yield $c_1 = 0$ and $c_2 = -\omega F_0/\omega_0\left(\omega_0^2 - \omega^2\right)$, so the solution of the IVP is

$$x(t) \;=\; \frac{F_0}{\omega_0\left(\omega_0^2-\omega^2\right)}\left(-\omega\sin\omega_0 t + \omega_0\sin\omega t\right)$$

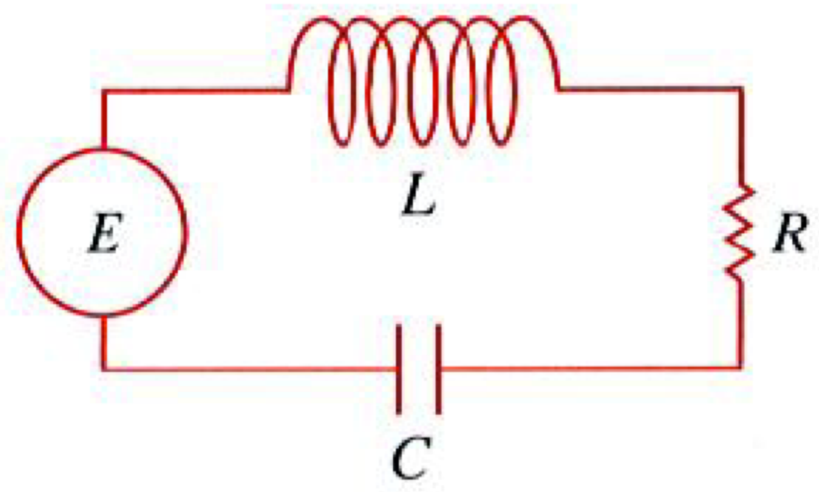
Though the equation is not defined for $\omega = \omega_0$, the limit $\omega \to \omega_0$ can be calculated using he L'Hospital rule giving

$$x(t) \;=\; \lim_{\omega\to\omega_0} F_0\frac{-\omega\sin\omega_0 t + \omega_0\sin\omega t}{\omega_0\left(\omega_0^2-\omega^2\right)} = \frac{F_0}{2\omega_0^2}\sin\omega_0 t - \frac{F_0}{2\omega_0}\,t\;\cos\omega_0 t$$

As time increases, so does the response of the system to the driving and the displacements become large. This is the phenomenon of **pure resonance**.



LRC-series electric circuit



$i(t)$ - the current in a circuit at time t

$q(t)$ - the charge on the capacitor at time t

L - inductance

C - capacitance

R - resistance

According to **Kirchhoff's second law**, the impressed voltage $E(t)$ must equal to the sum of the voltage drops in the loop.

$$V_L + V_C + V_R = E(t)$$

Inductor

$$V_L = L\frac{\mathrm{d}i}{\mathrm{d}t} = L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2}$$

Capacitor

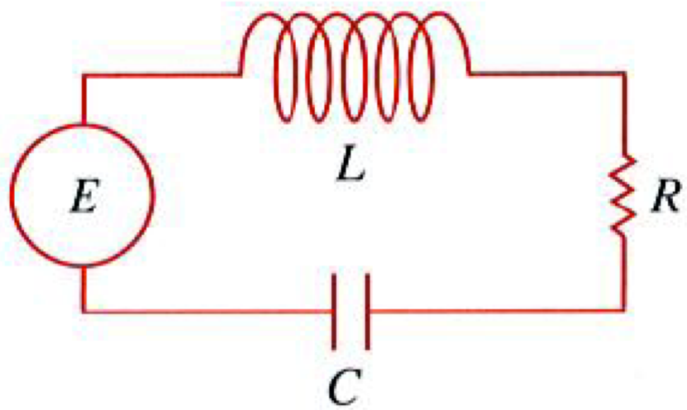
$$V_C = \frac{q}{C}$$

Resistor

$$V_R = Ri = R\frac{\mathrm{d}q}{\mathrm{d}t}$$

LRC circuit

$$L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}q = E(t)$$



Example: LRC circuit

Find the steady-state solution q_p and the **steady-state current** in an LRC-series circuit when the driving voltage is $E(t) = E_0 \sin \omega t$.

The steady-state solution q_p is a particular solution of the differential equation

$$L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}q = E(t)$$

Using the method of undetermined coefficients, we assume the particular solution of the form $q_p(t) = A \sin \omega t + B \cos \omega t$. Substituting this into the DE, simplifying and equating coefficients gives

$$A = \frac{E_0\left(L\omega - \frac{1}{C\omega}\right)}{-\omega\left(L^2\omega^2 - \frac{2L}{C} + \frac{1}{C^2\omega^2} + R^2\right)}, \quad B = \frac{E_0 R}{-\omega\left(L^2\omega^2 - \frac{2L}{C} + \frac{1}{C^2\omega^2} + R^2\right)}$$

It is convenient to express this using the **reactance** $X = L\omega - 1/(C\omega)$ and the **impedance** $Z = \sqrt{X^2 + R^2}$ (both measured in ohms). We get

$$A = \frac{E_0 X}{-\omega Z^2}, \qquad B = \frac{E_0 R}{-\omega Z^2}$$

so the steady state charge is

$$q_p(t) = -\frac{E_0 X}{\omega Z^2} \sin \omega t - \frac{E_0 R}{\omega Z^2} \cos \omega t$$

and the steady-state current $i_p(t) = q_p'(t)$

$$i_p(t) = \frac{E_0}{Z} \left(\frac{R}{Z} \sin \omega t - \frac{X}{Z} \cos \omega t \right)$$