Cauchy-Euler equation

In general, solving a linear equation with **variable coefficients** is more involved than solving a linear equation with **constant coefficients** and typically involves a solution in a form of an infinite series. However, the Cauchy-Euler equation is an exception of which a general solution can always be expressed in terms of powers of x, sines, cosines, logarithmic and exponential functions.

Any linear differential equation of the form

$$a_n\, x^n\, \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}\, x^{n-1}\, \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ \ldots\ + a_1\, x\, \frac{\mathrm{d}y}{\mathrm{d}x} + a_0\, y = g(x)$$

where the coefficients $a_n, a_{n-1}, \ldots, a_0$ are constants is known as **Cauchy-Euler equation**. Notice that the degree $k = n, n-1, \ldots, 1, 0$ of the monomial coefficient x^k matches the order k of the differenciation $\frac{\mathrm{d}^k y}{\mathrm{d}x^k}$.

To guarantee an existence of a unique solution, we will confine our attention to finding the general solution on the interval $(0, \infty)$. Solutions on the interval $(-\infty, 0)$ can be obtained by substituting $t = -x$ into the differential equation.

We will start with examination of the forms of the general solutions of the **homogeneous second-order equation**

$$a\, x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + b\, x \frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0.$$

Method of solution

We try a solution of the form $y = x^m$ where m is to be determined. After substituting $y = x^m$ into a Cauchy-Euler equation, the equation becomes a polynomial in m times x^m:

$$\begin{aligned} a_k\, x^k\, \frac{\mathrm{d}^k y}{\mathrm{d}x^k} &= a_k\, x^k\, m(m-1)(m-2)\ldots(m-k+1)\, x^{m-k} \\ &= a_k\, m(m-1)(m-2)\ldots(m-k+1)\, x^m. \end{aligned}$$

For example, by substituting x^m to the second order equation gives

$$a\, x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + b\, x \frac{\mathrm{d}y}{\mathrm{d}x} + cy = am(m-1)x^m + bmx^m + cx^m = (am(m-1) + bm + c)x^m.$$

Thus $y = x^m$ is a solution of the differential equation whenever m is a solution of the **auxiliary equation**:

$$am(m-1) + bm + c = 0 \qquad \text{or} \qquad am^2 + (b-a)m + c = 0.$$

There are three distinct cases to be considered depending on whether the roots are

(I) real and distinct,
(II) real and equal, or
(III) complex.

Case I: Distinct real roots

Let m_1 and m_2 denote the real roots of the auxiliary equation

$$am^2 + (b - a)m + c = 0$$

such that $m_1 \neq m_2$. Then $y_1 = x^{m_1}$ and $y_2 = x^{m_2}$ form a fundamental set of solutions. Hence the general solution is

$$y = c_1 x^{m_1} + c_2 x^{m_2}.$$

Example: Solve:

$$x^2 \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x \frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 0.$$

Assume $y = x^m$ and differentiate twice

$$\frac{\mathrm{d}y}{\mathrm{d}x} = mx^{m-1}, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = m(m-1)x^{m-2},$$

and substitute to the differential equation

$$\begin{aligned} x^2 \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x \frac{\mathrm{d}y}{\mathrm{d}x} - 4y &= x^2.m(m-1)x^{m-2} - 2x.mx^{m-1} - 4x^m \\ &= x^m(m(m-1) - 2m - 4) = x^m(m^2 - 3m - 4) = 0. \end{aligned}$$

The roots of the equation $m^2 - 3m - 4 = 0$ are $m_1 = -1, m_2 = 4$, so the general solution is

$$y = c_1x^{-1} + c_2x^4.$$

Case II: Repeated real roots

If the roots of the auxiliary equation

$$am^2 + (b - a)m + c = 0$$

are repeated, $m_1 = m_2$, then we obtain only one solution $y = x^{m_1}$. In this case, it follows from the quadratic formula that the root must be $m_1 = -(b - a)/2a$.

We can construct the second solution y_2 using the reduction of order method. We write the Cauchy-Euler equation in the standard form

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{b}{ax}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{c}{ax^2}y = 0$$

with $P(x) = b/ax$ and $\int (b/ax)dx = (b/a)\ln x$.

We get with $P(x) = b/ax$ and $\int(b/ax)dx = (b/a)\ln x$ the following

$$
\begin{aligned}
y_2 &= x^{m_1} \int \frac{e^{-(b/a)\ln x}}{x^{2m_1}} dx \\
&= x^{m_1} \int x^{-b/a}.x^{-2m_1} dx \\
&= x^{m_1} \int x^{-b/a}.x^{(b-a)/a} dx \\
&= x^{m_1} \int \frac{dx}{x} = x^{m_1} \ln x
\end{aligned}
$$

The general solution is then

$$
y = c_1 x^{m_1} + c_2 x^{m_1} \ln x.
$$

For higher-order equations, it can be shown that if m_1 is a root of multiplicity k, then

$$x^{m_1}, x^{m_1} \ln x, x^{m_1} (\ln x)^2, \ldots, x^{m_1} (\ln x)^{k-1},$$

are k linearly independent solutions and the general solution must contain a linear combination of these k solutions.

Example: Solve

$$4\,x^2\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 8x\,\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0.$$

The substitution $y = x^m$ yields

$$4\,x^2\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 8x\,\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^m((4m(m-1) + 8m + 1) = x^m(4m^2 + 4m + 1) = 0$$

when $4m^2 + 4m + 1 = (2m+1)^2 = 0$, so $m_1 = -\frac{1}{2}$ is a repeated root.

The general solution is then

$$y = c_1 x^{-1/2} + c_2 x^{-1/2} \ln x.$$

Case III: Conjugate complex roots

If the roots of the auxiliary equation

$$am^2 + (b - a)m + c = 0$$

are the conjugate pair $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ where α and $\beta > 0$ are real, then a solution is

$$y = C_1 x^{\alpha + i\beta} + C_2 x^{\alpha - i\beta}.$$

However, we wish to write the solution in terms of real functions only.

We use the identity

$$x^{i\beta} = \left(e^{\ln x}\right)^{i\beta} = e^{i\beta \ln x}$$

and using the Euler formula

$$\begin{aligned} x^{i\beta} &= \cos(\beta \ln x) + i \sin(\beta \ln x) \\ \text{and} & \\ x^{-i\beta} &= \cos(\beta \ln x) - i \sin(\beta \ln x) \end{aligned}$$

Adding and subtracting the last two results gives

$$x^{i\beta} + x^{-i\beta} = 2\cos(\beta \ln x) \quad \text{and} \quad x^{i\beta} - x^{-i\beta} = 2i \sin(\beta \ln x),$$

respectively.

Taking into account that $y = C_1 x^{\alpha+i\beta} + C_2 x^{\alpha-i\beta}$ is a solution for any values of the constants, we choose $C_1 = C_2 = 1$ and $C_1 = -C_2 = 1$and get

$$y_1 = x^{\alpha}\left(x^{i\beta} + x^{-i\beta}\right) \quad \text{and} \quad y_1 = x^{\alpha}\left(x^{i\beta} - x^{-i\beta}\right)$$

or

$$y_1 = 2x^{\alpha}\cos(\beta \ln x) \quad \text{and} \quad y_2 = 2ix^{\alpha}\sin(\beta \ln x)$$

are also solutions.

Since $W\left(x^{\alpha}\cos(\beta \ln x), x^{\alpha}\sin(\beta \ln x)\right) = \beta x^{2\alpha-1} \neq 0$, $\beta > 0$, on the interval $(0, \infty)$, we conclude that

$$y_1 = x^{\alpha}\cos(\beta \ln x) \quad \text{and} \quad y_2 = x^{\alpha}\sin(\beta \ln x)$$

constitute a fundamental set of solutions of the differential equation.

The general solution is then

$$y = x^{\alpha} \left[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x) \right].$$

Once we have solved the associated homogeneous equation, we can use **the method of variation of parameters** to find the solution of the original non-homogeneous equation.

Example: Initial value problem

$$4x^2y'' + 17y = 0, \qquad y(1) = -1, y'(1) = -\frac{1}{2}.$$

The solution $y = x^m$ yields

$$4x^2y'' + 17y = x^m(4m(m-1) + 17) = x^m(4m^2 - 4m + 17) = 0$$

when $4m^2 - 4m + 17 = 0$ has two conjugate complex roots $m_1 = \frac{1}{2} + 2i$ and $m_2 = \frac{1}{2} - 2i$, so $\alpha = \frac{1}{2}$ and $\beta = 2$.

The general solution is then

$$y = x^{1/2}\left[c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)\right].$$

By applying the initial conditions we find $c_1 = -1$ and $c_2 = 0$, so the solution of the IVP is

$$y = -x^{1/2} \cos(2 \ln x).$$

Example: Solve

$$x^2y'' - 3xy' + 3y = 2x^4e^x.$$

We first solve the associated homogeneous problem: from the auxiliary equation $(m-1)(m-3) = 0$ we find

$$y_c = c_1x + c_2x^3$$

We assume the particular solution in the form $y_p = u_1y_1 + u_2y_2$ and

$$u_1' = \frac{W_1}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W}$$

where W_1, W_2, and W are the determinants defined earlier and are derived under the assumption that the ODE has been put into the standard form by division by x^2

$$\begin{aligned} y'' + P(x)y' + Q(x)y &= f(x), \\ y'' - \frac{3}{x}y' + \frac{3}{x^2}y &= 2x^2e^x. \end{aligned}$$

With the identification $f(x) = 2x^2e^x$, and $y_1 = x$ and $y_2 = x^3$ we get

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3, \quad W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix} = -2x^5e^x, \quad W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix} = 2x^3e^x.$$

and we find that

$$u_1' = -\frac{2x^5e^x}{2x^3} = -x^2e^x \quad \text{and} \quad u_2' = \frac{2x^3e^x}{2x^3} = e^x.$$

The integral of the last expressions (for u_1 using integration by parts twice), we get the results

$$u_1 = -x^2e^x + 2xe^x - 2e^x \quad \text{and} \quad u_2 = e^x$$

so the particular solution is

$$y_p = u_1y_1 + u_2y_2 = \left(-x^2e^x + 2xe^x - 2e^x\right)x + e^xx^3 = 2x^2e^x - 2xe^x.$$

The general solution is then

$$y = y_c + y_p = c_1x + c_2x^3 + 2x^2e^x - 2xe^x.$$

Remark:

There is a similarity between the form of solutions of Cauchy-Euler equations and solutions of linear equations with constant coefficients.

For example when the roots of auxiliary equations for $ay'' + by' + cy = 0$ and $ax^2y'' + bxy' + cy = 0$ are distinct and real, the respective general solutions are

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \quad \text{and} \quad y = c_1 x^{m_1} + c_2 x^{m_2}, x > 0.$$

As a consequence of the identity $e^{\ln x} = x, x > 0$, the second solution can be expressed in the same form as the first solution:

$$y = c_1 e^{m_1 \ln x} + c_2 e^{m_2 \ln x} = c_1 e^{m_1 t} + c_2 e^{m_2 t},$$

where $t = \ln x$.

Any Cauchy-Euler equation can always be rewritten as a linear differential equation with constant coefficients by means of the substitution $x = e^t$. The new equation can be solved in terms of the variable t, and once the general solution is obtained, resubstitute $t = \ln x$.